COMPLEX

MATT KOSTER

1. Analytic

Definition 1.0.1. *For $U \subset \mathbb{C}$ open, a map $f : U \to \mathbb{C}$ is called* ***holomorphic at*** $a \in U$ *if:*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$$

exists and $f'(a)$ is called the ***(complex) derivative of*** f ***at*** a*. We say that f is* ***holomorphic*** *if it is holomorphic at every $a \in U$.*

As a simple consequence of this definition, if we take $h \in \mathbb{R}$, we have, for $f = u + iv$:

$$\begin{aligned} f'(a_1, a_2) &= \lim_{h \to 0} \frac{u(a_1 + h, a_2) + iv(a_1 + h, a_2) - (u(a_1, a_2) + iv(a_1, a_2))}{h} \\ &= \lim_{h \to 0} \frac{u(a_1 + h, a_2) - u(a_1, a_2)}{h} + i \lim_{h \to 0} \frac{v(a_1 + h, a_2) - v(a_1, a_2)}{h} \\ &= \frac{\partial u}{\partial x}(a_1, a_2) + i \frac{\partial v}{\partial x}(a_1, a_2) \end{aligned}$$

and taking $h \in i\mathbb{R}$ a similar calculation gives:

$$f'(a_1, a_2) = -i \left(\frac{\partial u}{\partial y}(a_1, a_2) + i \frac{\partial v}{\partial y}(a_1, a_2) \right).$$

Thus:

Theorem 1.0.2 (Cauchy-Riemann equations)**.** *Writing $f(x + iy) = u(x, y) + iv(x, y)$, f is holomorphic at $a = (a_1, a_2)$ if and only if $f \in C^1$ and:*

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \textit{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a)$$

If $f = u + iv$ is holomorphic and $v \in C^2$ then by Clairaut's theorem:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}$$

so $\Delta v = 0$. Assuming $u \in C^2$ an identical calculation reveals $\Delta u = 0$. It turns out (see "Cauchy's integral formula" below) that the real and imaginary parts of a holomorphic function are C^∞ so:

Corollary 1.0.3. *The real and imaginary parts of a holomorphic function are harmonic.*

Remark 1.0.4. A function f being holomorphic at a is equivalent to $d_a f : T_a \mathbb{R}^2 \to T_{f(a)} \mathbb{R}^2$ being complex-linear i.e. commuting with:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By a **complex structure** on a real vector space V of even dimension $2k$ we mean a linear transformation $J\colon V \to V$ such that $J^2 = -I$. Such a map gives V the structure of a complex vector space by declaring $(x+iy)v = xv + yJ(v)$ (the only axiom that needs verifying is associativity of scalar multiplication). As a complex vector space V has dimension k (begin with the observation that $J(v) = iv \in \operatorname{span}_{\mathbb{C}}(v)$).

Now fix $k = 1$ so $V = \mathbb{R}^2$ and let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. By definition $V \otimes_{\mathbb{R}} \mathbb{C}$ is a real vector space together with a bilinear map $\pi\colon V \times \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying the following universal property: for all vector spaces U and bilinear maps $T\colon V \times \mathbb{C} \to U$ there exists a unique linear map $\widehat{T}\colon V \otimes_{\mathbb{R}} \mathbb{C} \to U$ making the following diagram commute:

$$\begin{tikzcd} V \times \mathbb{C} \arrow[d, "\pi"] \arrow[dr, "T"] & \\ V \otimes_{\mathbb{R}} \mathbb{C} \arrow[r, "\widehat{T}"] & U \end{tikzcd}$$

We call the image of π **simple tensors** and use the notation $v \otimes \alpha := \pi(v, \alpha)$. Turn $V_{\mathbb{C}}$ into a complex vector space by $\alpha \cdot (v \otimes \beta) := v \otimes (\alpha\beta)$ and extend linearly.

