Vector fields

Exercise 1. Consider the smooth vector field $v = x^k \frac{\partial}{\partial x}$, $k \geq 0$ on the real line $X = \mathbb{R}$. The flow $\Phi_v(x,t)$ of the vector field v for time t starting at $x \in X$ is defined for (x,t) in an open subset $U \subset X \times \mathbb{R}$. Determine this open set precisely for each k.

Exercise 2. Let v be a vector field on the manifold M, and suppose it vanishes at the point $p \in M$. In coordinates $(x^1, \ldots, x^n)$ centered at p, we may write

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i},$$

and the following expression defines an endomorphism of T_pM:

$$d_p v = \sum_{i=1}^{n} (dv^i)|_p \otimes \left.\frac{\partial}{\partial x^i}\right|_p \in T_p^*M \otimes T_pM.$$

Prove that $d_p v$ does not depend on the choice of coordinates centered at p.

Exercise 3. Let v be a vector field on $M = \mathbb{R}^2$ with an isolated zero at the origin. For a sufficiently small circle $\gamma(t) = \varepsilon e^{it}$, the normalized vector field

$$\sigma(t) = \frac{v(\gamma(t))}{|v(\gamma(t))|} \tag{1}$$

defines a map $S^1 \to S^1$. The winding number of this map is called the *index* of the vector field at the origin.

1. Provide an explicit family of vector fields v_k on the plane with index k at the origin for $k \in \mathbb{Z}$.
2. Given a continuous family v_t of vector fields on $\mathbb{R}^2$ parametrized by $t \in \mathbb{R}$, such that v_t always has a single zero in the unit disc at the origin, prove that the index remains constant in the family. [This requires a basic understanding of what the fundamental group is, in particular the fact $\pi_1(S^1) = \mathbb{Z}$.]
3. Suppose that the vector field v on $\mathbb{R}^2$ is nonvanishing on the unit circle $\gamma(t) = e^{it}$, and suppose that the winding number of the map (1) is nonzero. Prove that v must have a zero somewhere in the unit disc.
4. Use the above to prove that S^2 cannot have a nowhere-vanishing vector field. Use the description of S^2 and its tangent bundle in terms of a pair of stereographic charts.

Transversality

Vector subspaces U, V of W are *transverse* when $U + V = W$. Two submanifolds K, L of the manifold M intersect *transversally* if at each point $p \in K \cap L$, the tangent spaces T_pK and T_pL are transverse in T_pM.

Exercise 4. Prove that if the submanifolds K, L of M intersect transversally, then $K \cap L$ is also a submanifold. Also, determine the dimension of the intersection.

For each $k = 0, 1, \ldots$ give an example of two transversally intersecting submanifolds L, K of $S^1 \times S^1$ which intersect in exactly k points.

Exercise 5. Sard's theorem states that for any smooth map, the set of critical *values* has measure zero in the codomain. In other words, the regular values are dense. Recall that for a point y in the codomain of f to be regular, each point in the preimage $f^{-1}(y)$ must be regular, i.e. have surjective derivative. (Important point: if $f^{-1}(y)$ is empty, then y is regular!).

1. If $f : M \to M$ is a smooth map from a compact manifold to itself, prove that there must be a point $y \in M$ with $f^{-1}(y)$ finite.

2. If $f : M \to S^n$ is a smooth map and $\dim M < n$, prove that f is smoothly homotopic to a constant map. 'Smoothly homotopic' in this case would mean that you have a smooth map

$$F : M \times [0, 1] \to S^n$$

with $F(-, 0) = f(-)$ and $F(-, 1)$ being a constant map.

Exercise 6. We say that a smooth map $f : K \to M$ is transverse to the submanifold $L \subset M$ if $Df(T_pK) + T_{f(p)}L = T_{f(p)}M$ for all $p \in f^{-1}(L)$. If f were an embedding of the submanifold K, we would recover the usual notion of transversality.

Let S be another manifold (think of it as a parameter space) and suppose that $F : K \times S \to M$ is a smooth map which is transverse to L. We would like to know if the individual maps $F(-, s) : K \to M$, where s is fixed, are transverse to L.

1. Prove that $Q = F^{-1}(L)$ is a smooth submanifold of $K \times S$.

2. Let $\pi : Q \to S$ be the projection map. Prove that if s is a regular value for π, then $F(-, s) : K \to M$ is transverse to L. Conclude that $F(-, s) : K \to M$ is transverse to L for almost all s.

Exercise 7. Let f be a smooth real-valued function on the compact manifold M such that df is transverse to the zero section, meaning that the image of the section $df \in \Gamma(M, T^*M)$ in T^*M defines a submanifold which intersects the image of the zero section transversally. Prove that f has finitely many critical points, at each of which its Hessian is nondegenerate.