

Having defined the integral, we wish to explain the duality between d and ∂ : A $n-1$ -form α on a n -manifold may be pulled back to the boundary ∂M and integrated. On the other hand, it can be differentiated and integrated over M . The fact that these are equal is Stokes' theorem, and is a generalization of the fundamental theorem of calculus.

First we must make some simple observations concerning the behaviour of forms in a neighbourhood of the boundary.

Recall the operation of contraction with a vector field X , which maps $\rho \in \Omega^k(M)$ to $i_X \rho \in \Omega^{k-1}(M)$, defined by the condition of being a graded derivation $i_X(\alpha \wedge \beta) = i_X \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge i_X \beta$ such that $i_X f = 0$ and $i_X df = X(f)$ for all $f \in C^\infty(M, \mathbb{R})$.

Proposition 4.13. *Let M be a manifold with boundary. If M is orientable, then so is ∂M . Furthermore, an orientation on M induces one on ∂M .*

Proof. We need a vector field X which is tangent to M , pointing outward everywhere along the boundary (and nonvanishing on the boundary). This can be constructed by taking a locally finite covering $\{U_i\}$ of the boundary of M by charts of M , choosing vector fields X_i in U_i corresponding to the outward pointing vector field $-\frac{\partial}{\partial x^m}$ in the coordinates, and forming $X = \sum_i \theta_i X_i$ for a partition of unity $\{\theta_i\}$.

Given an orientation $[v]$ of M , we can form $[j^*(i_X v)]$, for $j : \partial M \rightarrow M$ the inclusion of the boundary. This is then an orientation of ∂M . This depends only on $[v]$ and X being a nonvanishing outward vector field. \square

We now verify a local computation leading to Stokes' theorem. If

$$\alpha = \sum_i a_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^m$$

is a degree $m-1$ form with compact support in $U \subset H^m$, and if U does not intersect the boundary ∂H^m , then by the fundamental theorem of calculus,

$$\int_U d\alpha = \sum_i (-1)^{i-1} \int_U \frac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m = 0.$$

Now suppose that $V = U \cap \partial H^m \neq \emptyset$. Then

$$\begin{aligned} \int_U d\alpha &= \sum_i (-1)^{i-1} \int_U \frac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m \\ &= -(-1)^{m-1} \int_V a_m(x_1, \dots, x_{m-1}, 0) dx^1 \cdots dx^{m-1} \\ &= \int_V a_m(x_1, \dots, x_{m-1}, 0) i_{-\frac{\partial}{\partial x^m}} (dx^1 \wedge \cdots \wedge dx^m) \\ &= \int_V j^* \alpha, \end{aligned}$$

where the last integral is with respect to the orientation induced by the outward vector field.

Theorem 4.14 (Stokes' theorem). *Let M be an oriented manifold with boundary, and let the boundary be oriented with respect to an outward pointing vector field. Then for $\alpha \in \Omega_c^{m-1}(M)$ and $j : \partial M \rightarrow M$ the inclusion of the boundary, we have*

$$\int_M d\alpha = \int_{\partial M} j^* \alpha.$$

Proof. For a locally finite atlas (U_i, φ_i) , we have

$$\int_M d\alpha = \int_M d\left(\sum_i \theta_i \alpha\right) = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i \alpha)$$

By the local calculation above, if $\varphi_i(U_i) \cap \partial H^m = \emptyset$, the summand on the right hand side vanishes. On the other hand, if $\varphi_i(U_i) \cap \partial H^m \neq \emptyset$, we obtain (letting $\psi_i = \varphi_i|_{U_i \cap \partial H^m}$ and $j' : \partial H^m \rightarrow \mathbb{R}^n$), using the local result,

$$\begin{aligned} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i \alpha) &= \int_{\varphi_i(U_i) \cap \partial H^m} j'^* (\varphi_i^{-1})^* (\theta_i \alpha) \\ &= \int_{\varphi_i(U_i) \cap \partial H^m} (\psi_i^{-1})^* (j'^* (\theta_i \alpha)). \end{aligned}$$

This then shows that $\int_M d\alpha = \int_{\partial M} j^* \alpha$, as desired. \square

Corollary 4.15. *If $\partial M = \emptyset$, then for all $\alpha \in \Omega_c^{n-1}(M)$, we have $\int_M d\alpha = 0$.*