The bilinear map $V \times \mathbb{C} \to \mathbb{C}^2$ given by $(x, y, \alpha) \mapsto (\alpha x, \alpha y)$ descends to an isomorphism $V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}^2$ so $\dim_{\mathbb{C}} V \otimes_{\mathbb{R}} \mathbb{C} = 2$ and has as a basis $e_1 \otimes 1$, $e_2 \otimes 1$. Canonical inclusion $\iota\colon V \to V_{\mathbb{C}}$ where $\iota(v) = v \otimes 1$ identifies V as a subspace of $V_{\mathbb{C}}$ fixed by complex conjugation. But this identification of V as $v \otimes 1$ is not as a *complex* subspace - the real subspace $i(V)$ satisfies $V \cap i(V) = 0$. Hence we have a direct sum $V \oplus i(V)$ meaning every $z \in V_{\mathbb{C}}$ can be written uniquely in the form $z = u \otimes 1 + v \otimes i$ for $u, v \in V$. For brevity we will often drop the $\otimes$ symbols and write $z = u + iv$.

Let $\widetilde{J}\colon V \to V$ be the complex structure given by $\widetilde{J}(x, y) = (-y, x)$. This is left multiplication of $\begin{pmatrix} x \\ y \end{pmatrix}$ by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Complexifying $\widetilde{J}$ defines a complex linear map $J := \widetilde{J}_{\mathbb{C}}\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ so that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[d, "\widetilde{J}"] & V_{\mathbb{C}} \arrow[d, "J"] \\ V \arrow[r, "\iota"] & V_{\mathbb{C}} \end{tikzcd}$$

On simple tensors this looks like $J(v \otimes \alpha) = \widetilde{J}(v) \otimes \alpha$ so $J(e_1 \otimes 1) = -e_2 \otimes 1$ and $J(e_2 \otimes 1) = e_1 \otimes 1$ hence with respect to the ordered basis $e_1 \otimes 1$ and $e_2 \otimes 1$ the transformation J is again left multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $J^2 = -I$ we see J has eigenvalues $\pm i$. Let $V^{1,0}$ and $V^{0,1}$ be the $\lambda = i$ and $\lambda = -i$ eigenspaces of $V_{\mathbb{C}}$, respectively. More concretely:

$$V^{1,0} = \operatorname{span}_{\mathbb{C}}\{e_1 \otimes 1 - e_2 \otimes i\} =: \{2z\}, \quad V_{0,1} = \operatorname{span}_{\mathbb{C}}\{e_1 \otimes 1 + e_2 \otimes i\}$$

Note that complex conjugation on $V_{\mathbb{C}}$ gives a *real* isomorphism of $V^{1,0}$ with $V_{0,1}$ and this defines a basis $\{z, \overline{z}\}$ for $V_{\mathbb{C}}$, thus:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

and write $v = \alpha_1 z + \alpha_2 \overline{z}$ for $\alpha_1, \alpha_2 \in \mathbb{C}$. Note that $v \in V^{1,0}$ we have $J(v) = iv$ so $J^2(v) = -v$ giving an identification of $(V^{1,0}, J)$ with $(V, \widetilde{J})$ as complex vector spaces. The choice of basis played no role in this identification so this identification is canonical. Note that this is the *second* identification of V as a subspace of $V_{\mathbb{C}}$, but this time it is as a complex subspace.

Note that $\mathbb{R}$-bilinear maps $V \times \mathbb{C} \to \mathbb{C}$ are canonically identified with $\mathbb{C}$-linear maps $V_{\mathbb{C}} \to \mathbb{C}$, but are also canonically identified with $\mathbb{R}$-linear maps $V \to \mathbb{C}$. So $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^*$. Moreover, when V, W are finite dimensional, the bilinear map $V^* \times W \to \mathrm{Hom}(V, W)$ given by $(f, w) \mapsto (v \mapsto f(v)w)$ descends to an isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ so in particular $(V^*)_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = (V_{\mathbb{C}})^*$.

