2.1 Tangent morphism

The tangent bundle itself is only the result of applying the tangent functor to a manifold. We must explain how to apply the tangent functor to a morphism of manifolds. This is otherwise known as taking the "derivative" of a smooth map $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \longrightarrow \psi_\alpha(V_\alpha)$ which satisfy

$$\psi_{\alpha\beta} \circ f_{i\alpha} = f_{j\beta} \circ \varphi_{ij}.$$

Differentiating, we obtain

$$D\psi_{\alpha\beta} \circ Df_{i\alpha} = Df_{j\beta} \circ D\varphi_{ij},$$

and hence we obtain a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Tf : TM \longrightarrow TN$ (or sometimes just $Df : TM \longrightarrow TN$). Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Just as $\pi^{-1}(x) = T_xM \subset TM$ is a vector space for all x, making TM into a "bundle of vector spaces", the map $Tf : T_xM \longrightarrow T_{f(x)}N$ is a linear map and hence Tf is a "bundle of linear maps". The pair (f, Tf) is a morphism of vector bundles $(TM, \pi_M, M) \longrightarrow (TN, \pi_N, N)$.

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Tf \circ Tg = Th$. As a result, we obtain the following category-theoretic statement.

Proposition 2.2. *The map T which takes a manifold M to its tangent bundle TM, and which takes maps $f : M \longrightarrow N$ to the derivative $Tf : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself.*

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart (U_i, φ_i), we would say that a vector field X_i is simply a vector-valued function on U_i, i.e. a function $X_i : \varphi(U_i) \longrightarrow \mathbb{R}^n$. Of course if we had another vector field X_j on (U_j, φ_j), then the two would agree as vector fields on the overlap $U_i \cap U_j$ when $D\varphi_{ij} : X_i(p) \mapsto X_j(\varphi_{ij}(p))$. So, if we specify a collection $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue on overlaps, this would define a global vector field. This leads precisely to the following definition.

Definition 12. A smooth vector field on the manifold M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X : M \longrightarrow M$ is the identity. Essentially it is a smooth assignment of a unique tangent vector to each point in M.

Such maps X are also called *cross-sections* or simply *sections* of the tangent bundle TM, and the set of all such sections is denoted $C^\infty(M, TM)$ or sometimes $\Gamma^\infty(M, TM)$, to distinguish them from simply smooth maps $M \longrightarrow TM$.

Example 2.3. *From a computational point of view, given an atlas $(\tilde{U}_i, \varphi_i)$ for M, let $U_i = \varphi_i(\tilde{U}_i) \subset \mathbb{R}^n$ and let $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Then a global vector field $X \in \Gamma^\infty(M, TM)$ is specified by a collection of vector-valued functions $X_i : U_i \longrightarrow \mathbb{R}^n$ such that $D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x))$ for all $x \in \varphi_i(\tilde{U}_i \cap \tilde{U}_j)$.*

For example, if $S^1 = U_0 \sqcup U_1 / \sim$, with $U_0 = \mathbb{R}$ and $U_1 = \mathbb{R}$, with $x \in U_0 \backslash \{0\} \sim y \in U_1 \backslash \{0\}$ whenever $y = x^{-1}$, then $\varphi_{01} : x \mapsto x^{-1}$ and $D\varphi_{01}(x) : (x, v) \mapsto (x^{-1}, -x^{-2}v)$.

If we choose the coordinate vector field $X_0 = \frac{\partial}{\partial x}$ (in coordinates this is simply $x \mapsto (x, 1)$), then we see that $D\varphi_{01}(X_0) = (x^{-1}, -x^{-2} \cdot 1)$, i.e. the vector field $y \mapsto (y, -y^2)$, in other words $X_1 = -y^2 \frac{\partial}{\partial y}$.

Hence the following local vector fields glue to form a global vector field on S^1:

$$X_0 = \frac{\partial}{\partial x}$$

$$X_1 = -y^2 \frac{\partial}{\partial y}.$$

This vector field does not vanish in U_0 but vanishes to order 2 at a single point in U_1. Find the local expression in these charts for the rotational vector field on S^1 given in polar coordinates by $\frac{\partial}{\partial\theta}$.

A word of warning: it may be tempting to think that the assignment $M \mapsto \Gamma^\infty(M, TM)$ is a functor from manifolds to vector spaces; it is not, because there is no way to push forward or pull back vector fields. Nevertheless, if $f : M \longrightarrow N$ is a smooth map, it does define an equivalence relation between vector fields on M and N:

Definition 13. if $f : M \longrightarrow N$ smooth, then $X \in \Gamma^\infty(M, TM)$ is called f-related to $Y \in \Gamma^\infty(N, TN)$ when $f_*(X(p)) = Y(f(p))$, i.e. the diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"'] & N \arrow[u, "Y"'] \end{tikzcd}$$

So, another way to phrase the definition of a vector field is that they are local vector-valued functions which are φ_{ij}-related on overlaps.

2.2 Properties of vector fields

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 14. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$. Note that this makes sense for noncommutative algebras also.

In the following, we show that derivations of the algebra of functions actually correspond to vector fields.

The vector fields $\Gamma^\infty(M, TM)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless $\dim M = 0$). They also form a module over the ring of smooth functions $C^\infty(M, \mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$, we claim that $fX : x \mapsto f(x)X(x)$ defines a smooth vector field: this is clear from local considerations: A vector field $V = \sum_i v^i(x^1, \ldots x^n)\frac{\partial}{\partial x^i}$ is smooth precisely when the component functions v^i are smooth: the vector field fV then has components fv^i, still smooth.

The important property of vector fields which we are interested in is that they act as $\mathbb{R}$-derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i(\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M, \mathbb{R})$.

Proposition 2.4. *Let f be a smooth function on $U \subset \mathbb{R}^n$, and $X : U \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ a vector field. Then*

$$X(f) = \pi_2 \circ Df \circ X,$$

where $\pi_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is the second projection (i.e. projection to the fiber of $T\mathbb{R}^n$.) In local coordinates, we have $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$ whereas $Df : X(x) \mapsto (f(x), \sum_i \frac{\partial f}{\partial x^i} a^i)$, so that we obtain the result by projection.

Proposition 2.5. *Local partial differentiation extends to an injective map $\Gamma^\infty(M, TM) \longrightarrow \mathrm{Der}(C^\infty(M, \mathbb{R}))$.*

Proof. A global function is given by $f_i = f_j \circ \varphi_{ij}$. We verify that

$$\begin{aligned} X_i(f_i) &= \pi_2 \circ Df_i \circ X_i &(12)\\ &= \pi_2 \circ Df_j \circ D\varphi_{ij} \circ X_i &(13)\\ &= \pi_2 \circ Df_j \circ X_j \circ \varphi_{ij} &(14)\\ &= X_j(f_j) \circ \varphi_{ij}, &(15) \end{aligned}$$

showing that $\{X_i(f_i)\}$ defines a global function. Injectivity follows from the local fact that $V(f) = 0$ for all f would imply, for $V = \sum_i v^i \frac{\partial}{\partial x^i}$, that $V(x^i) = v^i = 0$ for all i, i.e. $V = 0$. $\square$