3.5 Associated vector bundles

Recall that the tangent bundle (TM, π_M, M) is a bundle of vector spaces

$$TM = \sqcup_p T_p M$$

which has local trivializations $\Phi : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ which preserve the projections to U, and which are linear maps for fixed p.

By the smoothness and linearity of these local trivializations, we note that, for charts (U_i, φ_i), we have

$$g_{ij} = (\Phi_j \circ \Phi_i^{-1}) = T(\varphi_j \circ \varphi_i^{-1}) : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k)$$

is a collection of smooth matrix-valued functions, called the "transition functions" of the bundle. These obviously satisfy the gluing conditions or "cocycle condition" $g_{ij} g_{jk} g_{ki} = 1_{k \times k}$ on $U_i \cap U_j \cap U_k$.

We will now use the tangent bundle to create other bundles, and for this we will use functors from the category of vector spaces $\mathbf{Vect}_{\mathbb{R}}$ to itself obtained from linear algebra.

Example 3.29 (Cotangent bundle)**.** *Consider the duality functor $V \mapsto V^* = \mathrm{Hom}(V, \mathbb{R})$ which is contravariant, i.e. if $A : V \longrightarrow W$ then $A^* : W^* \longrightarrow V^*$. Also, it is a smooth functor in the sense that the map $A \mapsto A^*$ is smooth map of vector spaces (in this case it is the identity map, essentially).*

The idea is to apply this functor to the bundle fibrewise, to apply it to the trivializations fiberwise, and to use the smoothness of the functor to obtain the manifold structure on the result.

Therefore we can form the set

$$T^*M = \sqcup_p (T_p M)^*,$$

which also has a projection map p_M to M. And, for each trivialization $\Phi : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^k$, we obtain bijections $F = (\Phi^*)^{-1} : {p_M}^{-1}(U) \longrightarrow U \times \mathbb{R}^k$. We use these bijections as charts for T^*M, and we check the smoothness by computing the transition functions:

$$(P_j \circ P_i^{-1}) = (g_{ij}^*)^{-1}.$$

Therefore we see that the transition functions for T^*M are the inverse duals of the transition functions for TM. Since this is still smooth, we obtain a smooth vector bundle. It is called the **cotangent bundle***.*

Example 3.30. *There is a well-known functor $\mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ given by $(V, W) \mapsto V \oplus W$. This is a smooth functor and we may apply it to our vector bundles to obtain new ones, such as $TM \oplus T^*M$. The transition functions for this particular example would be*

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & (g_{ij}^*)^{-1} \end{pmatrix}$$

Example 3.31 (Bundle of multivectors and differential forms)**.** *Recall that for any finite-dimensional vector space V, we can form the exterior algebra*

$$\wedge^\bullet V = \mathbb{R} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V,$$

for $n = \dim V$. The product is usually denoted $(a, b) \mapsto a \wedge b$, and it satisfies $a \wedge b = (-1)^{|a||b|} b \wedge a$. With this product, the algebra is generated by the degree 1 elements in V. So, $\wedge^\bullet V$ is a "finite dimensional $\mathbb{Z}$-graded algebra generated in degree 1".

If $(v_1, \ldots v_n)$ is a basis for V, then $v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $i_1 < \cdots < i_k$ form a basis for $\wedge^k V$. This space then has dimension $\binom{n}{k}$, hence the algebra $\wedge^\bullet V$ has dimension 2^n.

Note in particular that $\wedge^n V$ has dimension 1, is also called the determinant line $\det V$, and a choice of nonzero element in $\det V$ is called an "orientation" on the vector space V.

Recall that if $f : V \longrightarrow W$ is a linear map, then $\wedge^k f : \wedge^k V \longrightarrow \wedge^k W$ is defined on monomials via

$$\wedge^k f(a_1 \wedge \cdots \wedge a_k) = f(a_1) \wedge \cdots \wedge f(a_k).$$

In particular, if $A : V \longrightarrow V$ is a linear map, then for $n = \dim V$, the top exterior power $\wedge^n A : \wedge^n V \longrightarrow \wedge^n V$ is a linear map of a 1-dimensional space onto itself, and is hence given by a number, called $\det A$*, the determinant of A.*

We may now apply this functor to the tangent and cotangent bundles: we obtain new bundles $\wedge^\bullet TM$ and $\wedge^\bullet T^*M$, called the **bundle of multivectors** *and the* **bundle of differential forms***. Each of these bundles is a sum of degree k sub-bundles, called the k-multivectors and k-forms, respectively. We will be concerned primarily with sections of the bundle of k-forms, i.e.*

$$\Omega^k(M) = \Gamma(M, \wedge^k T^*M).$$

3.6 Coordinate representations

We are familiar with vector fields, which are sections of TM, and we know that a vector field is written locally in coordinates $(x^1, \ldots, x^n)$ as

$$X = \sum_i v^i \tfrac{\partial}{\partial x^i},$$

with coefficients v^i smooth functions.