Proposition 1.0.5. *If V is a vector space with complex structure J then $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$ has complex structure:*

$$J'(f)(v) = f(J(v))$$

Starting with a complex structure on V this proposition defines a complex structure on V^* which in turn provides a corresponding decomposition $(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$. Unravelling the definitions one sees that:

$$(V^*)^{1,0} = \{f : f(J(v)) = if(v)\}, \quad (V^*)^{0,1} = \{f : f(J(v)) = -if(v)\}$$

If $\{e_1, e_2\}$ is as above a basis for (V, J) satisfying $J(e_1) = e_2$ and $J(e_2) = -e_1$ then the dual basis $\{dx, dy\}$ for V^* satisfies $J'(dx) = -dy$ and $J'(dy) = dx$ so we see $(V^*)^{1,0}$ is spanned by $dx + idy$ and $(V^*)^{0,1}$ is spanned by $dx - idy$. Now, notice that $(dx + idy)(z) = (dx + idy)((e_1 - ie_2)/2) = 1$ and similarly $(dx - idy)(\overline{z}) = 1$, so:

$$(V^*)^{1,0} = \mathrm{span}\{dz\}, \quad (V^*)^{0,1} = \mathrm{span}\{d\overline{z}\}$$

i.e. $(V^*)_{1,0} = (V^{1,0})^*$ and $(V^*)_{0,1} = (V_{0,1})^*$.

The complex structure $J(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$, $J(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$ gives a decomposition $(T_z\mathbb{R}^2)_{\mathbb{C}} = (T_z\mathbb{R}^2)^{1,0} \oplus (T_z\mathbb{R}^2)^{0,1}$ with corresponding basis vectors:

$$\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\right\} = \left\{\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right\}$$

and a decomposition $(T_z^*\mathbb{R}^2)_{\mathbb{C}} = (T_z^*\mathbb{R}^2)^{1,0} \oplus (T_z^*\mathbb{R}^2)^{0,1}$ with corresponding basis vectors:

$$\{dz, d\overline{z}\} = \{dx + idy, dx - idy\}.$$

The same holds for any open $U \subseteq \mathbb{R}^2$. For brevity we use the notation $T_z^{1,0}U := (T_zU)^{1,0}$ and $T_z^{0,1}U := (T_zU)^{1,0}$. Patching together the complexified tangent spaces we define the **complexified tangent bundle**:

$$T_{\mathbb{C}}U := \bigsqcup_{z \in U} (T_zU)_{\mathbb{C}}.$$

$T_{\mathbb{C}}U$ is a complex vector bundle over U and is isomorphic (as complex vector bundles) to $TU \otimes \mathbb{C}$. The decompositions $(T_zU)_{\mathbb{C}} = T_z^{1,0}U \oplus T_z^{0,1}U$ patch together to the splitting $T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U$ as a Whitney sum. The same holds for:

$$T_{\mathbb{C}}^*U := \bigsqcup_{z \in U} (T_z^*U)_{\mathbb{C}} \cong T^*U \otimes \mathbb{C}$$

leading to the Whitney sum $T_{\mathbb{C}}^*U = (T^*U)^{1,0} \oplus (T^*U)^{0,1}$.

For $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ let $\pi_i(v_1, v_2) = v_i$. The derivative Df of a smooth map $f \colon U \to \mathbb{R}^2$ fits into the following commutative diagram:

$$\begin{array}{ccc} TU & \xrightarrow{Df} & \mathbb{R}^2 \times \mathbb{R}^2 \\ \downarrow & & \downarrow \pi_1 \\ U & \xrightarrow{f} & \mathbb{R}^2 \end{array}$$

Using again the identification $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ we identify $D_p f : T_p U \to T_{f(p)}\mathbb{C}$ with a complex linear map $T_p^{\mathbb{C}} U \to T_{f(p)}\mathbb{C}$ that fits into the commutative diagram:

$$\begin{array}{ccc} T_{\mathbb{C}}U & \xrightarrow{D^{\mathbb{C}} f} & \mathbb{C} \times \mathbb{C} \\ \downarrow & & \downarrow \pi_1 \\ U & \xrightarrow{f} & \mathbb{C} \end{array}$$

