Now we investigate the measure of the critical values of a map $f : M \longrightarrow N$ where $\dim M = \dim N$. Of course the set of critical points need not have measure zero, but we shall see that because the values of f on the critical set do not vary much, the set of critical *values* will have measure zero.

Theorem 3.11 (Equidimensional Sard)**.** *Let $f : M \longrightarrow N$ be a C^1 map of n-manifolds, and let $C \subset M$ be the set of critical points. Then $f(C)$ has measure zero.*

Proof. It suffices to show result for the unit cube. Let $f : I^n \longrightarrow \mathbb{R}^n$ a C^1 map and let $C \subset I^n$ be the set of critical points.

Since $f \in C^1(I^n, \mathbb{R}^n)$, we have that the linear approximation to f at $x \in I^n$, namely

$$f_x^{lin}(y) = f(x) + f_*(x)(y - x),$$

approximates f to second order, i.e. there is a positive function $b(\epsilon)$ with $b \to 0$ as $\epsilon \to 0$ such that

$$||f(y) - f_x^{lin}(y)|| \leq b(|y - x|)||y - x||.$$

Of course f is still Lipschitz so that

$$||f(y) - f(x)|| \leq a||y - x|| \quad \forall x, y \in I^n$$

Since f is Lipschitz, we know that $||y-x|| < \epsilon$ implies $||f(y)-f(x)|| < a\epsilon$. But if x is a critical point, then f_x^{lin} has image contained in a hyperplane P_x, which is of lower dimension and hence measure zero. This means the distance of $f(y)$ to P_x is less than $\epsilon b(\epsilon)$.

Therefore $f(y)$ lies in the cube centered at $f(x)$ of edge $a\epsilon$, but if we choose the cube to have a face parallel to P_x, then the edge perpendicular to P_x can be shortened to only $2\epsilon b\epsilon$. Therefore $f(y)$ is in a region of volume $(a\epsilon)^{n-1}2\epsilon b(\epsilon)$.

Now partition I^n into h^n cubes each of edge h^{-1}. Any such cube containing a critical point x is certainly contained in a ball around x of radius $r = h^{-1}\sqrt{n}$. The image of this ball then has volume $\leq (ar)^{n-1}2rb(r) = Ar^n b(r)$ for $A = 2a^{n-1}$. The total volume of all the images is then less than

$$h^n A r^n b(r) = A n^{n/2} b(r).$$

Note that A and n are fixed, while $r = h^{-1}\sqrt{n}$ is determined by the number h of cubes. By increasing the number of cubes, we may decrease their radius arbitrarily, and hence the above total volume, as required. ☐

The argument above will not work for $\dim N < \dim M$; we need more control on the function f. In particular, one can find a C^1 function from $I^2 \longrightarrow \mathbb{R}$ which fails to have critical values of measure zero (hint: $C + C = [0, 2]$ where C is the Cantor set). As a result, Sard's theorem in general requires more differentiability of f.

Theorem 3.12 (Big Sard's theorem)**.** *Let $f : M \longrightarrow N$ be a C^k map of manifolds of dimension m, n, respectively. Let C be the set of critical points, i.e. points $x \in U$ with*

$$\text{rank } Df(x) < n.$$

Then $f(C)$ has measure zero if $k > \frac{m}{n} - 1$.

Do not give proof in class, no time. As before, it suffices to show for $f : I^m \longrightarrow \mathbb{R}^n$.

Define $C_1 \subset C$ to be the set of points x for which $Df(x) = 0$. Define $C_i \subset C_{i-1}$ to be the set of points x for which $D^j f(x) = 0$ for all $j \leq i$. So we have a descending sequence of closed sets:

$$C \supset C_1 \supset C_2 \supset \cdots \supset C_k.$$

We will show that $f(C)$ has measure zero by showing

1. $f(C_k)$ has measure zero,

2. each successive difference $f(C_i \backslash C_{i+1})$ has measure zero for $i \geq 1$,

3. $f(C \backslash C_1)$ has measure zero.

Step 1: For $x \in C_k$, Taylor's theorem gives the estimate

$$f(x+t) = f(x) + R(x,t), \quad \text{with } ||R(x,t)|| \leq c||t||^{k+1},$$

where c depends only on I^m and f, and t sufficiently small.

