# Fourier transforms of Kirilov's Orbital integrals

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We compute simple expressions for the Fourier transforms of Orbital integrals on two co-adjoint orbits, namely the semisimple cone of $SL(2)$ acting on $\mathfrak{sl}(2)^*$ and the unit sphere of $SO(3)$ acting on $\mathfrak{so}(3)^*$. This was done under the supervision of professor Julia Gordon (UBC).

$G$ denotes a Lie group with Lie algebra $\mathfrak{g}$.
$X \in \mathfrak{g}^*, \mathcal{O}_X$ denotes the co-adjoint orbit of $X$.
$X \in \mathfrak{g}^*, Y \in \mathfrak{g}, \langle X, Y \rangle$ denotes $X(Y)$.
$\omega$ the Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}_X$
$\mathcal{S}(\mathfrak{g}^*)$ the Schwartz space on $\mathfrak{g}^*$
$\lambda$ denotes the Lebesgue measure

Given $G$, $\mathfrak{g}$, $X \in \mathfrak{g}^*$, fix $m_0 \in M = \mathcal{O}_X$. We have $\varphi_{m_0} : G \to \mathfrak{g}^*$ given by $\varphi_{m_0}(g) = Ad_g^*(m_0)$. The differential of $\varphi_{m_0}$ at $e \in G$ gives an identification $\Phi_{m_0} : \mathfrak{g}/\mathfrak{c}(X) \to T_{m_0}M \hookrightarrow T_{m_0}\mathfrak{g}^* \cong \mathfrak{g}^*$. This can be computed either by recalling that $\langle \Phi_{m_0}(X), Y \rangle = \langle m_0, [Y, X] \rangle$ or via the exponential map:

$$\Phi_{m_0}(X) = \frac{d}{dt}\Big|_{t=0} (\varphi_{m_0} \circ \exp)(tX).$$

The identification $\mathfrak{g}/\mathfrak{c}(X) \cong T_{m_0}M$ allows us to define a symplectic 2-form $\omega$ on $M$ by:

$$\omega_{m_0}(u, v) = \langle m_0, [\Phi_{m_0}^{-1}(u), \Phi_{m_0}^{-1}(v)] \rangle$$

We extend $\omega$ to a 2-form on $\mathfrak{g}^*$ by using the decomposition $T\mathfrak{g}^* = TM \oplus \nu(M)$ where $\nu(M)$ denotes the normal bundle to $M$. Given $p \in M$ we can thus write $v \in T_p\mathfrak{g}^*$ as $v = v^t + v^n$ where $v^t \in T_pM$ and $v^n \in \nu(M)$. Under this decomposition we define the extension $\widetilde{\omega}$ of $\omega$ at $p$ to be $\widetilde{\omega}_p(v_1, v_2) = \omega_p(v_1^t, v_2^t)$. One can further define $\omega$ at points $p \notin M$ using projection mappings and a partition of unity but we shall not need that here. In this way we can see $\omega$ as a form on $\mathfrak{g}^*$.

Letting $T : \mathcal{S}(\mathfrak{g}^*) \to \mathbb{R}$ denote the distribution on $\mathfrak{g}^*$ given by:

$$T(f) = \int_{Y \in \mathcal{O}_X} f(Y)\omega(Y)$$

we evaluate $\widehat{T}(f)$:

$$\widehat{T}(f) = \int_{Y \in \mathcal{O}_X} \left( \int_{X \in \mathfrak{g}} f(X) \exp(-2\pi i \langle Y, X \rangle) d\lambda(X) \right) \omega(Y).$$

With no formal justification we proceed in the following way:

$$\begin{aligned}
\widehat{T}(f) &= \int_{Y\in\mathcal{O}_X}\left(\int_{X\in\mathfrak{g}} f(X)\exp(-2\pi i\langle Y,X\rangle)d\lambda(X)\right)\omega(Y)\\
&= \int_{X\in\mathfrak{g}}\int_{Y\in\mathcal{O}_X} f(X)\exp(-2\pi i\langle Y,X\rangle)d\lambda(X)\omega(Y)\\
&= \int_{X\in\mathfrak{g}} f(X)\left(\int_{Y\in\mathcal{O}_X}\exp(-2\pi i\langle Y,X\rangle)\omega(Y)\right)d\lambda(X)\\
&= \int_{X\in\mathfrak{g}} f(X)\mu(X)d\lambda(X)
\end{aligned}$$

so that one may say $\widehat{T}$ is given by integration against $\mu$. It is a deep result of Harish-Chandra that $\mu$ is $L^1$ and analytic on the regular set of $\mathfrak{g}$. Below, we first compute simple expressions for $\omega$ as forms on $\mathfrak{g}^*$, and then use this to compute simple expressions for $\mu(X)$.

