Having the constant rank theorem in hand, we may also apply it to study manifolds *with boundary*. The following two results illustrate how this may easily be done.

Proposition 2.16. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold of M. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \dots, x_m) \mapsto x_m.$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. $\square$

Example 2.17. *Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \dots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.*

Example 2.18. *Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?*

Proposition 2.19. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a regular submanifold with boundary (i.e. locally modeled on $\mathbb{R}^k \subset \mathbb{R}^n$ or the inclusion $H^k \subset H^n$ given by $(x_1, \dots x_k) \mapsto (0, \dots, 0, x_1, \dots x_k)$.) Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a regular submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a regular submanifold of $\mathbb{R}^m$ of dimension $m - n$.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \dots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 2.16, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) \ : \ x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. $\square$

One important use of the above result is in a proof of the Brouwer fixed point theorem. But in order to use it, we need to know that most values are regular values, i.e. that regular values are generic. This is a result of transversality theory, known as Sard's theorem [next section].

Corollary 2.20. *Let M be a compact manifold with boundary. There is no smooth map $f : M \longrightarrow \partial M$ leaving ∂M pointwise fixed. Such a map is called a smooth retraction of M onto its boundary.*

Proof. Such a map f must have a regular value by Sard's theorem, let this value be $y \in \partial M$. Then y is obviously a regular value for $f|_{\partial M} = \text{Id}$ as well, so that $f^{-1}(y)$ must be a compact 1-manifold with boundary given by $f^{-1}(y) \cap \partial M$, which is simply the point y itself. Since there is no compact 1-manifold with a single boundary point, we have a contradiction. $\square$

For example, this shows that the identity map $S^n \longrightarrow S^n$ may not be extended to a smooth map $f : \overline{B(0, 1)} \longrightarrow S^n$.

Corollary 2.21. *Every smooth map of the closed n-ball to itself has a fixed point.*

Proof. Let $D^n = \overline{B(0,1)}$. If $g : D^n \longrightarrow D^n$ had no fixed points, then define the function $f : D^n \longrightarrow S^{n-1}$ as follows: let $f(x)$ be the point nearer to x on the line joining x and $g(x)$.

This map is smooth, since $f(x) = x + tu$, where

$$u = ||x - g(x)||^{-1}(x - g(x)),$$

and t is the positive solution to the quadratic equation $(x + tu) \cdot (x + tu) = 1$, which has positive discriminant $b^2 - 4ac = 4(1 - |x|^2 + (x \cdot u)^2)$. Such a smooth map is therefore impossible by the previous corollary. □

Theorem 2.22 (Brouwer fixed point theorem)**.** *Any continuous self-map of D^n has a fixed point.*

Not given in class, won't use it in class. The Weierstrass approximation theorem says that any continuous function on $[0, 1]$ can be uniformly approximated by a polynomial function in the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. In other words, the polynomials are dense in the continuous functions with respect to the supremum norm. The Stone-Weierstrass is a generalization, stating that for any compact Hausdorff space X, if A is a subalgebra of $C^0(X, \mathbb{R})$ such that A separates points $(\forall x, y, \exists f \in A : f(x) \neq f(y))$ and contains a nonzero constant function, then A is dense in C^0.

Given this result, approximate a given continuous self-map g of D^n by a polynomial function p' so that $||p' - g||_\infty < \epsilon$ on D^n. To ensure p' sends D^n into itself, rescale it via

$$p = (1 + \epsilon)^{-1} p'.$$

Then clearly p is a D^n self-map while $||p - g||_\infty < 2\epsilon$. If g had no fixed point, then $|g(x) - x|$ must have a minimum value μ on D^n, and by choosing $2\epsilon = \mu$ we guarantee that for each x,

$$|p(x) - x| \geq |g(x) - x| - |g(x) - p(x)| > \mu - \mu = 0.$$

Hence p has no fixed point. Such a smooth function can't exist and hence we obtain the result. □

3 Transversality

We shall now continue to use the inverse and constant rank theorems to produce more manifolds, except now these shall be cut out only locally by functions. We shall ask when the *intersection* of two submanifolds yields a submanifold. You should think that intersecting a given submanifold with another is the local imposing of a certain number of constraints.

Two subspaces $K, L \subset V$ of a vector space V are called *transversal* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that

$$\dim V = \dim K + \dim L - \dim K \cap L.$$

We may apply this to submanifolds as follows:

Definition 22. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM.$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Note: at this point, we have not defined the tangent bundle of a manifold, but we may understand tangent spaces locally, in each chart. We may make sense of this as follows: Let $k : K \longrightarrow M$ and $l : L \longrightarrow M$ be the inclusion maps. Then we may consider T_pK, T_pL to be the images of the derivatives of k and l, in charts for K, L, M. Transversality then requires that these images span $\mathbb{R}^m$, where $m = \dim M$.

Proposition 3.1. *If $K, L \subset M$ are transverse regular submanifolds then $K \cap L$ is also a regular submanifold, of dimension* $\dim K + \dim L - \dim M$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^k$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^l$, where K and L have codimension k, l respectively.

Now note that $K \cap L \cap U = (f, g)^{-1}(0)$, where $(f, g) : K \cap L \cap U \longrightarrow \mathbb{R}^{k+l}$. But 0 is a regular value for (f, g), since $\ker T(f, g) = \ker Tf \cap \ker Tg = T_pK \cap T_pL$, which has codimension $k + l$ by the transversality assumption. Hence the rank of $T(f, g)$ must be $k + l$, just because the rank of a linear map is always given by the codimension of its kernel. ☐

Example 3.2 (Exotic spheres)**.** *Consider the following intersections in $\mathbb{C}^5\backslash 0$:*

$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\mathrm{im}(Tk(a)) + \mathrm{im}(Tl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 23. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $Tf(T_aK) + Tg(T_bL) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.