There is an easy way to produce examples of 1-forms in $\Omega^1(M)$, using smooth functions f. We note that the action $X \mapsto X(f)$ defines a dual vector at each point of M, since $(X(f))_p$ depends only on the vector X_p and not the behaviour of X away from p. Recall that $X(f) = \pi_2 \circ Tf \circ X$.

Definition 27. The exterior derivative of a function f, denoted df, is the section of T^*M given by the fiber projection $\pi_2 \circ Tf$.

In a coordinate chart, we can apply d to the coordinates x^i; we obtain dx^i, which satisfy $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$. Therefore $(dx^1, \ldots, dx^n)$ is the dual basis to $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. Therefore, a section of T^*M has local expression

$$\xi = \sum_i \xi_i dx^i,$$

for ξ_i smooth functions, given by $\xi_i = \xi(\frac{\partial}{\partial x^i})$. In particular, the exterior derivative of a function df can be written

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

A general differential form $\rho \in \Omega^k(M)$ can be written

$$\rho = \sum_{i_1 < \cdots < i_k} \rho_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

4 Differential forms

There are several properties of differential forms which make them indispensible: first, the k-forms are intended to give a notion of k-dimensional volume (this is why they are multilinear and skew-symmetric, like the determinant) and in a way compatible with the boundary map (this leads to the exterior derivative, which we define below). Second, they behave well functorially, as we see now.

4.1 Pullback of forms

Given a smooth map $f : M \longrightarrow N$, we obtain bundle maps $f_* : TM \longrightarrow TN$ and hence $f^* := \wedge^k(f_*)^* : \wedge^k T^*N \longrightarrow \wedge^k T^*M$. Hence we have the diagram

$$\begin{tikzcd}
\wedge^k T^*M \arrow[d, "\pi_M"'] & \wedge^k T^*N \arrow[l, "f^*"'] \arrow[d, "\pi_N"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

The interesting thing is that if $\rho \in \Omega^k(N)$ is a differential form on N, then it is a section of π_N. Composing with f, f^*, we obtain a section $f^*\rho := f^* \circ \rho \circ f$ of π_M. Hence we obtain a natural map

$$\Omega^k(N) \xrightarrow{f^*} \Omega^k(M).$$

Such a natural map does not exist (in either direction) for multivector fields, for instance.

Suppose that $\rho \in \Omega^k(N)$ is given in a coordinate chart by $\rho = \sum \rho_{i_1\cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$. Now choose a coordinate chart for M with coordinates $x^1, \ldots x^m$. What is the local expression for $f^*\rho$? We need only compute f^*dy_i. We use a notation where f^k denotes the k^{th} component of f in the coordinates $(y^1, \ldots y^n)$, i.e. $f^k = y^k \circ f$.

$$\begin{aligned}
f^*dy_i(\tfrac{\partial}{\partial x^j}) &= dy_i(f_* \tfrac{\partial}{\partial x^j}) && (18)\\
&= dy_i(\sum_k \tfrac{\partial f^k}{\partial x^j} \tfrac{\partial}{\partial y_k}) && (19)\\
&= \tfrac{\partial f^i}{\partial x^j}. && (20)
\end{aligned}$$

Hence we conclude that

$$f^*dy_i = \sum_j \tfrac{\partial f^i}{\partial x^j} dx^j.$$

Finally we compute

$$\begin{aligned}
f^*\rho &= \sum_{i_1<\cdots<i_k} f^*\rho_{i_1\cdots i_k} f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}) && (21)\\
&= \sum_{i_1<\cdots<i_k} (\rho_{i_1\cdots i_k} \circ f) \sum_{j_1} \cdots \sum_{j_k} \tfrac{\partial f^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial f^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge \cdots dx^{j_k}. && (22)
\end{aligned}$$