Sections $X \in \Gamma(T_{\mathbb{C}}U)$ then act on $C^\infty(U, \mathbb{C})$ by $X(f) = \pi_2 \circ D_{\mathbb{C}} f \circ X$ i.e. $X(f)_p = (D_p^{\mathbb{C}} f)(X_p)$. More explicitly, the identification $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ is given by $T \mapsto (V \otimes \alpha) \mapsto \alpha T(v)$ so $D_p^{\mathbb{C}} f(v \otimes \alpha) = \alpha D_p f(v)$. Hence for $X(p) = \frac{\partial}{\partial x} \otimes \alpha_1 + \frac{\partial}{\partial y} \otimes \alpha_2$ we compute $X(f) = D_p f(\frac{\partial}{\partial x} \otimes \alpha_1 + \frac{\partial}{\partial y} \otimes \alpha_2) = \alpha_1 D_p f(\frac{\partial}{\partial x}) + \alpha_2 D_p f(\frac{\partial}{\partial x})$. In particular:

$$\begin{aligned} \frac{\partial}{\partial z} f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z) = \frac{1}{2}\left(D_z f(\frac{\partial}{\partial x}) - iD_z f(\frac{\partial}{\partial y})\right) &= \frac{1}{2}\left(\begin{pmatrix} \frac{\partial f_1}{\partial x}(z) \\ \frac{\partial f_2}{\partial x}(z) \end{pmatrix} - i\begin{pmatrix} \frac{\partial f_1}{\partial y}(z) \\ \frac{\partial f_2}{\partial y}(z) \end{pmatrix}\right) \\ &= \frac{1}{2}\left(\begin{pmatrix} \frac{\partial f_1}{\partial x}(z) \\ \frac{\partial f_2}{\partial x}(z) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_2}{\partial y}(z) \\ -\frac{\partial f_1}{\partial y}(z) \end{pmatrix}\right) \end{aligned}$$

and by a similar calculation:

$$\frac{\partial}{\partial \overline{z}} f(z) = \frac{1}{2}\left(\begin{pmatrix} \frac{\partial f_1}{\partial x}(z) \\ \frac{\partial f_2}{\partial x}(z) \end{pmatrix} + \begin{pmatrix} -\frac{\partial f_2}{\partial y}(z) \\ \frac{\partial f_1}{\partial y}(z) \end{pmatrix}\right)$$

where $f(z) = (f_1(z), f_2(z)) \in \mathbb{C}$. Thus we introduce the notations:

$$f(z) = f_1(z) + if_2(z), \quad \frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y}$$

so that:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

The Cauchy-Riemann equations are then equivalent to:

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

In Cartesian coordinates:

$$2\frac{\partial f}{\partial z} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \quad 2\frac{\partial f}{\partial \overline{z}} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Note that this satisfies:

$$D_a f(\alpha) = \frac{\partial f(a)}{\partial z}\alpha + \frac{\partial f(a)}{\partial \overline{z}}\overline{\alpha}$$

Definition 1.0.6. *A* ***complex*** k***-form*** *on U is a section of $\bigwedge^k (T_{\mathbb{C}}^* U)$.*

A 0-form is a section of $\bigwedge^0(T^*_{\mathbb{C}}U) = U \times \mathbb{C}$ i.e. a map $U \to \mathbb{C}$. A 1-form is a section of $\bigwedge^1(T^*_{\mathbb{C}}U) = T^*_{\mathbb{C}}U$ and can be written in the form $\alpha = f(z)dz + g(z)d\overline{z}$ for $f, g \colon U \to \mathbb{C}$. Finally, a 2-form is a section of $\bigwedge^1(T^*_{\mathbb{C}}U)$ and can be written in the form $\omega = f(z)dz \wedge d\overline{z}$ for $f \colon U \to \mathbb{C}$.

The usual exterior derivative d extends uniquely to a complex linear operator on complex forms that we continue to denote by d:

Proposition 1.0.7. *If f is a complex 0-form then:*

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

If $\omega = gdz + hd\overline{z}$ is a complex 1-form then:

$$d\omega = \left(\frac{\partial h}{\partial z} - \frac{\partial g}{\partial \overline{z}}\right) dz d\overline{z}$$

For $f(z) = z$ this computes $df = dz$ and for $f(z) = \overline{z}$ we see $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial \overline{z}} = 1$ so $df = d\overline{z}$. Notice that when f is holomorphic $df = f'(z)dz$. The usual definitions of closed and exact carry over to complex forms - ω is closed if $d\omega = 0$ and exact if $\omega = d\eta$.

