3 Transversality

In this section, we continue to use the inverse and constant rank theorems to produce more manifolds, except now these are cut out only locally by functions. We ask when the *intersection* of two submanifolds yields a submanifold. You should think that intersecting a given submanifold with another is the local imposing of a certain number of constraints.

Two subspaces $K, L \subset V$ of a vector space V are called *transversal* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that

$$\dim V = \dim K + \dim L - \dim K \cap L.$$

We may apply this to submanifolds as follows:

Definition 22. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM.$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Proposition 3.1. *If $K, L \subset M$ are transverse regular submanifolds then $K \cap L$ is also a regular submanifold, of dimension* $\dim K + \dim L - \dim M$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^k$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^l$, where K and L have codimension k, l respectively.

Now note that $K \cap L \cap U = (f, g)^{-1}(0)$, where $(f, g) : K \cap L \cap U \longrightarrow \mathbb{R}^{k+l}$. But 0 is a regular value for (f, g), since $\ker T(f, g) = \ker Tf \cap \ker Tg = T_pK \cap T_pL$, which has codimension $k + l$ by the transversality assumption. Hence the rank of $T(f, g)$ must be $k + l$, just because the rank of a linear map is always given by the codimension of its kernel. □

Example 3.2 (Exotic spheres)**.** *Consider the following intersections in $\mathbb{C}^5 \backslash 0$:*

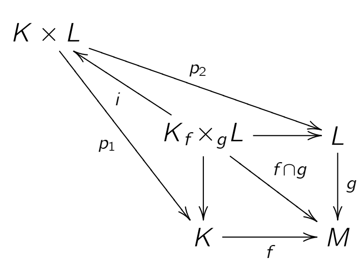
$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

This is a transverse intersection, and for $k = 1, \dots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We now phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse when, $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\mathrm{im}(Tk(a)) + \mathrm{im}(Tl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 23. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $Tf(T_aK) + Tg(T_bL) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 3.3. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K {}_f\!\times_g L = \{(a, b) \in K \times L : f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*



where i is the inclusion and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

The manifold $K_f \times_g L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection of submanifolds.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. Then we show that the following intersection of regular submanifolds is transverse:

$$\Gamma_{f\cap g} = (\Gamma_f \times \Gamma_g) \cap (K \times L \times \Delta_M),$$

where $\Delta_M = \{(p, p) \in M \times M \ : \ p \in M\}$ is the diagonal. To show this, let $f(k) = g(l) = m$ so that $x = (k, l, m, m) \in X$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))), \ v \in T_k K, \quad w \in T_l L\} \tag{16}$$

whereas we also have

$$T_x(K \times L \times \Delta_M) = \{((v, m), (w, m)) \ : \ v \in T_k K, \quad w \in T_l L, \quad m \in T_p M\} \tag{17}$$

By transversality of f, g, any tangent vector $m_i \in T_p M$ may be written as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)),$$

leading directly to the transversality of the spaces (16), (17). This shows that $\Gamma_{f\cap g}$ is a regular submanifold of $K \times L \times M \times M$. Actually since it sits inside $K \times L \times \Delta_M$, we may compose with the projection diffeomorphism to view it as a regular submanifold in $K \times L \times M$. Then we observe that the restriction of the projection onto $K \times L$ to the submanifold $\Gamma_{f\cap g}$ is an embedding with image exactly $K_f \times_g L$. Hence the fiber product is a smooth manifold and $\Gamma_{f\cap g}$ may then be viewed as the graph of a smooth map $f \cap g : K_f \times_g L \longrightarrow M$ which makes the diagram above commute by definition. ☐

Example 3.4. *If $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$, we may view both K_i as smooth fiber bundles over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".*

Example 3.5. *Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \backslash \{0\}$ with the projection $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product*

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.

The following result is an exercise: just as we may take the product of a manifold with boundary K with a manifold without boundary L to obtain a manifold with boundary $K \times L$, we have a similar result for fiber products.

Proposition 3.6. *Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \longrightarrow M$ and $g : L \longrightarrow M$ are smooth maps such that both f and ∂f are transverse to g. Then the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.*

3.1 Stability

We wish to understand the intuitive notion that "transversality is a stable condition", which in some sense means that if true, it remains so under small perturbations (of the submanifolds or maps involved). After this, we will go much further using Sard's theorem, and show that not only is it *stable*, it is actually *generic*, meaning that even if it is not true, it can be made true by a small perturbation. In this sense, stability says that transversal maps form an open set, and genericity says that this open set is dense in the space of maps. To make this precise, we would introduce a topology on the space of maps, something which we leave for another course.

A property of a smooth map $f_0 : M \longrightarrow N$ is *stable* under perturbations when for any smooth homotopy f_t of f_0, i.e. a smooth map $f : [0,1] \times M \longrightarrow N$ with $f|_{\{0\}\times M} = f_0$, the property holds for all $f_t = f|_{\{t\}\times M}$ with $t < \epsilon$ for some $\epsilon > 0$.

Proposition 3.7. *Let M be a compact manifold and $f_0 : M \longrightarrow N$ a smooth map. Then the property of being an immersion or submersion are each stable under perturbations. If M' is compact, then the transversality of $f_0 : M \longrightarrow N$, $g_0 : M' \longrightarrow N$ is also stable under perturbations of f_0, g_0.*

As an exercise, show that local diffeomorphisms, diffeomorphisms, and embeddings are also stable.

