Morse theory of the Grassmannian

Recall that $\mathrm{Gr}(2,4)$ is the Grassmannian of 2-dimensional complex linear subspaces $\Lambda \subset \mathbb{C}^4$. We have a natural Hermitian inner product on $\mathbb{C}^4$ given by

$$\langle x, y\rangle = x \cdot \overline{y} = \sum x_i \overline{y_i}. \tag{1}$$

The symmetry group of this inner product is the unitary group

$$U(4) = \{X \in \mathrm{GL}(4,\mathbb{C}) \ : \ XX^* = \mathbf{1}\}, \tag{2}$$

where $X^* = \overline{X}^\top$.

Exercise 1. *Prove that* $U(4)$ *acts transitively on* $\mathrm{Gr}(2,4)$ *and determine the stabilizer subgroup associated to the point* $\Lambda_0 = \mathrm{Span}(e_1, e_2)$*.*

The Lie algebra of $U(4)$, i.e. the tangent space at the identity element, is the real vector space (of what dimension?) of skew-adjoint matrices

$$\mathfrak{u}(4) = \{X \in \mathrm{End}(\mathbb{C}^4) \ : \ A + A^* = 0\}. \tag{3}$$

This is called a Lie algebra because the commutator $[A_1, A_2]$ of two elements of $\mathfrak{u}(4)$ is also in $\mathfrak{u}(4)$. There is a natural embedding of $\mathrm{Gr}(2,4)$ inside $\mathfrak{u}(4)$ which goes as follows:

Any $\Lambda \in \mathrm{Gr}(2,4)$ has an orthogonal complement $\Lambda^\perp$ using the Hermitian inner product, and from the decomposition of any element $x \in \mathbb{C}^4$ as $x = x_\Lambda + x_{\Lambda^\perp}$ with $x_\Lambda \in \Lambda$ and $x_{\Lambda^\perp} \in \Lambda^\perp$, we can define a projection operator $P_\Lambda : \mathbb{C}^4 \to \Lambda$ by

$$P_\Lambda : x \mapsto x_\Lambda. \tag{4}$$

Exercise 2. *Prove that* $\iota : \Lambda \mapsto iP_\Lambda$ *is an embedding of* $\mathrm{Gr}(2,4)$ *into* $\mathfrak{u}(4)$*, and that for* $g \in U(4)$*, we have*

$$\iota(g \cdot \Lambda) = g\iota(\Lambda)g^{-1}, \tag{5}$$

where on the left hand side we have the usual action of g *on a subspace of* $\mathbb{C}^4$ *and on the right hand side we have the conjugation action of* $U(4)$ *on* $\mathfrak{u}(4)$*, known as the adjoint action. Conclude that* $\iota(\mathrm{Gr}(2,4))$ *consists of a single adjoint orbit* $\mathcal{O}$ *and describe the point* $\iota(\Lambda_0)$ *in this orbit.*

Let $\mathfrak{h} \subset \mathfrak{u}(4)$ be the diagonal skew-adjoint matrices (a 4-dimensional real subspace). Elements in $\mathfrak{h}$ with distinct eigenvalues are called *regular* elements of $\mathfrak{h}$, and the others are called *singular*. Regular elements have the following nice property: any element of $\mathfrak{u}(4)$ which commutes with a regular diagonal element must itself be diagonal.

Exercise 3. *Let* $\mathfrak{h} \subset \mathfrak{u}(4)$ *be the diagonal matrices (a 4-dimensional real subspace). Determine the intersection of* $\mathrm{Gr}(2,4) \cong \mathcal{O}$ *with* $\mathfrak{h}$*.*

The real Lie algebra $\mathfrak{u}(4)$ has a Euclidean inner product

$$\langle A, B \rangle = \mathrm{Tr}(AB^*) = -\mathrm{Tr}(AB). \tag{6}$$

This inner product is invariant under the adjoint action of $U(4)$, and also satisfies

$$\langle [A, B], C \rangle + \langle B, [A, C] \rangle = 0,$$

for all $A, B, C \in \mathfrak{u}(4)$. We define a Morse function on the Grassmannian as follows: Choose real numbers $a_1 > \cdots > a_4$, so that $Z = \mathrm{diag}(ia_1, \ldots ia_4)$ is a regular element in $\mathfrak{h}$. Then we define

$$f(\Lambda) = \langle \iota(\Lambda), Z \rangle \, . \tag{7}$$

Exercise 4. *Prove that f is the restriction of the square of the distance function on $\mathfrak{u}(4)$ to the adjoint orbit $\mathcal{O} \cong \mathrm{Gr}(2,4)$: use the polarization identity to prove that*

$$f(\Lambda) = -\tfrac{1}{2} || \iota(\Lambda) - Z ||^2 + C, \tag{8}$$

for some constant C independent of Λ.

Exercise 5. *Use the above results to determine the critical points of f; a fact which will be useful is the following: the tangent space to $\mathcal{O}$ at a point $X \in \mathcal{O}$, translated to the origin of $\mathfrak{u}(4)$, coincides with the image of the infinitesimal adjoint action:*

$$T_X \mathcal{O} = \{ [X, Y] \ : \ Y \in \mathfrak{u}(4) \}. \tag{9}$$

Bonus 1. *Determine the Morse indices of the critical points of f.*

Bonus 2. *Determine the descending manifolds of f.*