If we now subdivide I^m into h^m cubes with edge h^{-1}, suppose that x sits in a specific cube I_1. Then any point in I_1 may be written as $x+t$ with $||t|| \leq h^{-1}\sqrt{m}$. As a result, $f(I_1)$ lies in a cube of edge $ah^{-(k+1)}$, where $a = 2cm^{(k+1)/2}$ is independent of the cube size. There are at most h^m such cubes, with total volume less than

$$h^m(ah^{-(k+1)})^n = a^n h^{m-(k+1)n}.$$

Assuming that $k > \frac{m}{n} - 1$, this tends to 0 as we increase the number of cubes.
Step 2: For each $x \in C_i \backslash C_{i+1}$, $i \geq 1$, there is a $i+1^{th}$ partial $\partial^{i+1} f_j / \partial x_{s_1} \cdots \partial x_{s_{i+1}}$ which is nonzero at x. Therefore the function

$$w(x) = \partial^k f_j / \partial x_{s_2} \cdots \partial x_{s_{i+1}}$$

vanishes at x but its partial derivative $\partial w / \partial x_{s_1}$ does not. WLOG suppose $s_1 = 1$, the first coordinate. Then the map

$$h(x) = (w(x), x_2, \ldots, x_m)$$

is a local diffeomorphism by the inverse function theorem (of class C^k) which sends a neighbourhood V of x to an open set V'. Note that $h(C_i \cap V) \subset \{0\} \times \mathbb{R}^{m-1}$. Now if we restrict $f \circ h^{-1}$ to $\{0\} \times \mathbb{R}^{m-1} \cap V'$, we obtain a map g whose critical points include $h(C_i \cap V)$. Hence we may prove by induction on m that $g(h(C_i \cap V)) = f(C_i \cap V)$ has measure zero. Cover by countably many such neighbourhoods V.
Step 3: Let $x \in C \backslash C_1$. Then there is some partial derivative, wlog $\partial f_1 / \partial x_1$, which is nonzero at x. the map

$$h(x) = (f_1(x), x_2, \ldots, x_m)$$

is a local diffeomorphism from a neighbourhood V of x to an open set V' (of class C^k). Then $g = f \circ h^{-1}$ has critical points $h(V \cap C)$, and has critical values $f(V \cap C)$. The map g sends hyperplanes $\{t\} \times \mathbb{R}^{m-1}$ to hyperplanes $\{t\} \times \mathbb{R}^{n-1}$, call the restriction map g_t. A point in $\{t\} \times \mathbb{R}^{m-1}$ is critical for g_t if and only if it is critical for g, since the Jacobian of g is

$$\begin{pmatrix} 1 & 0 \\ * & \frac{\partial g_t^i}{\partial x_j} \end{pmatrix}$$

By induction on m, the set of critical values for g_t has measure zero in $\{t\} \times \mathbb{R}^{n-1}$. By Fubini, the whole set $g(C')$ (which is measurable, since it is the countable union of compact subsets (critical values not necessarily closed, but critical points are closed and hence a countable union of compact subsets, which implies the same of the critical values.) is then measure zero. To show this consequence of Fubini directly, use the following argument:

First note that for any covering of $[a, b]$ by intervals, we may extract a finite subcovering of intervals whose total length is $\leq 2|b-a|$. Why? First choose a minimal subcovering $\{I_1, \ldots, I_p\}$, numbered according to their left endpoints. Then the total overlap is at most the length of $[a, b]$. Therefore the total length is at most $2|b-a|$.

Now let $B \subset \mathbb{R}^n$ be compact, so that we may assume $B \subset \mathbb{R}^{n-1} \times [a, b]$. We prove that if $B \cap P_c$ has measure zero in the hyperplane $P_c = \{x^n = c\}$, for any constant $c \in [a, b]$, then it has measure zero in $\mathbb{R}^n$.

If $B \cap P_c$ has measure zero, we can find a covering by open sets $R_c^i \subset P_c$ with total volume $< \epsilon$. For sufficiently small α_c, the sets $R_c^i \times [c-\alpha_c, c+\alpha_c]$ cover $B \cap \bigcup_{z \in [c-\alpha_c, c+\alpha_c]} P_z$ (since B is compact). As we vary c, the sets $[c-\alpha_c, c+\alpha_c]$ form a covering of $[a, b]$, and we extract a finite subcover $\{I_j\}$ of total length $\leq 2|b-a|$.

Let R_j^i be the set R_c^i for $I_j = [c-\alpha_c, c+\alpha_c]$. Then the sets $R_j^i \times I_j$ form a cover of B with total volume $\leq 2\epsilon|b-a|$. We can make this arbitrarily small, so that B has measure zero. □

We now proceed with the first step towards showing that transversality is generic.