For $f \colon U \to \mathbb{C}$ define $\int_U f = \int_U f_1 + i\int_U f_2$ where $f = f_1 + if_2$ (this is the usual Bochner integral).

For a path $\gamma : [a, b] \to \mathbb{R}^2$ with components $\gamma(t) = (x(t), y(t))$ integration of the complex 1-form $fdz + gd\overline{z}$ along γ is defined to be:

$$\begin{aligned}\int_\gamma fdz + gd\bar{z} &= \int_a^b f(\gamma(t))(x'(t) + iy'(t))dt + \int_a^b g(\gamma(t))(x'(t) - iy'(t))dt \\ &= \int_a^b (f(\gamma(t)) + g(\gamma(t)))x'(t)dt + i\int_a^b (f(\gamma(t)) - g(\gamma(t)))y'(t)dt\end{aligned}$$

Introducing the notation $\gamma'(t) = x'(t) + iy'(t)$ this becomes:

$$\int_\gamma fdz + gd\bar{z} = \int_a^b f(\gamma(t))\gamma'(t)dt + \int_a^b g(\gamma(t))\overline{\gamma}'(t)dt$$

Definition 1.0.8. *If $f : U \to \mathbb{C}$ is holomorphic and $\gamma : [a, b] \to U$ is a curve we define the **integral with respect to arc length**:*

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt$$

*and the **length** of γ by:*

$$\ell(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|dt$$

Corollary 1.0.9 (Mean value principle)**.** *If $f : U \to \mathbb{C}$ is holomorphic at $0 \in U$ and $B_r(0) \subset U$ then:*

$$f(0) = \frac{1}{2\pi}\int_{\partial B_r(0)} fdS = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta = \frac{1}{2\pi r}\int_\gamma f(z)|dz|$$

where $\gamma : [0, 2\pi] \to U$ is $\gamma(t) = re^{it}$.

Corollary 1.0.10 (Maximum modulus principle). *Suppose U is connected and $f : U \to \mathbb{C}$ is continuous and satisfies the conclusion of the mean value principle. If $a \in U$ is a local max for f then f is constant in U.*

Lemma 1.0.11 (Schwarz lemma). *If $f : \mathbb{D} \to \overline{\mathbb{D}}$ is holomorphic and $f(0) = 0$ then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If there exists some $z \in \mathbb{D} \setminus \{0\}$ such that $|f(z)| = |z|$ or if $|f'(0)| = 1$ then there exists $a \in S^1$ with $f(z) = az$.*

Theorem 1.0.12 (Cauchy's theorem). *If $f : U \to \mathbb{C}$ is holomorphic then $f(z)dz$ is closed.*

Proof. If $\omega = f(z)dz$ then:

$$d\omega = (\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}) \wedge dz = 0$$

□

Theorem 1.0.13 (Cauchy's integral formula). *If $f : U \to \mathbb{C}$ is holomorphic, $z_0 \in U$ and $|z - z_0| \leq r \subset U$, then for all $a \in |z - z_0| \leq r$:*

$$f(a) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-a}dz$$

Proof. Define $g : U \to \mathbb{C}$ by:

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

Then g is continuous on U and holomorphic on $U \setminus \{a\}$ so $g(z)dz$ is closed hence:

$$0 = \int_{|z-z_0|=r} g(z)dz = \int_{|z-z_0|=r} \frac{f(z)-f(a)}{z-a}$$

ie:

$$\int_{|z-z_0|=r} \frac{f(z)}{z-a} = \int_{|z-z_0|=r} \frac{f(a)}{z-a} = \int_{|z-z_0|=r} \frac{1}{z-a} = 2\pi i f(a)$$

□

Corollary 1.0.14 (Cauchy's inequality). *If $f : U \to \mathbb{C}$ is holomorphic at a, then:*

$$|a_n| \leq \frac{M(r)}{r^n}$$

where $M(r) = \sup_{|z-a|=r} f(z)$.