Corollary 4.16. *Let M be orientable and compact, and let $v \in \Omega^n(M)$ be nonvanishing. Then $\int_M v > 0$, when M is oriented by $[v]$. Hence, v cannot be exact, by the previous corollary. This tells us that the class $[v] \in H_{dR}^n(M)$ cannot be zero. In this way, integration of a closed form may often be used to show that it is nontrivial in de Rham cohomology.*

4.4 The Mayer-Vietoris sequence

Decompose a manifold M into a union of open sets $M = U \cup V$. We wish to relate the de Rham cohomology of M to that of U and V separately, and also that of $U \cap V$. These 4 manifolds are related by obvious inclusion maps as follows:

$$U \cup V \longleftarrow U \sqcup V \begin{matrix} \xleftarrow{\partial_U} \\ \xrightarrow{\partial_V} \end{matrix} U \cap V$$

Applying the functor Ω^\bullet , we obtain morphisms of complexes in the other direction, given by simple restriction (pullback under inclusion):

$$\Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \begin{matrix} \xrightarrow{\partial_V^*} \\ \xleftarrow{\partial_U^*} \end{matrix} \Omega^\bullet(U \cap V)$$

Now we notice the following: if forms $\omega \in \Omega^\bullet(U)$ and $\tau \in \Omega^\bullet(V)$ come from a single global form on $U \cup V$, then they are killed by $\partial_V^* - \partial_U^*$. Hence we obtain a complex of (morphisms of cochain complexes):

$$0 \longrightarrow \Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\partial_V^* - \partial_U^*} \Omega^\bullet(U \cap V) \longrightarrow 0 \quad (23)$$

It is clear that this complex is exact at the first position, since a form must vanish if it vanishes on U and V . Similarly, if forms on U, V agree on $U \cap V$, they must glue to a form on $U \cup V$. Hence the complex is exact at the middle position. We now show that the complex is exact at the last position.

Theorem 4.17. *The above complex (of de Rham complexes) is exact. It may be called a “short exact sequence” of cochain complexes.*

Proof. Let $\alpha \in \Omega^q(U \cap V)$. We wish to write α as a difference $\tau - \omega$ with $\tau \in \Omega^q(U)$ and $\omega \in \Omega^q(V)$. Let (ρ_U, ρ_V) be a partition of unity subordinate to (U, V) . Then we have $\alpha = \rho_U \alpha - (-\rho_V \alpha)$ in $U \cap V$. Now observe that $\rho_U \alpha$ may be extended by zero in V (call the result τ), while $-\rho_V \alpha$ may be extended by zero in U (call the result ω). Then we have $\alpha = (\partial_V^* - \partial_U^*)(\tau, \omega)$, as required. \square

It is not surprising that given an exact sequence of morphisms of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

we obtain maps between the cohomology groups of the complexes

$$H^k(A^\bullet) \xrightarrow{f_*} H^k(B^\bullet) \xrightarrow{g_*} H^k(C^\bullet).$$

And it is not difficult to see that this sequence is exact at the middle term: Let $[\rho] \in H^k(B^\bullet)$, for $\rho \in B^k$ such that $d_B \rho = 0$. Suppose that $g(\rho)$ vanishes in cohomology, meaning $g(\rho) = d_C \gamma$ in C^k . Then by surjectivity of g , there exists $q \in B^{k-1}$ with $g(q) = \gamma$, and then $g(\rho - d_B q) = 0$, so that by middle exactness, there exists $\tau \in A^k$ with $f(\tau) = \rho - d_B q$. Then since f is a morphism of complexes, it follows that $f(d_A \tau) = d_B f(\tau) = d_B \rho = 0$. Since $f : A^{k+1} \rightarrow B^{k+1}$ is injective, this implies that $d_A \tau = 0$, so we have an actual cohomology class $[\tau]$ such that $f_*[\tau] = [\rho]$, as required.