4.2 The exterior derivative

Differential forms are equipped with a natural differential operator, which extends the exterior derivative of functions to all forms: $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. The exterior derivative is uniquely specified by the following requirements: first, it satisfies $d(df) = 0$ for all functions f. Second, it is a graded derivation of the algebra of exterior differential forms of degree 1, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta.$$

This allows us to compute its action on any 1-form $d(\xi_i dx^i) = d\xi_i \wedge dx^i$, and hence, in coordinates, we have

$$d(\rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_k \tfrac{\partial \rho}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Extending by linearity, this gives a local definition of d on all forms. Does it actually satisfy the requirements? this is a simple calculation: let $\tau_p = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\tau_q = dx^{j_1} \wedge \cdots \wedge dx^{j_q}$. Then

$$d((f\tau_p) \wedge (g\tau_q)) = d(fg\tau_p \wedge \tau_q) = (gdf + fdg) \wedge \tau_p \wedge \tau_q = d(f\tau_p) \wedge g\tau_q + (-1)^p f\tau_p \wedge d(g\tau_q),$$

as required.

Therefore we have defined d, and since the definition is coordinate-independent, we can be satisfied that d is well-defined.

Definition 28. d is the unique degree $+1$ graded derivation of $\Omega^\bullet(M)$ such that $df(X) = X(f)$ and $d(df) = 0$ for all functions f.

Example 4.1. *Consider $M = \mathbb{R}^3$. For $f \in \Omega^0(M)$, we have*

$$df = \tfrac{\partial f}{\partial x^1}dx^1 + \tfrac{\partial f}{\partial x^2}dx^2 + \tfrac{\partial f}{\partial x^3}dx^3.$$

Similarly, for $A = A_1dx^1 + A_2dx^2 + A_3dx^3$, we have

$$dA = (\tfrac{\partial A_2}{\partial x^1} - \tfrac{\partial A_1}{\partial x^2})dx^1 \wedge dx^2 + (\tfrac{\partial A_3}{\partial x^1} - \tfrac{\partial A_1}{\partial x^3})dx^1 \wedge dx^3 + (\tfrac{\partial A_3}{\partial x^2} - \tfrac{\partial A_2}{\partial x^3})dx^2 \wedge dx^3$$

Finally, for $B = B_{12}dx^1 \wedge dx^2 + B_{13}dx^1 \wedge dx^3 + B_{23}dx^2 \wedge dx^3$, we have

$$dB = (\tfrac{\partial B_{12}}{\partial x^3} - \tfrac{\partial B_{13}}{\partial x^2} + \tfrac{\partial B_{23}}{\partial x^1})dx^1 \wedge dx^2 \wedge dx^3.$$

Definition 29. The form $\rho \in \Omega^\bullet(M)$ is called *closed* when $d\rho = 0$ and *exact* when $\rho = d\tau$ for some τ.

Example 4.2. *A function $f \in \Omega^0(M)$ is closed if and only if it is constant on each connected component of M: This is because, in coordinates, we have*

$$df = \tfrac{\partial f}{\partial x^1}dx^1 + \cdots + \tfrac{\partial f}{\partial x^n}dx^n,$$

and if this vanishes, then all partial derivatives of f must vanish, and hence f must be constant.

Theorem 4.3. *The exterior derivative of an exact form is zero, i.e. $d \circ d = 0$. Usually written $d^2 = 0$.*

Proof. The graded commutator $[d_1, d_2] = d_1 \circ d_2 - (-1)^{|d_1||d_2|}d_2 \circ d_1$ of derivations of degree $|d_1|, |d_2|$ is always (why?) a derivation of degree $|d_1| + |d_2|$. Hence we see $[d, d] = d \circ d - (-1)^{1 \cdot 1}d \circ d = 2d^2$ is a derivation of degree 2 (and so is d^2). Hence to show it vanishes we must test on functions and exact 1-forms, which locally generate forms since every form is of the form $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

But $d(df) = 0$ by definition and this certainly implies $d^2(df) = 0$, showing that $d^2 = 0$. $\square$

The fact that $d^2 = 0$ is dual to the fact that $\partial(\partial M) = \emptyset$ for a manifold with boundary M. We will see later that Stokes' theorem explains this duality. Because of the fact $d^2 = 0$, we have a very special algebraic structure: we have a sequence of vector spaces $\Omega^k(M)$, and maps $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ which are such that any successive composition is zero. This means that the image of d is contained in the kernel of the next d in the sequence. This arrangement of vector spaces and operators is called a *cochain complex* of vector spaces [2]. We often simply refer to this as a "complex" and omit the term "cochain". The reason for the "co" is that the differential increases the degree k, which is opposite to the usual boundary map on manifolds, which decreases k. We will see chain complexes when we study homology.

