

Say, $P := (P_1, \dots, P_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$, assume $\#P^{-1}(0) < \infty$

$$\infty > \dim_{\mathbb{C}} \frac{\mathbb{C}[x]}{(P) \cdot \mathbb{C}[x]} = \sum_{a \in P^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathbb{C}[x]_a}{(P) \cdot \mathbb{C}[x]_a}$$

Let $\mathcal{L}P_j(x) := HP_j(0, x)$, where $HP_j(x_0, x) := x_0^{\deg P_j} \cdot P_j(\frac{x}{x_0})$.

Assume also $\{\mathcal{L}P_1(x) = \dots = \mathcal{L}P_n(x) = 0\} = \{0\} (*)$

(so called “no solutions at ∞ ”) for $\mathbb{C}^n \hookrightarrow \mathbb{P}^n \Rightarrow$

Bezout's Theorem

a. $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is proper finite to one

$$b. \sum_{a \in P^{-1}(0)} \deg_a^{\mathbb{C}} P = \deg P = \prod_{1 \leq j \leq n} \deg P_j$$

Proof of $(*) \Rightarrow a$.

Say, $\xi_k \in \mathbb{C}^n$, assume $\xi_k \nearrow \infty$ and all $|P_j(\xi_k)| \leq C < \infty$

$$a_k := \frac{\xi_k}{\|\xi_k\|} \in S^1 \Rightarrow \exists \text{ subsequence } a_k \rightarrow a \neq 0$$

$$\rho_k := \frac{1}{\|\xi_k\|} \rightarrow 0, 0 < d_j = \deg P_j$$

Then it follows $\forall j, P_j(\xi_k) = [P_j(\frac{a_k}{\rho_k}) \cdot \rho_k^{d_j}] \cdot \frac{1}{\rho_k^{d_j}} = HP_j(\rho_k, a_k) \cdot \frac{1}{\rho_k^{d_j}}$

Therefore, $|HP_j(\rho_k, a_k)| \leq C \cdot \rho_k^{d_j} \Rightarrow \mathcal{L}P_j(a) = 0 \Rightarrow ?! \checkmark$

Now prove b. in three steps.

Step 1 is $\deg_0^{\mathbb{C}} \mathcal{L}P = \deg_0^{\mathbb{C}} HP$, where

$$\mathcal{L}P := (\mathcal{L}P_1, \dots, \mathcal{L}P_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$$

$$HP: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$$

$$HP(x_0, x) := (x_0, HP_1(x_0, x), \dots, HP_n(x_0, x))$$

y is a regular value for $\mathcal{L}P \Rightarrow (0, y)$ is regular value for HP

$\Rightarrow$ (obviously) $\#HP^{-1}(0, y) = \#\mathcal{L}P^{-1}(y)$, which suffices.

Indeed

$$\frac{\partial HP}{\partial(x_0, x)}|_{x_0=0} = \begin{pmatrix} \underline{1} & * & * & * \\ 0 & & & \\ 0 & & \frac{\partial HP}{\partial x} & \\ 0 & & & \end{pmatrix} \Big|_{x_0=0}$$

$$\text{and } \frac{\partial HP}{\partial x}(0, x)|_{x_0=0} = \frac{\partial \mathcal{L}P}{\partial x}(x)$$

Step 1. completed. $\checkmark$

Step 2 is $\deg P = \deg_0^{\mathbb{C}} HP$



Step 3 is $\deg_0^{\mathbb{C}} \mathcal{L}P = \prod_j \deg P_j$. Let $\phi := \mathcal{L}P \Rightarrow \phi^{-1}(0) = \{0\}$

Proof is split into

Thm. $0 \neq \mathcal{J}_\phi = \deg_0^{\mathbb{C}} \phi \cdot \delta(x, 0) \pmod{(\phi) \cdot \mathbb{C}\{x\}}$

(with $\delta(x, y)$ from 2 below). Claim. $\mathcal{J}_\phi = \prod_{1 \leq j \leq n} d_j \cdot \delta(x, 0)$.

Proof of Thm

1. Using Mumford's Lemma, $\exists$ open $U \ni 0$, $V \ni 0$ s.th.

$\phi := \phi|_U: U \rightarrow V$ proper. (and $U \cap \phi^{-1}(0) = \{0\}$)

Lemma

$\Rightarrow Z := \text{Cr.Val}(\phi) := \phi(\{\mathcal{J}_\phi = 0\}) \subset V$ closed analytic.

$$2. A(x, y) := \int_0^1 \frac{\partial \phi}{\partial x}(tx + (1-t)y) dt \Rightarrow A(x, y) \cdot (x - y) =$$

$$\phi(x) - \phi(y) \text{ and } \mathcal{J}_\phi(x) = \delta(x, x), \text{ where } \delta(x, y) := \det A(x, y)$$

$$3. \text{ Let } h(x, z) := \sum_{\{y: \phi(y)=z\}} \delta(x, y) \Rightarrow h \text{ is } \mathbb{C}\text{-analytic on } V \setminus Z$$

$$\text{and bounded} \Rightarrow h \text{ analytic on } V \text{ (Riemann Extension Thm)}$$

$$4. h(x, 0) = \deg_0^{\mathbb{C}} \phi \cdot \delta(x, 0) \text{ due to } \phi|_U^{-1}(0) = \{0\}.$$

$$5. \mathcal{J}_\phi(x) - h(x, z) = 0 \text{ if } z = \phi(x) \notin Z$$

$$\Rightarrow \mathcal{J}_\phi(x) - h(x, z) \in (z - \phi(x)) \cdot \mathbb{C}\{x, z\}$$

$$6. \text{ Set } z = 0 \Rightarrow \mathcal{J}_\phi(x) - (\deg_0^{\mathbb{C}} \phi) \cdot \delta(x, 0) \in (\phi) \cdot \mathbb{C}\{x\}, \text{ as required.}$$

