Due date: **November 19, 2009**
Please contact me if there are any errors.

Exercise 1. A smooth map $f : X \longrightarrow Y$ is said to be transverse to an embedded submanifold $Z \subset Y$ when f is transverse to the inclusion map $\iota : Z \hookrightarrow Y$. Assuming X is compact and Z closed, show that the transversality of f to Z is stable under perturbations of f.

Exercise 2. Let M, N be embedded submanifolds of a manifold X, and suppose that their intersection $M \cap N$ is an embedded submanifold. Then M, N are said to have *clean* intersection when, for each $p \in M \cap N$, we have $T_p(M \cap N) = T_pM \cap T_pN$.

i) Can the intersection of embedded submanifolds be transverse but not clean? Can it be clean but not transverse? Give examples or proofs as necessary.

ii) If K, K' are cleanly intersecting submanifolds of X, of codimension k, k' respectively, show that there is a neighbourhood U of any point $p \in K \cap K'$, and coordinates $x^1, \ldots, x^n$ on U such that $K \cap U = \{x^1 = \cdots = x^k = 0\}$ and $L \cap U = \{x^j = \cdots = x^{j+k'-1} = 0\}$, for some $j \leq k+1$.

iii) Let M, N intersect cleanly in X, and let $g : G \longrightarrow M$ and $h : H \longrightarrow N$ be submersions. Is $G \times_X H := G_{\iota_M \circ g} \times_{\iota_N \circ h} H$ an embedded submanifold of $G \times H$? (here ι_M, ι_N are the inclusions). Justify your claim.

Exercise 3.
Compute the mod 2 self-intersection number of the zero section $X \longrightarrow TX$ for the manifolds $X \in \{S^1, S^2, \mathbb{R}P^2\}$. Deduce that every smooth vector field on $\mathbb{R}P^2$ must have a zero.

Exercise 4. Let X be compact and $f : X \longrightarrow Y$ smooth with $\dim X = \dim Y$ and Y connected. Recall that the mod 2 degree of f is defined in terms of the mod 2 intersection number as follows: $\deg_2(f) = I_2(f, \iota)$, where $\iota : y \mapsto Y$ is the inclusion map of a point $y \in Y$.

i) Prove that $\deg_2(f)$ is independent of the point $y \in Y$.

ii) If Y is non-compact, prove that $\deg_2(f) = 0$.

iii) A map $f : X \longrightarrow Y$ is called *essential* when it is not homotopic to a constant map. Prove that if $\deg_2(f) = 1$, then f is essential.

iv) Give example of a smooth surjective map $f : S^2 \longrightarrow S^2$ with $\deg_2(f) = 0$.

v) Can there exist a smooth map $f : S^2 \longrightarrow T^2$ with $\deg_2(f) = 1$? [Hint: consider two embedded circles C_1, C_2 in T^2 intersecting transversally at a single point.] Can there exist a smooth map of $\deg_2(f) = 1$ in the opposite direction? In each case, give proofs.

Exercise 5. Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and consider the 1-form $\theta = dx + y dz \in \Omega^1(\mathbb{R}^3)$. Does there exist a smooth immersion f from a neighbourhood of the origin in $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $f^*\theta = 0$? [Hint: see illustration on course webpage.] What if, instead, $\theta = x dx + y dy + z dz$? Justify this result also.

Exercise 6. Consider S^n and its two stereographic coordinate charts φ_S, φ_N to $\mathbb{R}^n$. Using standard coordinates on $\mathbb{R}^n$, write down the coordinate expressions for a smooth, nowhere-vanishing n-form on S^n.