**I** :

Let $G = SL(2;\mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}(2;\mathbb{R})$, and let $\beta = \{\mathbf{e},\mathbf{f},\mathbf{h}\}$ the basis for $\mathfrak{g}$ given by:

$$\mathbf{e} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Define $\beta^* = \{\mathbf{e}^*,\mathbf{f}^*,\mathbf{h}^*\}$ to be the basis for $\mathfrak{g}^*$ dual to $\beta$ under the Killing form. Explicitly this means that $\mathbf{e}^*(\mathbf{f}) = 4$, $\mathbf{f}^*(\mathbf{e}) = 4$, and $\mathbf{h}^*(h) = 8$. Denote by $M$ the co-adjoint orbit of $\mathbf{e}^*$. Finally let $\{dx, dy, dz\}$ be the basis for $\Omega^1(\mathfrak{g}^*)$ dual to $\{\frac{\partial}{\partial\mathbf{e}^*},\frac{\partial}{\partial\mathbf{f}^*},\frac{\partial}{\partial\mathbf{h}^*}\}$.

We start by computing $M$ as a submanifold of $\mathfrak{g}^*$. First we note that $O(\mathbf{e})$ can be computed as:

$$\begin{aligned}
O(\mathbf{e}) = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}, \mid ad-bc=1\} &= \{\begin{pmatrix}-ac & a^2\\ -c^2 & ac\end{pmatrix} \mid ad-bc=1\}\\
&= \{a^2\mathbf{e} - c^2\mathbf{f} - ac\mathbf{h}\}\\
&= \{(x,y,z)\in\mathfrak{g} \mid y<0<x, z^2+xy=0\}
\end{aligned}$$

From which it follows that:

$$M = \{(x,y,z)\in\mathfrak{g}^* \mid x<0<y, z^2+xy=0\}$$

Fix $m_0 \in M$ and write $m_0 = (x,y,z)$ in $\beta^*$ coordinates. Let $\omega(m_0) = f_1(m_0)dx\wedge dy + f_2(m_0)dx\wedge dz + f_3(m_0)dy\wedge dz$. Then we see that at $m_0$ with respect to $\{\frac{\partial}{\partial\mathbf{e}^*},\frac{\partial}{\partial\mathbf{f}^*},\frac{\partial}{\partial\mathbf{h}^*}\}\big|_{m_0}$ coordinates for $T_{m_0}\mathfrak{g}^*$ we have:

$$\begin{aligned}
(1,0,0)^t &= (x^2+4z^2, -xy, -2yz)\\
(0,1,0)^t &= (-xy, y^2+4z^2, -2xz)\\
(0,0,1)^t &= (-2yz, -2xz, x^2+y^2)
\end{aligned}$$

A brief computation shows that with respect to $\beta$, $\beta^*$ coordinates the matrix representation for $\Phi_{m_0}$ is given by:

$$\begin{pmatrix} -2z & 0 & 2x \\ 0 & 2z & -2y \\ y & -x & 0 \end{pmatrix}$$

thus we see that:

$$\begin{aligned} \Phi_{m_0}(-2z, 0, \frac{x}{2}) &= \frac{\partial}{\partial \mathbf{e}^*}^t \\ \Phi_{m_0}(0, 2z, -\frac{y}{2}) &= \frac{\partial}{\partial \mathbf{f}^*}^t \\ \Phi_{m_0}(y, -x, 0) &= \frac{\partial}{\partial \mathbf{h}^*}^t \end{aligned}$$

Now we are ready to compute:

$$\begin{aligned} f_1(m_0) &= \omega_{m_0}(\frac{\partial}{\partial \mathbf{e}^*}, \frac{\partial}{\partial \mathbf{f}^*}) = \langle m_0, [(-2z, 0, \frac{x}{2}), (0, 2z, -\frac{y}{2})]\rangle = -8z(x^2 + y^2 + 4z^2) = -8z|v|^2 \\ f_2(m_0) &= \omega_{m_0}(\frac{\partial}{\partial \mathbf{e}^*}, \frac{\partial}{\partial \mathbf{h}^*}) = \langle m_0, [(-2z, 0, \frac{x}{2}), (y, -x, 0)]\rangle = 4x(x^2 + y^2 + 4z^2) = 4x|v|^2 \\ f_3(m_0) &= \omega_{m_0}(\frac{\partial}{\partial \mathbf{f}^*}, \frac{\partial}{\partial \mathbf{h}^*}) = \langle m_0, [(0, 2z, -\frac{y}{2}), (y, -x, 0)]\rangle = -4y(x^2 + y^2 + 4z^2) = -4y|v|^2 \end{aligned}$$

where $|v|^2 = x^2 + y^2 + 4z^2$. Thus:

$$\omega = 4|v|^2(-2z\ dx \wedge dy + x\ dx \wedge dz - y\ dy \wedge dz)$$

Parametrizing with $\rho : (0, \infty) \times [0, 2\pi) \to \mathfrak{g}^*$ given by $\rho(t, \theta) = (t(\cos\theta - 1), t(\cos\theta + 1), t\sin\theta)$ we conclude:

$$\rho^*\omega = -4\ dt \wedge d\theta$$

Putting this into the definition of $\mu(X)$:

$$\begin{aligned} \mu(X) &= \mu(a, b, c) \\ &= \int_{(x,y,z)\in M} \exp(-2\pi i\langle(x, y, z), (a, b, c)\rangle)\ \omega \\ &= -4 \int_0^\infty \int_0^{2\pi} \exp(-8\pi i t(b(\cos\theta + 1) + a(\cos\theta - 1) + 2c\sin\theta))\ d\theta\ dt \\ &= -4 \int_0^\infty \int_0^{2\pi} \exp(-8\pi i t((a + b)\cos\theta + 2c\sin\theta + b - a)\ d\theta\ dt \\ &= -8\pi \int_0^\infty \exp(-8\pi i(b - a)t) J_0(8\pi t\sqrt{(a + b)^2 + 4c^2})\ dt \\ (*) &= -\frac{\sigma(a - b)}{2\sqrt{ab + c^2}} \end{aligned}$$

The equality (*) needs some justification. The following argument was shown to me by Saminul Haque. First we note that $\int_0^\infty t^{2m}\exp(-at)dt = \frac{(2m)!}{a^{2m+1}}$. To see this:

$$\begin{aligned}\int_0^\infty t^{2m}\exp(-at)dt &= \int_0^\infty \frac{t^{2m}a^{2m}\exp(-at)}{a^{2m}}dt\\ &= \int_0^\infty \frac{u^{2m}\exp(-u)}{a^{2m+1}}du\\ &= \frac{1}{a^{2m+1}}\int_0^\infty u^{2m}\exp(-u)du\\ &= \frac{\Gamma(2m+1)}{a^{2m+1}}\\ &= \frac{(2m)!}{a^{2m+1}}\end{aligned}$$

Now we see:

$$\begin{aligned}\int_0^\infty \exp(-Pt)J_0(Qt)dt &= \int_0^\infty \exp(-Pt)\sum_{m=0}^\infty \frac{(-1)^m Q^{2m}t^{2m}}{(m!)^2 4^m}dt\\ &= \sum_{m=0}^\infty \int_0^\infty \frac{(-1)^m Q^{2m}\exp(-Pt)t^{2m}}{(m!)^2 4^m}dt\\ &= \sum_{m=0}^\infty \frac{(-1)^m Q^{2m}}{(m!)^2 4^m}\int_0^\infty t^{2m}\exp(-Pt)dt\\ &= \frac{1}{P}\sum_{m=0}^\infty \frac{(-1)^m (2m)!}{(m!)^2 4^m}(\frac{Q^2}{P^2})^m\end{aligned}$$

where the final equality follows from the previous computation. Finally we use that:

$$\sum_{k=0}^\infty \frac{(-1)^k(2k)!}{4^k(k!)^2}x^k = \frac{1}{\sqrt{x+1}}$$

to conclude:

$$\int_0^\infty \exp(-Pt)J_0(Qt)dt = \frac{1}{P\sqrt{\frac{Q^2}{P^2}+1}}$$

and so:

$$-8\pi\int_0^\infty \exp(-8\pi i(b-a)t)J_0(8\pi t\sqrt{(a+b)^2+4c^2})dt = -\frac{\sigma(a-b)}{2\sqrt{ab+c^2}}$$

by the above formula with $P = 8\pi i(b-a)$ and $Q = 8\pi\sqrt{(a+b)^2+4c^2}$, where $\sigma$ denotes the sign function.

**II:**
Let $G = SO(3)$, $\mathfrak{g} = \mathfrak{so}(3)$ and let $\beta = \{X, H, Y\}$ be the basis for $\mathfrak{g}$ where:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\beta^* = \{X^*, H^*, Y^*\}$ be the dual basis to $\beta$ under the Killing form. Explicitly this means $X^*(X) = H^*(H) = Y^*(Y) = -2$. Denote by $M$ the co-adjoint orbit of $H^*$. Then a brief calculation shows that:

$$M = \{(x, y, z) \in \mathfrak{g}^* \mid x^2 + y^2 + z^2 = 1\}$$

.