The interesting thing is that the maps g_* are not necessarily surjective, nor are f_* necessarily injective. In fact, there is a natural map $\delta : H^k(C^\bullet) \rightarrow H^{k+1}(A^\bullet)$ (called the connecting homomorphism) which extends the 3-term sequence to a full complex involving all cohomology groups of arbitrary degree:

If $[\alpha] \in H^k(C^\bullet)$, where $d_C \alpha = 0$, then there must exist $\xi \in B^k$ with $g(\xi) = \alpha$, and $g(d_B \xi) = d_C(g(\xi)) = d_C \alpha = 0$, so that there must exist $\beta \in A^{k+1}$ with $f(\beta) = d_B \xi$, and $f(d_A \beta) = d_B(f(\beta)) = 0$. Hence this determines a class $[\beta] \in H^{k+1}(A^\bullet)$, and one can check that this does not depend on the choices made. We then define $\delta([\alpha]) = [\beta]$.

Exercise: with this definition of δ , we obtain a “long exact sequence” of vector spaces as follows:

$$\begin{array}{ccc} H^\bullet(A) & \xrightarrow{f_*} & H^\bullet(B) \\ & \nwarrow \delta^{+1} \quad \nearrow g_* & \\ & H^\bullet(C) & \end{array}$$

Therefore, from the complex of complexes (23), we immediately obtain a long exact sequence of vector spaces, called the Mayer-Vietoris sequence:

$$\cdots \rightarrow H^k(U \cup V) \rightarrow H^k(U) \oplus H^k(V) \rightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(U \cup V) \rightarrow \cdots,$$

where the first map is simply a restriction map, the second map is the difference of the restrictions $\delta_V^* - \delta_U^*$, and the third map is the connecting homomorphism δ , which can be written explicitly as follows:

$$\delta[\alpha] = [\beta], \quad \beta = -d(\rho_V \alpha) = d(\rho_U \alpha).$$

(notice that β has support contained in $U \cap V$.)

4.5 Examples of cohomology computations

Example 4.18 (Circle). Here we present another computation of $H_{dR}^\bullet(S^1)$, by the Mayer-Vietoris sequence. Express $S^1 = U_0 \cup U_1$ as before, with $U_i \cong \mathbb{R}$, so that $H^0(U_i) = \mathbb{R}$, $H_{dR}^1(U_i) = 0$ by the Poincaré lemma. Since $U_0 \cap U_1 \cong \mathbb{R} \sqcup \mathbb{R}$, we have $H^0(U_0 \cap U_1) = \mathbb{R} \oplus \mathbb{R}$ and $H^1(U_0 \cap U_1) = 0$. Since we know that $H_{dR}^2(S^1) = 0$, the Mayer-Vietoris sequence only has 4 a priori nonzero terms:

$$0 \rightarrow H^0(S^1) \rightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta_1^* - \delta_0^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1(S^1) \rightarrow 0.$$

The middle map takes $(c_1, c_0) \mapsto c_1 - c_0$ and hence has 1-dimensional kernel. Hence $H^0(S^1) = \mathbb{R}$. Furthermore the kernel of δ must only be 1-dimensional, hence $H^1(S^1) = \mathbb{R}$ as well. Exercise: Using a partition of unity, determine an explicit representative for the class in $H_{dR}^1(S^1)$, starting with the function on $U_0 \cap U_1$ which takes values 0,1 on each respective connected component.

Example 4.19 (Spheres). To determine the cohomology of S^2 , decompose into the usual coordinate charts U_0, U_1 , so that $U_i \cong \mathbb{R}^2$, while $U_0 \cap U_1 \sim S^1$. The first line of the Mayer-Vietoris sequence is

$$0 \longrightarrow H^0(S^2) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}.$$

The third map is nontrivial, since it is just the subtraction. Hence this first line must be exact, and $H^0(S^2) = \mathbb{R}$ (not surprising since S^2 is connected). The second line then reads (we can start it with zero since the first line was exact)

$$0 \longrightarrow H^1(S^2) \longrightarrow 0 \longrightarrow H^1(S^1) = \mathbb{R},$$

where the second zero comes from the fact that $H^1(\mathbb{R}^2) = 0$. This then shows us that $H^1(S^2) = 0$. The last term, together with the third line now give

$$0 \longrightarrow H^1(S^1) = \mathbb{R} \longrightarrow H^2(S^2) \longrightarrow 0,$$

showing that $H^2(S^2) = \mathbb{R}$.

Continuing this process, we obtain the de Rham cohomology of all spheres:

$$H_{dR}^k(S^n) = \begin{cases} \mathbb{R}, & \text{for } k = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$