In fact, vector fields provide all possible derivations of the algebra $A = C^\infty(M, \mathbb{R})$:

Theorem 2.6. *The map* $\Gamma^\infty(M, TM) \longrightarrow \mathrm{Der}(C^\infty(M, \mathbb{R}))$ *is an isomorphism.*

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y)) g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i) g_i(z) - \sum_i (x^i(z) - x^i(y)) Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i the associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p) D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside U_i; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. □

Since vector fields are derivations, we deduce that they have all the properties that derivationsdo, and we also have a natural source of examples, coming from infinitesimal automorphisms of M.

Definition 15. For any algebra A, the derivations $\mathrm{Der}(A)$ form a Lie algebra via the bracket $[X, Y](f) = X(Y(f)) - Y(X(f))$. For vector fields ($A = C^\infty(M, \mathbb{R})$), this bracket is called the Lie bracket.

Example 2.7. *Let* φ_t *: be a smooth family of diffeomorphisms of* M *with* $\varphi_0 = \mathrm{Id}$. *That is, let* $\varphi : (-\epsilon, \epsilon) \times M \longrightarrow M$ *be a smooth map and* $\varphi_t : M \longrightarrow M$ *a diffeomorphism for each* t. *Then* $X(f)(p) = \frac{d}{dt}|_{t=0}(\varphi_t^* f)(p)$ *defines a smooth vector field. A better way of seeing that it is smooth is to rewrite it as follows: Let* $\frac{\partial}{\partial t}$ *be the coordinate vector field on* $(-\epsilon, \epsilon)$ *and observe* $X(f)(p) = \frac{\partial}{\partial t}(\varphi^* f)(0, p)$.

In many cases, a smooth vector field may be expressed as above, i.e. as an infinitesimal automorphism of M, but this is not always the case. In general, it gives rise to a "local 1-parameter group of diffeomorphisms", as follows:

Definition 16. A local 1-parameter group of diffeomorphisms is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map

$$\begin{aligned} \Phi : & U \longrightarrow M \\ & (t, x) \mapsto \varphi_t(x) \end{aligned}$$

such that $\mathbb{R} \times \{x\} \cap U$ is connected, $\varphi_0(x) = x$ for all x and if $(t, x), (t + t', x), (t', \varphi_t(x))$ are all in U then $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$ (note that this last fact indicates that φ_t are all diffeomorphisms, having inverses φ_{-t}).

Then the local existence and uniqueness of solutions to systems of ODE implies that every smooth vector field $X \in \Gamma^\infty(M, TM)$ gives rise to a local 1-parameter group of diffeomorphisms (U, Φ) such that the curve $\gamma_x : t \mapsto \varphi_t(x)$ is such that $(\gamma_x)_*(\frac{d}{dt}) = X(\gamma_x(t))$ (this means that γ_x is an integral curve or "trajectory" of the "dynamical system" defined by X). Furthermore, if (U', Φ') are another such data, then $\Phi = \Phi'$ on $U \cap U'$.

Definition 17. A vector field $X \in \Gamma^\infty(M, TM)$ is called *complete* when it has a local 1-parameter group of diffeomorphisms with $U = \mathbb{R} \times M$.

Theorem 2.8. *If M is compact, then every smooth vector field is complete. Similarly any compactly-supported vector field is complete.*

Example 2.9. *The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on $\{1 - tx > 0\}$.*

Remark 3. *If ϕ_t and ψ_t are families of automorphisms of A with $\phi_0 = \psi_0 = \mathrm{Id}$, then they correspond to derivations $X = \frac{d}{dt}|_{t=0}\phi_t$ and $Y = \frac{d}{dt}|_{t=0}\psi_t$, and the family of automorphisms $\gamma_t = \phi_t\psi_t\phi_t^{-1}\psi_t^{-1}$ has $\frac{d}{dt}|_{t=0}\gamma_t = 0$ and $\frac{d^2}{dt^2}|_{t=0}\gamma_t = [X, Y]$.*

2.3 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ), (V, ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem*.

Theorem 2.10 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^m$ an open set and $f : U \longrightarrow \mathbb{R}^m$ a smooth map such that $Df(p)$ is an invertible linear operator. Then there is a neighbourhood $V \subset U$ of p such that $f(V)$ is open and $f : V \longrightarrow f(V)$ is a diffeomorphism. furthermore, $D(f^{-1})(f(p)) = (Df(p))^{-1}$.*

Proof not given in class – this is the standard proof seen in first analysis course. Without loss of generality, assume that U contains the origin, that $f(0) = 0$ and that $Df(p) = \mathrm{Id}$ (for this, replace f by $(Df(0))^{-1} \circ f$. We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define g so that $f(x) = x + g(x)$. Hence $g(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - g(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $\phi(y)$ by the Banach fixed point theorem (Look it up!), and so $y - g(\phi(y)) = \phi(y)$, i.e. ϕ is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0, r)$ where $||Dh_y|| \leq \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0, r)$,

$$||h_y(x) - h_y(x')|| \leq \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \leq ||h_y(x) - h_y(0)|| + ||h_y(0)|| \leq \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0, r)}$. Let $\phi(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned} ||\phi(y) - \phi(y')|| &= ||h_y(\phi(y)) - h_{y'}(\phi(y'))|| \\ &\leq ||g(\phi(y)) - g(\phi(y'))|| + ||y - y'|| \\ &\leq \tfrac{1}{2}||\phi(y) - \phi(y')|| + ||y - y'||, \end{aligned}$$

so that we have $||\phi(y) - \phi(y')|| \leq 2||y - y''||$, as required.