Write $\omega = f_1 dX^* \wedge dH^* + f_2 dX^* \wedge dY^* + f_3 dH^* \wedge dY^*$ as in the previous paragraph. We aim to compute $f_1, f_2, f_3$. Given $m_0 \in M$ write $m_0 = (x_1, x_2, x_3)$ in $\beta^*$ coordinates. Then at $m_0$ we see that with respect to $\{\frac{\partial}{\partial X^*}, \frac{\partial}{\partial H^*}, \frac{\partial}{\partial Y^*}\}\big|_{m_0}$ coordinates for $T_{m_0}\mathfrak{g}^*$ we have:

$$\begin{aligned} (1,0,0)^t &= (1 - x_1^2, -x_1x_2, -x_1x_3) \\ (0,1,0)^t &= (-x_1x_2, 1 - x_2^2, -x_2x_3) \\ (0,0,1)^t &= (-x_1x_3, -x_2x_3, 1 - x_3^2) \end{aligned}$$

A brief computation shows that with respect to $\beta, \beta^*$ coordinates, the matrix representation for $\Phi_{m_0}$ is given by:

$$\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

.

Thus we see that:

$$\begin{aligned} \Phi_{m_0}(0, x_3, -x_2) &= \frac{\partial}{\partial X^*}^t \\ \Phi_{m_0}(-x_3, 0, x_1) &= \frac{\partial}{\partial H^*}^t \\ \Phi_{m_0}(x_2, -x_1, 0) &= \frac{\partial}{\partial Y^*}^t \end{aligned}$$

Now we are ready to compute $f_1, f_2, f_3$:

$$\begin{aligned} f_1(m_0) &= \omega_{m_0}(\frac{\partial}{\partial X^*}, \frac{\partial}{\partial H^*}) = \langle m_0, [(0, x_3, -x_2), (-x_3, 0, x_1)] \rangle = -2(x_1^2x_3 + x_2^2x_3 + x_3^3) = -2x_3 \\ f_2(m_0) &= \omega_{m_0}(\frac{\partial}{\partial X^*}, \frac{\partial}{\partial Y^*}) = \langle m_0, [(0, x_3, -x_2), (x_2, -x_1, 0)] \rangle = 2(x_1^2x_2 + x_2^3 + x_2x_3^2) = 2x_2 \\ f_3(m_0) &= \omega_{m_0}(\frac{\partial}{\partial H^*}, \frac{\partial}{\partial Y^*}) = \langle m_0, [(-x_3, 0, x_1), (x_2, -x_1, 0)] \rangle = -2(x_1^3 + x_1x_2^2 + x_1x_3^2) = -2x_1 \end{aligned}$$

thus:

$$\omega(x,h,y) = -2x\ dH^* \wedge dY^* + 2h\ dX^* \wedge dY^* - 2y\ dX^* \wedge dH^*$$
$$= -2(x\ dH^* \wedge dY^* - h\ dX^* \wedge dY^* + y\ dX^* \wedge dH^*)$$

.

$$\begin{aligned}
\mu(v_1,v_2,v_3) &= \int_{q\in S^2} \exp(-2\pi i\langle q,v\rangle)d\omega(y) \\
&= \int_{(x,y,z)\in S^2} \exp(-2\pi i\langle (x,y,z),v\rangle)(x\ dy\wedge dz - ydx\wedge dz + zdx\wedge dy) \\
&= \int_{(x,y,z)\in S^2} \exp(4\pi i(xv_1+yv_2+zv_3))(x\ dy\wedge dz - ydx\wedge dz + zdx\wedge dy) \\
&= \int_0^{2\pi}\int_0^{\pi} \exp(4\pi i(v_1\sin\theta\cos\phi + v_2\sin\theta\sin\phi + v_3\cos\theta))\sin\theta\ d\theta d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} \exp(4\pi i(v_1\sin\theta\cos\phi + v_2\sin\theta\sin\phi + v_3\cos\theta))\sin\theta\ d\phi d\theta \\
&= 2\pi\int_0^{\pi} \exp(4\pi i v_3\cos(\theta))J_0(4\pi\sqrt{v_1^2+v_2^2}\sin\theta)\sin\theta\ d\theta
\end{aligned}$$

In the final equation we see that $\mu$ is a function only of $v_1^2+v_2^2+v_3^2$ hence $\mu(v_1,v_2,v_3) = \mu(0,0,\sqrt{v_1^2+v_2^2+v_3^2})$ so we may as well compute $\mu(0,0,v_3)$ and thus:

$$\begin{aligned}
\mu(0,0,v_3) &= 2\pi\int_0^{\pi}\exp(4\pi i v_3\cos\theta)\sin\theta\ d\theta \\
&= \frac{2\pi\sin(4\pi v_3)}{v_3}
\end{aligned}$$

For $v_3 \neq 0$. It is easily seen from the original equation that $\mu(0) = \int_{S^2}\omega = 4\pi$. Then to simplfiy we can write $v_1^2+v_2^2+v_3^2 = r^2$ and conclude:

$$\mu(r) = \frac{2\pi\sin(4\pi r)}{r}$$

This calculation is probably done a bit simpler in $SU(2)$.