Theorem 3.13 (Transversality theorem)**.** *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds where only X has boundary. Suppose that F and ∂F are transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ and ∂f_s are transverse to g.*

Proof. The fiber product $W = (X \times S) \times_Y Z$ is a regular submanifold (with boundary) of $X \times S \times Z$ and projects to S via the usual projection map π. We show that any $s \in S$ which is a regular value for both the projection map $\pi : W \longrightarrow S$ and its boundary map $\partial\pi$ gives rise to a f_s which is transverse to g. Then by Sard's theorem the s which fail to be regular in this way form a set of measure zero.

Suppose that $s \in S$ is a regular value for π. Suppose that $f_s(x) = g(z) = y$ and we now show that f_s is transverse to g there. Since $F(x, s) = g(z)$ and F is transverse to g, we know that

$$\mathrm{im} DF_{(x,s)} + \mathrm{im} Dg_z = T_y Y.$$

Therefore, for any $a \in T_y Y$, there exists $b = (w, e) \in T(X \times S)$ with $DF_{(x,s)}b - a$ in the image of Dg_z. But since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)}W$. Hence we observe that

$$(Df_s)(w - w') - a = DF_{(x,s)}[(w, e) - (w', e)] - a = (DF_{(x,s)}b - a) - DF_{(x,s)}(w', e),$$

where both terms on the right hand side lie in $\mathrm{im} Dg_z$.

Precisely the same argument (with X replaced with ∂X and F replaced with ∂F) shows that if s is regular for $\partial\pi$ then ∂f_s is transverse to g. This gives the result. ☐

The previous result immediately shows that transversal maps to $\mathbb{R}^n$ are generic, since for any smooth map $f : M \longrightarrow \mathbb{R}^n$ we may produce a family of maps

$$F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

via $F(x, s) = f(x) + s$. This new map F is clearly a submersion and hence is transverse to any smooth map $g : Z \longrightarrow \mathbb{R}^n$. For arbitrary target manifolds, we will imitate this argument, but we will require a (weak) version of Whitney's embedding theorem for manifolds into $\mathbb{R}^n$.

3.3 Whitney embedding

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2\dim M$ which Whitney showed (we shall reach $2\dim M + 1$ and possibly at the end of the course, show $N = 2\dim M$.)

Theorem 3.14 (Compact Whitney embedding in $\mathbb{R}^N$)**.** *Any* compact *manifold may be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 1.19 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M+1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_x M$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). □

Theorem 3.15 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$). *Any* compact *n-manifold may be embedded in* $\mathbb{R}^{2n+1}$.

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n + 1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M)\backslash\Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n + 1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n + 1$. □

Corollary 3.16. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in* $\mathbb{R}^N$*, then we can take* $N = 2n$ *instead of* $2n + 1$.

We now use Whitney embedding to prove genericity of transversality for all target manifolds, not just $\mathbb{R}^n$. We do this by embedding the manifold M into $\mathbb{R}^N$, translating it, and projecting back onto M.

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_y Y = \{v \in \mathbb{R}^N \ : \ v \perp T_y Y\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y, v) \in Y \times \mathbb{R}^N \ : \ v \in N_y Y\}.$$

This is an embedded submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension N, and it has a projection map $\pi : (y, v) \mapsto y$ such that (NY, π, Y) is a vector bundle. We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x, v) = x + v.$$

Definition 24. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y, v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N \ : \ \exists y \in Y \text{ with } |x - y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r \ : \ B(y, r) \subset U\}.$$

Theorem 3.17 (Tubular neighbourhood theorem)**.** *Every embedded submanifold of $\mathbb{R}^N$ has a tubular neighbourhood.*

Corollary 3.18. *Let X be a manifold with boundary and $f : X \longrightarrow Y$ be a smooth map to a manifold Y. Then there is an open ball $S = B(0, 1) \subset \mathbb{R}^N$ and a smooth map $F : X \times S \longrightarrow Y$ such that $F(x, 0) = f(x)$ and for fixed x, the map $f_x : s \mapsto F(x, s)$ is a submersion $S \longrightarrow Y$. In particular, F and ∂F are submersions.*

Proof. Embed Y in $\mathbb{R}^N$, and let $S = B(0, 1) \subset \mathbb{R}^N$. Then use the tubular neighbourhood to define

$$F(y, s) = (\pi \circ E^{-1})(f(y) + \epsilon(y)s),$$

□

The transversality theorem then guarantees that given any smooth $g : Z \longrightarrow Y$, for almost all $s \in S$ the maps $f_s, \partial f_s$ are transverse to g. We improve this slightly to show that f_s may be chosen to be *homotopic* to f.