A complex of vector spaces is usually drawn as a linear sequence of symbols and arrows as follows: if $f : U \longrightarrow V$ is a linear map and $g : V \longrightarrow W$ is a linear map such that $g \circ f = 0$, then we write

$$U \xrightarrow{f} V \xrightarrow{g} W$$

In general, this simply means that $\mathrm{im} f \subset \ker g$, and to measure the difference between them we look at the quotient $\ker g/\mathrm{im} f$, which is called the **cohomology** of the complex at the position V (or homology, if d decreases degree). If we are lucky, and the complex has no cohomology at V, meaning that $\ker g$ is precisely

[2] since this complex appears for $\Omega^\bullet(U)$ for any open set $U \subset M$, this is actually a cochain complex of *sheaves* of vector spaces, but this won't concern us right away.

equal to $\mathrm{im} f$, then we say that the complex is **exact** at V. If the complex is exact everywhere, we call it an exact sequence (and it has no cohomology!) In our case, we have a longer cochain complex:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

There is a bit of terminology to learn: we have seen that if $d\rho = 0$ then ρ is called *closed*. But these are also called **cocycles** and denoted $Z^k(M)$. Similarly the exact forms $d\alpha$ are also called **coboundaries**, and are denoted $B^k(M)$. Hence the cohomology groups may be written $H^k_{dR}(M) = Z^k_{dR}(M)/B^k_{dR}(M)$.

Definition 30. The de Rham complex is the complex $(\Omega^\bullet(M), d)$, and its cohomology at $\Omega^k(M)$ is called $H^k_{dR}(M)$, the de Rham cohomology.

Exercise: Check that the graded vector space $H^\bullet_{dR}(M) = \bigoplus_{k\in\mathbb{Z}} H^k(M)$ inherits a product from the wedge product of forms, making it into a $\mathbb{Z}$-graded ring. This is called the de Rham cohomology ring of M, and the product is called the *cup product*.

It is clear from the definition of d that it commutes with pullback via diffeomorphisms, in the sense $f^* \circ d = d \circ f^*$. But this is only a special case of a more fundamental property of d:

Theorem 4.4. *Exterior differentiation commutes with pullback: for $f : M \longrightarrow N$ a smooth map, $f^* \circ d_N = d_M \circ f^*$.*

Proof. We need only check this on functions g and exact 1-forms dg: let X be a vector field on M and $g \in C^\infty(N, \mathbb{R})$.

$$f^*(dg)(X) = dg(f_*X) = \pi_2 g_* f_* X = \pi_2(g \circ f)_* X = d(f^*g)(X),$$

giving $f^*dg = df^*g$, as required. For exact 1-forms we have $f^*d(dg) = 0$ and $d(f^*dg) = d(df^*g) = 0$ by the result for functions. $\square$

This theorem may be interpreted as follows: The differential forms give us a $\mathbb{Z}$-graded ring, $\Omega^\bullet(M)$, which is equipped with a differential $d : \Omega^k \longrightarrow \Omega^{k+1}$. This sequence of vector spaces and maps which compose to zero is called a *cochain complex*. Beyond it being a cochain complex, it is equipped with a wedge product.

Cochain complexes $(C^\bullet, d_C)$ may be considered as objects of a new category, whose morphisms consist of a sum of linear maps $\psi_k : C^k \longrightarrow D^k$ commuting with the differentials, i.e. $d_D \circ \psi_k = \psi_{k+1} \circ d_C$. The previous theorem shows that pullback f^* defines a morphism of cochain complexes $\Omega^\bullet(N) \longrightarrow \Omega^\bullet(M)$; indeed it even preserves the wedge product, hence it is a morphism of differential graded algebras.

Corollary 4.5. *We may interpret the previous result as showing that $\Omega^\bullet$ is a functor from manifolds to differential graded algebras (or, if we forget the wedge product, to the category of cochain complexes). As a result, we see that the de Rham cohomology $H^\bullet_{dR}$ may be viewed as a functor, from smooth manifolds to $\mathbb{Z}$-graded commutative rings.*