Exercise 1. Consider the 3-sphere $S^3 \subset \mathbb{R}^4$. Using the isomorphism $\mathbb{R}^4 \cong \mathbb{C}^2$, we obtain the inclusion $\iota : S^3 \to \mathbb{C}^2 \setminus \{0\}$. Composing with the projection map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$, we obtain

$$p = \pi \circ \iota : S^3 \to \mathbb{C}P^1,$$

known as the "Hopf fibration".

Using the coordinate charts given in class for S^3 and $\mathbb{C}P^1$, compute p in coordinates (one chart on each of the domain and codomain should suffice). Also, describe explicitly the preimage of a pair of points in $\mathbb{C}P^1$ under p, using a coordinate chart for S^3.

Exercise 2. Consider the function $f = \frac{1}{2}(|x|^2 - |y|^2)$ on $\mathbb{R}^{p+q}$, where $x = (x^1, \dots, x^p)$ and $y = (y^1, \dots, y^q)$ together form a standard coordinate system. Describe explicitly all paths of steepest descent as parametrized curves. Recall that a path of steepest descent is a path $\gamma : \mathbb{R} \to X$ which is an integral curve for the vector field $V = -\nabla f$, i.e. it satisfies

$$\dot{\gamma} = -\nabla f(\gamma). \tag{1}$$

Plot these paths for all (p, q) such that $p + q = 2$.

Exercise 3. Let X be the 2-dimensional torus, expressed as $\mathbb{R}^2/\Gamma$, where Γ is the translation action of $\mathbb{Z}^2$, i.e. $(a, b) \cdot (x, y) = (x + a, y + b)$. In standard coordinates (x, y) for $\mathbb{R}^2$, consider the function

$$h(x, y) = \sin^2(\pi x) + \sin^2(\pi y).$$

i) Determine the critical points of the function h.

ii) Compute the Hessian at each critical point and determine its signature.

iii) Determine and draw the paths of steepest descent as in the exercise above.

iv) Describe how the above responses change when h is replaced with

$$h(x, y) = \sin^2(2\pi x) + \sin^2(\pi y).$$

Exercise 4. Let $X = \mathbb{C}P^n$, with projective coordinates $[z_0 : \cdots : z_n]$. Fix real numbers $\lambda_0 < \ldots < \lambda_n$ and define the function

$$f([z_0 : \cdots : z_n]) = \frac{\sum_{i=0}^n \lambda_i |z_i|^2}{\sum_{i=0}^n |z_i|^2}.$$

As above, determine the critical points as well as Hessian signatures. Give a qualitiative description of the paths of steepest descent. Does the above work for real projective space as well? Plot the paths of steepest descent for $\mathbb{R}P^2$. It is possible to work on the sphere S^2 using the usual notion of gradient for functions on S^2 (this uses the usual metric on the sphere).

Exercise 5. Recall that $\mathsf{Gr}(k, V)$ is the set of k-dimensional subspaces of the vector space V. Normally we take $V = \mathbb{C}^n$ and write $\mathsf{Gr}(k, n)$. We will focus on $\mathsf{Gr}(2, 4)$ in this exercise. Fix a basis (e_1, e_2, e_3, e_4) for the purpose of defining a full flag F in V, meaning a sequence of subspaces

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = V,$$

where $F_i = \mathrm{Span}(e_1, \dots, e_i)$. Any point $\Lambda \in \mathsf{Gr}(2, 4)$ may then be intersected with the flag, giving subspaces

$$\Lambda \cap F_1 \subset \Lambda \cap F_2 \subset \Lambda \cap F_3 \subset \Lambda. \tag{2}$$

For most (or an open dense set of, or generic choice of) Λ, the first two subspaces in the above sequence would be zero, and $\Lambda \cap F_3$ would be 1-dimensional. This set of Λ is called $W_{0,0}$, the largest "Schubert cell" and is the "usual case". In general, for special elements $\Lambda \in \mathsf{Gr}(2, 4)$, the 1-dimensional intersection in (2) may happen a_1 places earlier than usual, and the 2-dimensional intersection may happen a_2 places earlier than usual, where $a_1 \geq a_2$. So we may define subsets W_{a_1,a_2} for any such pair (a_1, a_2). For example,

$$W_{1,1} = \{\Lambda \in \mathsf{Gr}(2, 4) \ : \ \dim(\Lambda \cap F_1) = 0 \text{ and } \dim(\Lambda \cap F_2) = 1 \text{ and } \dim(\Lambda \cap F_3) = 2.\}$$

i) A point $\Lambda \in \mathsf{Gr}(2, 4)$ may be described by giving a spanning pair of vectors for the 2-dimensional subspace it represents. We express this pair of vectors in terms of the chosen basis above, and arrange them as the rows of a 2×4 matrix

$$\lambda = \begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix},$$

with the understanding that two matrices λ, λ' represent the same element of the Grassmannian when $\lambda = g\lambda'$ for some $g \in \mathsf{GL}(2, \mathbb{C})$. Prove that $W_{1,1}$ is in bijection with matrices of the form

$$\lambda = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix},$$

and obtain analogous statements for the other Schubert cells $W_{a,b}$, including the largest one.

ii) There is a partial order on the Schubert cells, so that $W_{a,b} < W_{c,d}$ when $W_{a,b}$ is contained in the closure of $W_{c,d}$. Determine what the order relation is between the Schubert cells in $\mathsf{Gr}(2, 4)$ and draw the associated Hasse diagram.

iii) By projectivizing $\mathbb{C}^4$ to produce $\mathbb{C}P^3$, we see that 2-dimensional subspaces $\Lambda \in \mathbb{C}^4$ correspond to projective lines in the 3-dimensional projective space $\mathbb{C}P^3$. In this way we see that $\mathsf{Gr}(2, 4)$ may be viewed as the space $\mathbb{G}(1, 3)$ of lines in $\mathbb{C}P^3$. The flag F then becomes a projective flag, consisting of a choice of point p, projective line ℓ, and projective plane P in $\mathbb{C}P^3$ such that $p \in \ell$ and $\ell \subset P$. Give a geometric description of the *closure* of each of the Schubert cells in $\mathsf{Gr}(2, 4)$, using this projective model.