Definition 1.0.15. *A map f is called **entire** if it is defined and holomorphic at every $z \in \mathbb{C}$.*

Theorem 1.0.16 (Liouville's theorem). *If f is entire and bounded then it is constant.*

Proof. By Cauchy's inequality $|a_n| \leq \frac{M(r)}{r^n}$ and since f is bounded, $M = \sup_r M(r) < \infty$ so:

$$|a_n| \leq \lim_{r\to\infty} \frac{M}{r^n} = 0$$

except when $n = 0$, ie. f is constant.

□

Theorem 1.0.17. *If $f : U \to \mathbb{C}$ is holomorphic at a then f is analytic at a.*

Proof. By Cauchy's integral formula:

$$\begin{aligned} f(a) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)\left(\frac{1}{z}\sum_{n=0}^{\infty}\frac{a^n}{z^n}\right)dz &= \frac{a^n}{2\pi i}\int_{|z-z_0|=r}\left(\sum_{n=0}^{\infty}\frac{f(z)}{z^{n+1}}\right)dz \\ &= \sum_{n=0}^{\infty}(\int_{|z-z_0|=r}\frac{f(z)}{2\pi i z^{n+1}}dz)a^n \\ &= \sum_{n=0}^{\infty} c_n a^n \end{aligned}$$

□

Theorem 1.0.18 (Morera's theorem)**.** *If $\omega = f(z)dz$ is closed in U then f is holomorphic in U.*

Proof. For $a \in U$, since $f(z)dz$ is closed in U there exists $g : U \to \mathbb{C}$ holomorphic at a such that $dg = f(z)dz$ locally near a. But $dg = g'dz$, so $f(a) = g'(a)$. By theorem 1.1.8, g is analytic hence $g' = f$ is analytic ie. holomorphic at a. □

Theorem 1.0.19 (Identity theorem)**.** *If $f, g : U \to \mathbb{C}$ are holomorphic on the domain U (a connected open set) and $f \equiv g$ on a nonempty open subset $V \subseteq U$ then $f \equiv g$ everywhere in U.*

2. Analytic Functions

Definition 2.0.1. *The **exponential map** is defined by* $\exp(z) = e^z := \sum_{n=0}^{\infty}\frac{z^n}{n!}$.

As a consequence of the results of the previous section the domain of exp is an entire function. Moreover we have $\exp(z+w) = \exp(z)\exp(w)$.

Definition 2.0.2. *For $z, w \in \mathbb{C}$ we say z is a **logarithm** of w if $e^z = w$ and define* $\log(w) = \{z \in \mathbb{C} : z \text{ is a logarithm of } z\} = \{z \in \mathbb{C} : e^z = w\} = \exp^{-1}(w)$.

If $e^z = w$ then $e^{z+2\pi i} = e^z e^{2\pi i} = e^z = w$ so $\log(w)$ is a countably infinite discrete set where any two elements differ by an integer multiple of $2\pi i$. Indeed, writing $z = x + iy$ we have $w = e^z = e^{x+iy} = e^x e^{iy}$ if and only if $w \neq 0$, $|w| = e^x$, and $e^{iy} = w/|w|$. Since $w \neq 0$ we immediately get that $x = \log|w|$ (where the right hand side of this equation is the usual logarithm $\mathbb{R}_{>0} \to \mathbb{R}$).

Definition 2.0.3. *For $z, w \in \mathbb{C}$ we define* $z^w := \exp(w\log(z))$. *Note that $\log(z)$ is a set so z^w is also a set and by $w\log(z)$ we mean $\{wz' : z' \in \log(z)\}$.*

Example 2.0.4. $\log(1) = \{z : e^z = 1\} = \{2\pi i n : n \in \mathbb{Z}\}$ so:

$$1^{\frac{1}{2}} = \exp(\frac{1}{2}\log(1)) = \exp(\{\pi i n : n \in \mathbb{Z}\}) = \{\pm 1\}$$

Definition 2.0.5. *For $w \in \mathbb{C}\setminus 0$ we define the **argument** $\arg w$ to be the imaginary part of $\log(w)$. Note again that this is a set with elements differing by integer multiples of 2π (it is the set of y such that $e^{iy} = w/|w|$).*