Recall.

$$\deg \mathcal{L}P_j = \deg P_j = d_j, \quad \phi_j := \mathcal{L}P_j \Rightarrow \frac{\partial \phi_j}{\partial x_i}(tx) = t^{d_j-1} \frac{\partial \phi_j}{\partial x_i}(x)$$

Claim $\mathcal{J}_\phi = d_1 \cdots d_n \cdot \delta(x, 0)$

Cor. Since $\mathcal{J}_\phi \notin (\phi) \cdot \mathbb{C}\{x\}$ (Slides 9-11) $\Rightarrow \deg_0^{\mathbb{C}} \phi = d_1 \cdots d_n$

Proof. Explicit calculation

$$\begin{aligned} A(x, 0) &:= \int_0^1 \left\{ \frac{\partial \phi_j}{\partial x_i}(tx) \right\} dt = \int_0^1 \begin{bmatrix} t^{d_1-1} & & 0 \\ & \ddots & \\ 0 & & t^{d_n-1} \end{bmatrix} \left(\frac{\partial \phi}{\partial x}(x) \right) dt \\ &= \begin{bmatrix} \frac{1}{d_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{d_n} \end{bmatrix} \left(\frac{\partial \phi}{\partial x}(x) \right) \quad \checkmark \end{aligned}$$

Thm. $\phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0), \phi^{-1}(0) = \{0\} \Rightarrow \mathcal{J}_\phi(x) \notin (\phi) \cdot \mathbb{C}\{x\}$

Thm $\Leftarrow$ Claim. $h \in \mathbb{C}\{x\} \Rightarrow H_h(z) := \sum_{x \in \phi^{-1}(z)} \frac{h(x)}{\mathcal{J}_\phi(x)} \in \mathbb{C}\{z\}$

Indeed, $H_{\mathcal{J}_\phi} \equiv \deg_0^{\mathbb{C}} \phi \neq 0$, while $g = \sum_j \phi_j \cdot h_j \in (\phi) \cdot \mathbb{C}\{x\} \Rightarrow$

$H_g(z) = \sum_j z_j \cdot H_{h_j}(z) \in (z) \cdot \mathbb{C}\{z\}$, as required.

Proof of Claim $X := \phi|_U(\{\mathcal{J}_\phi|_U = 0\})$.

(Note $X = \{\prod_{x \in \phi|_U^{-1}(z)} \mathcal{J}_\phi(x) = 0\} \subset V$)

$\phi|_{\mathcal{J}_\phi=0} : \{\mathcal{J}_\phi = 0\} \rightarrow X$ proper fin. to one $\Rightarrow \dim_{\mathbb{C}} Z < \dim_{\mathbb{C}} X$

where $Z = \phi|_U(\text{Sing}\{\mathcal{J}_\phi = 0\}) \cup \text{Sing}X \cup \text{Cr.Val}(\phi|_{\mathcal{J}_\phi=0})$ analytic.

Step 1. $\forall a \in \{\mathcal{J}_{\phi|_U} = 0\} \setminus \phi|_U^{-1}(Z)$, $\exists$ local coord. changes s.th.

$$a = 0, \phi(a) = 0, \{\mathcal{J}_{\phi} = 0\} = \{x_1 = 0\}, X = \{z_1 = 0\}$$

$$\Rightarrow \phi(0, x_2, \dots, x_n) = (0, x_2, \dots, x_n). \text{ Also } \mathcal{J}_{\phi} \approx x_1^k$$

$$\text{Say } \phi_1(x_1, x_2, \dots, x_n) := x_1^d \cdot g(x), g(x) \notin (x_1) \cdot \mathbb{C}\{x\} \quad (d \geq 1)$$

$$\text{and } \theta(x) := \det\left[\frac{\partial(\phi_2, \dots, \phi_n)}{\partial(x_2, \dots, x_n)}\right]_{|x_1=0} = 1 \Rightarrow k = d - 1, g(0) \neq 0.$$

$$\text{Say, } h(x)^d = g(x) \Rightarrow (\text{ coord. change } x \mapsto (x_1 h(x), \phi_2, \dots, \phi_n))$$

$$\text{Near } a, \text{ we may assume } \phi(x) = (x_1^d, x_2, \dots, x_n)$$

Step 2. $h \in \text{Hol}(U) \Rightarrow H_h \in \text{Hol}(V - X).$

$$\text{But moreover, } \sum_{x_1: x_1^d = z_1} x_1^{k-d+1} = 0 \quad \forall k+1 \notin d \cdot \mathbb{Z}_+$$

with $h(x) = \sum_{k=0}^{\infty} h_k(x_2, \dots, x_n) x_1^k$ near a ($\mathcal{J}_\phi(x) = x_1^{d-1} \cdot d$) $\Rightarrow$

Near a , $\sum_{x \in \phi^{-1}(z)} \frac{h(x)}{\mathcal{J}_\phi(x)} = \sum_{k=0}^{\infty} h_k(z_2, \dots, z_n) \sum_{x_1^d = z_1} \frac{x_1^{k-d+1}}{d} =$

$$= \sum_{(k-d+1) \equiv d} h_k(z_2, \dots, z_n) \cdot z_1^{\frac{k-d+1}{d}}$$

$\Rightarrow H_h(z)$ extends as holom. to $V \setminus Z$, $\dim_{\mathbb{C}} Z \leq n-2$

(Hartog's Thm) $H_h \in \text{Hol}(V)$, done. $\checkmark$