Due date: **December 3, 2009** (in class) I realize that the time available is short, but these questions provide a good study of the last module we covered in class, and will be useful for the final exam.
Please contact me if there are any errors.

Exercise 1. Let z be the standard complex coordinate on $\mathbb{C}$, i.e. $z = x + iy$, and form the complex differential form $\frac{dz}{z}$. Where is this well-defined? Decompose it into real and imaginary parts. Are these closed forms? Compute the integrals

$$\int_{S^1} \iota^* \mu,$$

for $\iota : S^1 \longrightarrow \mathbb{C}$ the standard inclusion and μ the real or imaginary part of $\frac{dz}{z}$. Explain, using Stokes' theorem, why the value of the integral does not depend on the radius of the circle.

Exercise 2. Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by

$$(r, \phi, \theta) \mapsto (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),$$

where (r, ϕ, θ) are standard Cartesian coordinates on $\mathbb{R}^3$.

- Compute $\varphi^* dx, \varphi^* dy, \varphi^* dz$ where (x, y, z) are Cartesian coordinates for $\mathbb{R}^3$.
- Compute $\varphi^*(dx \wedge dy \wedge dz)$.
- For any vector field X, define ι_X to be the unique degree -1 (i.e. it reduces degree by 1) derivation (i.e. $\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \iota_X(\beta)$) of the algebra of differential forms such that $i_X(f) = 0$ and $i_X df = X(f)$ for $f \in \Omega^0(M)$. Compute the integral

$$\int_{S_r^2} \iota_X(dx \wedge dy \wedge dz),$$

for the vector field $X = \varphi_* \frac{\partial}{\partial r}$, where S_r^2 is the sphere of radius r.

Exercise 3. Compute the de Rham cohomology groups (Using Mayer-Vietoris if necessary) of the following spaces, for all degrees.

- $\mathbb{R}^3 - \{p\}$, for $p \in \mathbb{R}^3$ a point.
- $\mathbb{R}^3 - \{p_1 \cup p_2\}$ where p_i are distinct points?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$ where ℓ_i are non-intersecting lines?
- $\mathbb{R}^3 - \{\ell_1 \cup \ell_2\}$, assuming that l_1 intersects l_2 in exactly one point?

This question is an easier version of the one John Nash asked in class in the movie "A Beautiful Mind".