Due date: **November 5, 2009**
Warning: Start your work early, as late assignments are not accepted. You are encouraged to discuss the problems with the class. Please ask me to clarify any questions which you find confusing.

Exercise 1. Show that the orthogonal group $O(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ and the unitary group $U(n) \subset GL(n,\mathbb{C})$ are embedded submanifolds, and compute their dimensions. Is $U(n)$ diffeomorphic to $SU(n) \times S^1$? Are they isomorphic as groups?

Exercise 2. If K is an embedded submanifold of L and L is an embedded submanifold of M, is K necessarily an embedded submanifold of M? Justify your answer.

Exercise 3. If K, K' are embedded, transverse submanifolds of X of codimension k, k' respectively, show that there is a neighbourhood U of any point $p \in K \cap K'$, and coordinates $x^1, \ldots, x^n$ on U such that $K \cap U = \{x^1 = \cdots = x^k = 0\}$ and $L \cap U = \{x^n = \cdots = x^{n-k'+1} = 0\}$, with $k < n - k' + 1$.

Exercise 4. Let $f : M \longrightarrow M$ be a smooth map and suppose p is a fixed point under f, i.e. $f(p) = p$. The point p is called a *Lefschetz fixed point* when the derivative map $f_* : T_pM \longrightarrow T_pM$ does not have $+1$ as an eigenvalue.

Show that if M is compact and all fixed points for f are Lefschetz, then there are only finitely many fixed points for f.

Exercise 5. Prove that there are no smooth functions on a compact manifold M without critical points.

Exercise 6. Let $f : M \longrightarrow N$ be a smooth map of manifolds with the same dimension, and suppose M is compact. Let $\iota : [0,1] \longrightarrow N$ be an embedding such that both ι and $\partial\iota$ are transverse to f. Show first that $f^{-1}(\iota(0))$ and $f^{-1}(\iota(1))$ are finite sets and second that $\#(f^{-1}(\iota(0))) \equiv \#(f^{-1}(\iota(1)))$ (mod 2). Hint: What are the possible compact 1-dimensional manifolds with boundary?

For any $m \equiv n$ (mod 2), give an example of a map $f : S^1 \longrightarrow S^1$ such that $f^{-1}(1)$ has n elements and $f^{-1}(-1)$ has m elements, and give an example of an embedding ι as above with $\iota(0) = 1$ and $\iota(1) = -1$.

Exercise 7. Show that a compact n-manifold M cannot be embedded in $\mathbb{R}^n$. Can it be immersed? Can it be submerged?

Exercise 8. Consider the function $\mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}$ given by $f(z_1, z_2) = z_1^p + z_2^q$, for integers p, q which are relatively prime and ≥ 2. Describe the regular and critical points and values of this map.

Show that the intersection $K = f^{-1}(0) \cap S^3$ is transverse, where we view $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$, and identify the manifold K. By considering the 2-tori $\{(z_1, z_2) \ : \ |z_1| = c_1 \text{ and } |z_2| = c_2\}$ for constants c_1, c_2, describe (informally, using a diagram) the way in which K is embedded in S^3.