To see that ϕ is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = \phi(y)$ and $x' = \phi(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0, r)}$, which is not a problem.

$$\begin{aligned} ||\phi(y) - \phi(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))|| \\ &\leq ||(Df(x))^{-1}|| ||(Df(x))(x - x') - (f(x) - f(x'))|| \\ &\leq o(||x - x'||), \text{ using differentiability of } f \\ &\leq o(||y - y'||), \text{ using continuity of } \phi. \end{aligned}$$

Now that we have shown ϕ is differentiable with derivative $(Df)^{-1}$, we use the fact that Df is C^∞ and inversion is C^∞, implying that $D\phi$ is C^∞ and hence ϕ also. □

This theorem immediately provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.11 (Constant rank theorem)**.** *If $f : M \longrightarrow N$ is a smooth map of manifolds of dimension m, n respectively, and if Tf has constrant rank k in some open set $U \subset M$ then for each point $p \in U$ there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

Proof. Begin by choosing coordinates near $p, f(p)$ on M and N. Since $\mathrm{rk}\,(Tf) = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \ldots, x_k, y_1, \ldots y_{m-k})$ on V and $(u_1, \ldots u_k, v_1, \ldots, v_{n-k})$ on W.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \ldots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, \tilde{R} = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k \times k} & 0 \\ \frac{\partial \tilde{R}}{\partial x} & \frac{\partial \tilde{R}}{\partial y} \end{pmatrix}$$

and since we know that Tf must have rank k *in a neighbourhood of* p, we conclude that $\frac{\partial \tilde{R}}{\partial y} = 0$ in a neighbourhood of p, meaning that $\tilde{R}$ is a function $S(x)$ only of the variables x.

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)).$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0),$$

as required. □

Some special cases of the above theorem have special names:

$$\begin{aligned} \text{local immersion: } & (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0) \\ \text{local submersion: } & (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^k) \\ \text{local diffeomorphism: } & (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m) \end{aligned}$$

Definition 18. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Tf(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Tf(p)$ is injective at all points $p \in M$.

For linear maps $A : V \longrightarrow W$, we obtain new vector spaces as subspaces $\ker(A) \subset V$ and $\mathrm{im}(A) \subset W$. The same thing occurs for smooth maps, assuming that they satisfy the conditions of the theorem above.

Definition 19. An *embedded submanifold* (sometimes called regular submanifold) of dimension k in an n-manifold M is a subspace $S \subset M$ such that $\forall s \in S$, there exists a chart (U, φ) for M, containing s, and with

$$S \cap U = \varphi^{-1}(x_{k+1} = \cdots = x_n = 0).$$

In other words, the inclusion $S \subset M$ is locally isomorphic to the vector space inclusion $\mathbb{R}^k \subset \mathbb{R}^n$.

Of course, the remaining coordinates $\{x_1, \ldots, x_k\}$ define a smooth manifold structure on S itself, justifying the terminology.

Proposition 2.12 (analog of kernel)**.** *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Tf(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is an embedded submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k embedded submanifold. $\square$

Example 2.13. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is an embedded submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.9.*

If Tf has maximal rank at a point $p \in M$, this is a special case, because then it will have maximal rank in a neighbourhood of p, and the local normal form will hold.

Definition 20. A point $p \in M$ for which $Tf(p)$ has maximal rank is called a *regular* point. Otherwise it is called a critical point. Values $q \in N$ for which $f^{-1}(q)$ are all regular points are called *regular values* (including points for which $f^{-1}(q) = \emptyset$). Other values are called critical values. Warning: even if q is a critical value, $f^{-1}(q)$ may contain regular points.

Proposition 2.14 (maximal rank special case)**.** *If $f : M \longrightarrow N$ is a smooth map of manifolds and $q \in N$ is a regular value, then $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. $\square$

Warning: An immersion locally defines an embedded submanifold. But globally, it may not be injective, and it also may not be a homeomorphism onto its image (examples: figure 8 embedding of S^1 in $\mathbb{R}^2$ and number 9 immersion of $\mathbb{R}$ in $\mathbb{R}^2$.)

Definition 21. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*

Proposition 2.15. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $fx(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Then we can restrict the charts $(V', \psi|_{V'})$, $(U' = f^{-1}(V'), \varphi_{U'})$ so that we see the embedding. $\square$

Remark 4. *If $\iota : M \longrightarrow N$ is an embedding of M into N, then $T\iota : TM \longrightarrow TN$ is also an embedding, and hence $T^k\iota : T^kM \longrightarrow T^kN$ are all embeddings.*