Due date: **October 20, 2009**
Warning: Start your work early, as late assignments are not accepted. You are encouraged to discuss the problems with the class. Please ask me to clarify any questions which you find confusing.

Chapter 3 in Lee is a particularly thorough treatment of the tangent bundle. I recommend reading it over and over. Make sure you understand the section "computations in coordinates" which is very useful.

Exercise 1.

i) Prove that the group $SO(3,\mathbb{R})$ of 3×3 real special orthogonal matrices, i.e. $SO(3,\mathbb{R}) = \{T \in SL(3,\mathbb{R}) \ : \ TT^\top = 1\}$ is a smooth submanifold of the vector space of 3×3 matrices.

ii) Consider the subset of $T\mathbb{R}^3$ consisting of the vectors tangent to the 2-sphere $S^2 \subset \mathbb{R}^3$ and of unit length (we use the usual Euclidean length on $\mathbb{R}^3$, and the fact that $T\mathbb{R}^3 \cong \mathbb{R}^3\times\mathbb{R}^3$). Prove this subset is a regular submanifold.

iii) Show that the intersection of the sphere $|z_1|^2+|z_2|^2+|z_3|^2 = 1$ in $\mathbb{C}^3$ with the complex cone $z_1^2 + z_2^2 + z_3^2 = 0$ is a regular submanifold.

iv) Are any of the above three manifolds diffeomorphic to each other? Prove such assertions.

Bonus: Are any of the above manifolds diffeomorphic to $\mathbb{R}P^3$? Prove your claim.

Exercise 2. Prove that a smooth manifold with boundary is the disjoint union of its interior and its boundary. I.e. show that $M = \mathrm{Int}(M)\sqcup\partial M$.

Exercise 3. Construct, using the stereographic charts for S^2 given in class, a smooth vector field on S^2 which vanishes exactly at 2 points, and another vector field which vanishes at exactly 1 point.

Exercise 4.

For any vector space V, we have a natural diffeomorphism $TV \cong V\times V$, where the projection $\pi_V : TV \longrightarrow V$ corresponds to the first projection $\pi_1 : V\times V\longrightarrow V$ given by $(a,b)\mapsto a$.

Let M be a smooth manifold, and choose a chart (U,φ) on M. By applying the tangent functor, and using the canonical isomorphism $T\mathbb{R}^n \cong \mathbb{R}^n\times\mathbb{R}^n$, we obtain a chart $(TU, T\varphi)$ on the manifold TM. Repeating this procedure, we obtain a chart $(T(TU), T(T\varphi))$ on the manifold $T(TM)$.

Now define a diffeomorphism $J_U : T(TU)\longrightarrow T(TU)$ by the composition $(T(T\varphi))^{-1}\circ j_U\circ T(T\varphi)$, where j_U is the automorphism of $(\varphi(U)\times\mathbb{R}^n)\times(\mathbb{R}^n\times\mathbb{R}^n)$ given by

$$j_U : ((x,v),(u,w)) \mapsto ((x,u),(v,w)).$$

i) Show that for any atlas $\{(U_i,\varphi_i)\}$, we have $J_{U_i} = J_{U_j}$ on the overlap $T(T(U_i\cap U_j))$. Deduce that this defines a global diffeomorphism $J : T(TM)\longrightarrow T(TM)$ and show that it is independent of the atlas used to construct it.

ii) Consider the tangent bundle projection $\pi_M : TM\longrightarrow M$. Applying the tangent functor, we obtain the smooth map $T\pi_M : T(TM)\longrightarrow TM$. What is the relationship between this map and the tangent bundle projection $\pi_{TM} : T(TM)\longrightarrow TM$? [Hint: Write both maps in coordinates]

Bonus: Show that J defines a natural transformation from the functor $T\circ T$ to itself, and that this natural transformation is an equivalence.

Exercise 5. Let M be a manifold and let X be a smooth vector field on M. Viewing X as a map $X : M\longrightarrow TM$, we may differentiate it, obtaining a map $TX : TM\longrightarrow T(TM)$.

i) Is TX a vector field on TM?

ii) Using the automorphism J from Exercise 4, is $J\circ TX$ a vector field on TM?

iii) Is $X : M\longrightarrow TM$ an embedding? Justify your claim.