1.6 Smooth functions and partitions of unity

The set $C^\infty(M,\mathbb{R})$ of smooth functions on M inherits much of the structure of $\mathbb{R}$ by composition. $\mathbb{R}$ is a ring, having addition $+ : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and multiplication $\times : \mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ which are both smooth. As a result, $C^\infty(M,\mathbb{R})$ is as well: One way of seeing why is to use the smooth diagonal map $\Delta : M \longrightarrow M\times M$, i.e. $\Delta(p) = (p, p)$.

Then, given functions $f, g \in C^\infty(M,\mathbb{R})$ we have the sum $f + g$, defined by the composition

$$M \xrightarrow{\Delta} M\times M \xrightarrow{f\times g} \mathbb{R}\times\mathbb{R} \xrightarrow{+} \mathbb{R}\,.$$

We also have the product fg, defined by the composition

$$M \xrightarrow{\Delta} M\times M \xrightarrow{f\times g} \mathbb{R}\times\mathbb{R} \xrightarrow{\times} \mathbb{R}\,.$$

Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$, given by $f \mapsto f\circ\varphi$. This is called the pullback of functions, and defines a homomorphism of rings since $\Delta\circ\varphi = (\varphi\times\varphi)\circ\Delta$.

The association $M \mapsto C^\infty(M,\mathbb{R})$ and $\varphi\mapsto\varphi^*$ takes objects and arrows of C^∞-Man to objects and arrows of the category of rings, respectively, in such a way which respects identities and composition of morphisms. Such a map is called a functor. In this case, it has the peculiar property that it switches the source and target of morphisms. It is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M,\mathbb{R})$, but actually all automorphisms are of this form (Why?). Also, if M is a compact manifold, then an ideal $\mathcal{I} \subset C^\infty(M,\mathbb{R})$ is maximal if and only if it is the vanishing ideal $\{f \in C^\infty(M,\mathbb{R}) \;:\; f(p) = 0\}$ of a point $p\in M$ (Why? Also, Why must M be compact?).

The key tool for understanding the ring $C^\infty(M,\mathbb{R})$ is the partition of unity. This will allow us to *go from local to global*, i.e. to glue together objects which are defined locally, creating objects with global meaning.

Definition 8. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x\in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 9. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 1.18. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i,\varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0,3)$ and $\{V_i = \varphi_i^{-1}(B(0,1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x\in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P}_1$, and

$$K_{i+1} = \overline{P_1\cup\cdots\cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1\cup\cdots\cup P_r$.

Now note that M is the union of ring-shaped sets $K_i\backslash K_{i-1}^\circ$, each of which is compact. If $p\in A_\alpha$, then $p\in K_{i+2}\backslash K_{i-1}^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha},\varphi_{p,\alpha})$ with $U_{p,\alpha}\subset K_{i+2}\backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p,α vary, these neighbourhoods cover the compact set $K_{i+1}\backslash K_i^\circ$ without leaving the band $K_{i+2}\backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k},\varphi_{i,k})$ is the desired locally finite refinement. ☐

Definition 10. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\text{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R}\backslash\{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \quad \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\text{supp} f_\alpha \subset U_i$ for some i.

Theorem 1.19. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and supp$f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support supp$g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \dots.$$

□

Corollary 1.20 (Existence of bump functions)**.** *Let $A \subset M$ be any closed subset of a manifold, and let U be any open neighbourhood of A. Then there exists a smooth function $f_U : M \longrightarrow \mathbb{R}$ with $f_U \equiv 1$ on A and supp$f_U \subset U$.*

Proof. Consider the open cover $\{U, M\backslash A\}$ of M. Choose a regular subcover (U_i, φ_i) with subordinate partition of unity f_i. Then let f_U be the sum of all f_i with support contained in U. □

One interesting application of partitions of unity is to the extension of any chart $\varphi_i : U_i \subset M \longrightarrow \mathbb{R}^n$ to a smooth mapping $\varphi : M \longrightarrow \mathbb{R}^n$. If ψ is a bump function supported in U_i, then take

$$\varphi(x) = \begin{cases} 0 \text{ for } x \in M\backslash U_i \\ \psi(x)\varphi_i(x) \text{ for } x \in U_i \end{cases}$$

2 The tangent functor

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M, which is made up of the disjoint union of all tangent spaces to all points in M. Usually we think of tangent spaces as subspaces of Euclidean space which approximate a curved subset, but interestingly, the tangent space does not require an ambient space in order to be defined. In other words, the tangent space is "intrinsic" to the manifold and does not depend on any embedding.

As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

Definition 11. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM, and forms a real vector space of dimension $\dim M$.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_pM,$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 2.1. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n,$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

A more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \;\Leftrightarrow\; y = \varphi_j \circ \varphi_i^{-1}(x) \;\text{ and }\; v = D(\varphi_j \circ \varphi_i^{-1})_x u,$$

and verify the conditions of the general gluing construction (Assignment 1), obtaining a manifold $TM_{\mathcal{A}}$. Then show that the result is independent of the chosen atlas in the smooth structure: for a different atlas $\mathcal{A}'$, one obtains a diffeomorphism $\varphi_{\mathcal{A}\mathcal{A}'} : TM_{\mathcal{A}} \longrightarrow TM_{\mathcal{A}'}$ which itself satisfies $\varphi_{\mathcal{A}'\mathcal{A}''} \circ \varphi_{\mathcal{A}\mathcal{A}'} = \varphi_{\mathcal{A}\mathcal{A}''}$.

It is easy to see from the definition that (TM, π_M, M) is a fiber bundle with fiber type $\mathbb{R}^n$; in fact there is slightly more structure involved: the tangent spaces T_pM have a natural vector space structure and the given local trivializations $\Phi : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ preserve the vector space structure on each fiber, i.e. $\Phi|_{T_pM} : T_pM \longrightarrow \{p\} \times \mathbb{R}^n$ is a linear map for all p. This makes (TM, π_M, M) into a vector bundle.