Corollary 3.19 (Transverse deformation of maps)**.** *Given any smooth maps $f : X \longrightarrow Y$, $g : Z \longrightarrow Y$, where only X has boundary, there exists a smooth map $f' : X \longrightarrow Y$ homotopic to f with $f', \partial f'$ both transverse to g.*

Proof. Let S, F be as in the previous corollary. Away from a set of measure zero in S, the functions $f_s, \partial f_s$ are transverse to g, by the transversality theorem. But these f_s are all homotopic to f via the homotopy $X \times [0, 1] \longrightarrow Y$ given by

$$(x, t) \mapsto F(x, ts).$$

□

The last theorem we shall prove concerning transversality is a very useful extension result which is essential for intersection theory:

Theorem 3.20 (Transverse deformation of homotopies)**.** *Let X be a manifold with boundary and $f : X \longrightarrow Y$ a smooth map to a manifold Y. Suppose that ∂f is transverse to the closed map $g : Z \longrightarrow Y$. Then there exists a map $f' : X \longrightarrow Y$, homotopic to f and with $\partial f' = \partial f$, such that f' is transverse to g.*

Proof. First observe that since ∂f is transverse to g on ∂X, f is also transverse to g there, and furthermore since g is closed, f is transverse to g in a neighbourhood U of ∂X. (if $x \in \partial X$ but x not in $f^{-1}(g(Z))$ then since the latter set is closed, we obtain a neighbourhood of x for which f is transverse to g. If $x \in \partial X$ and $x \in f^{-1}(g(Z))$, then transversality at x implies transversality near x.)

Now choose a smooth function $\gamma : X \longrightarrow [0,1]$ which is 1 outside U but 0 on a neighbourhood of ∂X. (why does γ exist? exercise.) Then set $\tau = \gamma^2$, so that $d\tau(x) = 0$ wherever $\tau(x) = 0$. Recall the map $F : X \times S \longrightarrow Y$ we used in proving the transversality homotopy theorem 3.19 and modify it via

$$F'(x,s) = F(x, \tau(x)s).$$

Then F' and $\partial F'$ are transverse to g, and we can pick s so that $f' : x \mapsto F'(x,s)$ and $\partial f'$ are transverse to g. Finally, if x is in the neighbourhood of ∂X for which $\tau = 0$, then $f'(x) = F(x,0) = f(x)$. ☐

Corollary 3.21. *if $f : X \longrightarrow Y$ and $f' : X \longrightarrow Y$ are homotopic smooth maps of manifolds, each transverse to the closed map $g : Z \longrightarrow Y$, then the fiber products $W = X_f \times_g Z$ and $W' = X_{f'} \times_g Z$ are cobordant.*

Proof. if $F : X \times [0,1] \longrightarrow Y$ is the homotopy between $\{f, f'\}$, then by the previous theorem, we may find a (homotopic) homotopy $F' : X \times [0,1] \longrightarrow Y$ which is transverse to g. Hence the fiber product $U = (X \times [0,1])_{F'} \times_g Z$ is the cobordism with boundary $W \sqcup W'$. ☐

3.4 Intersection theory

The previous corollary allows us to make the following definition:

Definition 25. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X compact, g closed, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \sharp(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g, and where we assume the fiber product to consist of a finite number of points (this is always guaranteed, e.g. if g is proper, or if g is a closed embedding).

Example 3.22. *If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.*

Example 3.23. *If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle \cos(\theta/2)e_1 + \sin(\theta/2)e_2 \rangle$ and $\iota_2 : \theta \mapsto \langle \cos(\theta/2)e_2 + \sin(\theta/2)e_3 \rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2 \rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points. In particular, $\mathbb{R}P^2$ has a noncontractible loop.*

Example 3.24. *Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2 \dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.*

Example 3.25. *Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.*

Example 3.26. *By the previous argument, any compact manifold is not contractible.*

Example 3.27. *Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.*

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 3.28. *Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-cobordant, we cannot deform the planes to remove all intersection.*

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 26. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

Example 3.29. *Let $f : S^1 \longrightarrow S^1$ be given by $z \mapsto z^k$. Then $\deg_2(f) = k \pmod 2$.*