Due date: **October 1, 2009**
Warning: Start your work early, as late assignments are not accepted. You are encouraged to discuss the problems with the class. Please ask me to clarify any questions which you find confusing.

Exercise 1.

i) Let $\tilde{M} = S^n$ and define the equivalence relation $x \sim -x$. Show that the quotient $M = \tilde{M}/\sim$ is a topological manifold which inherits a unique smooth structure such that $q : \tilde{M} \longrightarrow M$ is smooth. Show that M is diffeomorphic to $\mathbb{R}P^n$ as defined in class.

ii) Let $\tilde{M} = \mathbb{C}^n - \{0\}$ and define the equivalence relation $x \sim y$ if and only if $y = 2^k x$ for $k \in \mathbb{Z}$. Prove that the quotient $M = \tilde{M}/\sim$ inherits a unique smooth structure for which the quotient map is smooth; show also that M is diffeomorphic to $S^{2n-1} \times S^1$.

Exercise 2. Consider the 3-sphere $S^3 \subset \mathbb{R}^4$. Using the isomorphism $\mathbb{R}^4 \cong \mathbb{C}^2$, we obtain the inclusion $\iota : S^3 \longrightarrow \mathbb{C}^2 - \{0\}$. Composing with the projection map $\pi : \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{C}P^1$, we obtain

$$p = \pi \circ \iota : S^3 \longrightarrow \mathbb{C}P^1,$$

known as the "Hopf fibration".

i) Using the coordinate charts given in class for S^3 and $\mathbb{C}P^1$, give a complete description of p in coordinates, and verify that it is indeed smooth [1]

ii) Show that $(S^3, p, \mathbb{C}P^1)$ is a fiber bundle.

Exercise 3. Begin with a countable collection $\mathcal{A} = \{U_i\}$ of open subsets $U_i \subset \mathbb{R}^n$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and "gluing" maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \ , \tag{1}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be *homeomorphisms*. Next, we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij} \cap U_{ik}}$ for all i, j, k. Finally, we require that $\forall p \in \partial U_{ij} \subset U_i$ and $\forall q \in \partial U_{ji} \subset U_j$, there exist neighbourhoods $V_p \subset U_i$ and $V_q \subset U_j$ of p, q respectively with $\varphi_{ij}(V_p \cap U_{ij}) \cap V_q = \emptyset$.

Then form the topological quotient

$$M = \frac{\coprod U_i}{\sim}, \tag{2}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, and for all i, j.

i) Prove that M inherits a unique topological n-manifold structure for which the quotient map is continuous, and that it has a distinguished atlas of coordinate charts, given by the inclusion maps $\iota_i : U_i \longrightarrow \mathbb{R}^n$.

ii) Prove that any topological manifold is homeomorphic to one constructed in the above way. Therefore we have a general method of constructing any topological manifold out of pieces.

iii) How would we modify the construction to guarantee that M inherits a unique smooth structure such that the quotient map is smooth? Would we also obtain a distinguished smooth atlas?

Exercise 4.

i) Give examples of compact, connected smooth 3-manifolds with boundary X_1, X_2 such that $\partial X_1 = S^2 \sqcup T^2$ and $\partial X_2 = S^2 \sqcup S^2 \sqcup S^2$.

ii) Prove that $S^2, S^1 \times S^1, S^3$, and $S^1 \times S^2$ are each null-cobordant, i.e. cobordant with the empty set.

Bonus 1. Prove that $\mathbb{R}P^3$ is null-cobordant.

[1] there are other ways of showing this map is smooth, but it is useful to be able to do things in coordinates.