Proof. Let $f_t, t \in [0,1]$ be a smooth homotopy of f_0, and suppose that f_0 is an immersion. This means that at each point $p \in M$, the jacobian of f_0 in some chart has a $m \times m$ submatrix with nonvanishing determinant, for $m = \dim M$. By continuity, this $m \times m$ submatrix must have nonvanishing determinant in a neighbourhood around $(0,p) \in [0,1] \times M$. $\{0\} \times M$ may be covered by a finite number of such neighbourhoods, since M is compact. Choose ϵ such that $[0,\epsilon) \times M$ is contained in the union of these intervals, giving the result.

The proof for submersions is identical. The condition that f_0 be transversal to g_0 is equivalent to the fact that $\Gamma_{f_0} \times \Gamma_{g_0}$ is transversal to $C = M \times Z \times \Delta_N$. Choosing coordinate charts adapted to C, we may express this locally as a submersion condition. Hence by the previous result we have stability. □

3.2 Genericity of transversality

The fundamental idea which allows us to prove that transversality is a generic condition is a the theorem of Sard showing that critical values of a smooth map $f : M \longrightarrow N$ (i.e. points $q \in N$ for which the map f and the inclusion $\iota : q \hookrightarrow N$ fail to be transverse maps) are *rare*. The following proof is taken from Milnor, based on Pontryagin.

The meaning of "rare" will be that the set of critical values is of *measure zero*, which means, in $\mathbb{R}^m$, that for any $\epsilon > 0$ we can find a sequence of balls in $\mathbb{R}^m$, containing $f(C)$ in their union, with total volume less than ϵ. Some easy facts about sets of measure zero: the countable union of measure zero sets is of measure zero, the complement of a set of measure zero is dense.

We begin with an elementary lemma which shows that "measure zero" is a property preserved by diffeomorphisms.

Lemma 3.8. *Let $A \subset \mathbb{R}^m$ have measure zero and let $F : A \longrightarrow \mathbb{R}^n$ be a C^1 map with $m \leq n$. Then $F(A)$ has measure zero.*

Proof. F has an extension to a neighbourhood W of A. Let $\overline{B}$ be a closed ball in W. We then show that $F(A \cap \overline{B})$ has measure zero, and since $F(A)$ is the union of countably many such sets, we obtain $F(A)$ of measure zero.

Since F is C^1, we have the mean value theorem stating for all $x, y \in \overline{B}$

$$f(y) - f(x) = [\int_0^1 F_*((1-t)x + ty)dt](y - x),$$

where the integral of the matrix is done component-wise. Then we have the estimate

$$\begin{aligned}||f(y)-f(x)|| &= ||[\int_0^1 F_*((1-t)x+ty)dt](y-x)|| \\ &\leq \int_0^1 ||F_*((1-t)x+ty)(y-x)||dt \\ &\leq \int_0^1 ||F_*((1-t)x+ty)|| \cdot ||y-x||dt \\ &= C||y-x||.\end{aligned}$$

Then the image of a ball of radius r contained in $\overline{B}$ would be contained in a ball of radius at most Cr, which would have volume proportional to r^n.

A is of measure zero, hence for each ϵ we have a countable covering of A by balls of radius r_k with total volume $c_m \sum_k r_k^m < \epsilon$. We deduce that $f(A_i)$ is covered by balls of radius Cr_k with total volume $\leq C^n c_n \sum_k r_k^n$ and since $n \geq m$ this is certainly arbitrarily small. We conclude that $f(A)$ is of measure zero. □

Remark 5. *If we considered the case $n < m$, the resulting sum of volumes may be larger in $\mathbb{R}^n$. For example, the projection map $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $(x, y) \mapsto x$ clearly takes the set of measure zero $y = 0$ to one of positive measure.*

A subset $A \subset M$ of a manifold is said to have measure zero when its image in any coordinate chart has measure zero. Since manifolds are second countable and we may choose a countable basis V_i such that $\overline{V}_i \subset U_i$ are compact subsets of coordinate charts (any coordinate neighbourhood is a countable union of closed balls), it follows that a subset $A \subset M$ of measure zero may be expressed as a countable union of subsets $A_k \subset \overline{V}_i$ with $\varphi_i(A_k)$ satisfying the Lemma. We therefore obtain

Proposition 3.9. *Let $f : M \longrightarrow N$ be a C^1 map of manifolds where $\dim M \leq \dim N$. Then the image $f(A)$ of a set $A \subset M$ of measure zero also has measure zero.*

Corollary 3.10 (Baby Sard)**.** *Let $f : M \longrightarrow N$ be a C^1 of manifolds where $\dim M < \dim N$. Then $f(M)$ (i.e. the set of critical values) has measure zero in N.*

Proof. We could form $\tilde{M} = M \times \mathbb{R}$ and consider $F : \tilde{M} \longrightarrow N$ given by $F(x, t) = f(x)$. Then $f(M) = F(M \times \{0\})$. Since $M \times \{0\} \subset M \times \mathbb{R}$ is measure zero, and $\dim \tilde{M} \leq \dim N$, so is the image. □