Definition 2.0.6. *For $f : \mathbb{C} \to \mathcal{P}(\mathbb{C})$ (we call such f a **multi-valued function**), let $D_f = \mathbb{C} \setminus f^{-1}(\emptyset)$. A **branch** of f is a subset W of $\mathbb{C}$ such that $f(z) \cap W$ has cardinality 1 for all $z \in D_f$. After a branch of f has been chosen we consider f as a map from D_f to W in the natural way.*

Example 2.0.7. For $f = \log$, $D_f = \mathbb{C} \setminus \{0\}$. For $w \in D_f$, $\log(w) = \log|w| + i\arg w$ so a branch of log is equivalent to a map $D_f \to \mathbb{Z}$. In particular, for $k \in \mathbb{R}$, $W_k := \{z \in \mathbb{C} : k < \operatorname{Im}(z) \leq k + 2\pi\}$ defines a branch of log.

Definition 2.0.8. *If f is a multi-valued function and a branch W has been chosen, a* ***branch cut*** *is an open subset U of D_f such that $f|_U \to W$ is continuous.*

Example 2.0.9. For $f = \log$ with branch $W_{-\pi} = \{z \in \mathbb{C} : -\pi < \text{Im}(z) \le \pi\}$, the set $U = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ defines a branch cut. Since $\log(U)$ is open and $\log|_U = \exp|_{\log(U)}^{-1}$ we conclude $\log|_U$ is a biholomorphism between U and $\log(U) = \{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\}$.

We use the above to write $\log(w) = \log|w| + i \arg w$ where $\log|w|$ is the usual logarithm $\mathbb{R}_{>0} \to \mathbb{R}$, keeping in mind that $i \arg w$ is a set unless a branch of log has been chosen.

3. Meromorphic

Definition 3.0.1. *A map $f : U \to \mathbb{C}$ is called* ***meromorphic in*** *U if f is there exists a discrete set $K \subset U$ such that f is holomorphic on $U \setminus K$. Elements of K are called* ***poles*** *of f.*

Definition 3.0.2. *If $f : U \to \mathbb{C}$ is meromorphic and $a \in U$, the* ***Laurent expansion*** *for f around a is:*

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where $\gamma \colon [c,d] \to U$ is any simple closed curve whose image lies in an open annulus centered at a in which f is holomorphic. The Laurent expansion is unique and f is equal to the Laurent expansion everywhere in the annulus. The number a_{-1} in the Laurent expansion for f is called the ***residue of*** *f* ***at*** *a, denoted $Res(f,a)$.*

Theorem 3.0.3 (Residue theorem)**.** *If $f : U \to \mathbb{C}$ is meromorphic and $\gamma : [a,b] \to U$ is a simple closed curve oriented positively (with respect to the standard orientation on $\mathbb{R}^2$) whose image does not pass through any of the poles of f, then:*

$$\int_\gamma f(z)dz = 2\pi i \sum_k Res(f, a_k)$$

Theorem 3.0.4 (Argument principle)**.** *If $f : U \to \mathbb{C}$ is meromorphic and $\gamma : [a,b] \to U$ is a simple closed curve that does not intersect any of the zeroes or poles of f then:*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z - P$$

where Z is the number of zeroes and P is the number of poles contained in the region bounded by γ.

Corollary 3.0.5 (Rouche's theorem)**.** *Let $f, g : U \to \mathbb{C}$ be two holomorphic functions. If γ is a simple closed curve in U and $|g(z)| < |f(z)|$ for all z on γ then f and $f+g$ have the same number of zeroes in the region bounded by γ.*

4. Several Complex Variables.

Definition 4.0.1. *A map $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}^n$ is open is called* ***holomorphic*** *if the Cauchy-Riemann equations are satisfied for all $z_j = x_j + iy_j$.*

Proposition 4.0.2. *There is a Cauchy integral formula for several variables:*

$$f(z) = \frac{1}{2\pi i} \int_{|\xi_i - \eta_i| = \epsilon_i} \frac{f(\xi)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi$$

Theorem 4.0.3. *If $f : U \to \mathbb{C}$ is holomorphic in each variable separately then f is holomorphic.*

Corollary 4.0.4. *If f is holomorphic then f is analytic.*

Corollary 4.0.5. *The maximum principle, identity theorem, and Liouville theorem all hold in several variables.*