The Coxeter-Killing element and the Poincare polynomial

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0 Introduction.

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$ and let $T \leq G$ be a maximal torus (a maximal compact, abelian subgroup) with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Then there exists a basis of weight/root vectors for $\mathfrak{g}^{\mathbb{C}}$ such that the matrix representations of Ad_g $(g \in G)$ are diagonal so the structure of $\mathfrak{g}^{\mathbb{C}}$ is essentially determined by the adjoint action of T on $\mathfrak{g}^{\mathbb{C}}$.

Many questions about G thus become simpler to answer questions about $\mathfrak{g}^{\mathbb{C}}$. The classical problem considered here asks to what end is the cohomology on G determined by the roots of $\mathfrak{g}^{\mathbb{C}}$. The amazing result is that the Poincare polynomial for G has a simple form determined entirely by the eigenvalues of a distinguished element $\gamma \in N_G(T)/T$.

Let h denote the order of γ and let ζ denote a primitive h'th root of unity. Then the eigenvalues for γ are $\{\zeta^{m_i}\}_{i=1}^{\ell}$. Let $d_i = 2m_i + 1$. Then the Poincare polynomial is given by:

$$p_G(t) = \prod_{i=1}^{\ell}(1 + t^{d_i})$$

Sections $1 - 3$ give the basic preliminary definitions and results (generating functions, singular cohomology with coefficients in $\mathbb{R}$, Betti numbers, de Rham's theorem). Section 4 briefly develops the structure of $\mathfrak{g}^{\mathbb{C}}$ induced by the adjoint action of T. Definitions given include maximal torus, rank, weights, roots, Weyl group, Coxeter-Killing transformation, and Coxeter plane. Section 5 covers the basic theory of the Coxeter-Killing transformation, the main result being theorem 5.7. Section 6 applies the theory of section 5 to prove the formula for the Poincare polynomial. Section 7 gives some interesting corollaries of the formula for the Poincare polynomial.

1 Generating Functions.

Given a collection $C = \{C_1, C_2, ...\}$ of finite sets C_n define $f(n) = |C_n|$. One calls f a "counting function". There are many explicit determinations of such f, the utility of each depending heavily on the objects being counted.

1.1 Examples.

i) Let $C_n = \mathcal{P}(n)$ be the powerset of $\{1, ..., n\}$. Then $f(n) = 2^n$. This is arguably the most useful expression for this function. We call such an expression a *closed-form* for f.

ii) Let $C_n \subset \mathcal{P}(n)$ be the subsets of $\{1, ..., n\}$ that do not contain two consecutive integers. One finds a closed form expression for f given by:

$$f(n) = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2\sqrt{5}}.$$

However, we also notice that $f(n) = f(n-1) + f(n-2)$. We call such an expression a *recurrence relation* for f. One could argue that the recurrence relation for this f is more useful than the closed form expression.

iii) Let $C_n = S_n$ be the n'th symmetric group (bijections from $\{1, ..., n\}$ to itself). Then:

$$f(n) \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

where $\sim$ is meant to say that the two expressions are asymptotic (the quotient of their limits as $n \to \infty$ is 1). We call such an expression an *asymptotic expression* for f (this particular formula is Stirling's asymptotic expression for $n!$).

iv) We call P a partition of n if $P = (P_1, P_2, ...)$ where $P_i \geq P_{i+1}$ and $\sum_k P_k = n$. Let C_n be the set of partitions of n such that for every $P \in C_n$, all the nonzero P_i are odd and distinct. For example, $P = (5, 3, 1, 0, 0, ...)$ would be such a partition of 9. Write:

$$F(x) = (1+x)(1+x^3)(1+x^5)\cdots = \sum_{n=0}^{\infty} a_n x^n$$

One can show that in fact $f(n) = a_n$. We call such an expression (ie. where $f(n)$ is the coefficient of x^n in a formal power series) a **generating function** for f. The utility of generating functions cannot be understated. As another example let:

$$F(x) = \frac{1}{1 - x - x^2}.$$

Notice that $F(0) = 1$, $F'(0) = 1$, $F''(0) = 2 \cdot 2$, $F'''(0) = 3 \cdot 6$, $F^{(4)}(0) = 5 \cdot 24$, $F^{(5)}(0) = 8 \cdot 120$, etc. In this way we see that $f(n) = \frac{F^{(n)}(0)}{n!}$ is the n'th Fibonacci number (in fact this is the same $f(n)$ from example ii). Hence we can write:

$$F(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} f(n) x^n$$

as a generating function for the Fibonacci numbers (ignoring issues of convergence).

2 Singular Homology.

Given a topological space X, a *singular n-simplex* in X is a continuous function $f : \Delta^n \to X$ where $\Delta^n \subset \mathbb{R}^{n+1}$ is the *standard n-simplex* given by:

$$\Delta^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \ \sum_{i=0}^{n} x_i = 1\}$$

Define $C_n(X)$ to be the real vector space generated by the singular n-simplices in X. An element of $C_n(X)$ is a formal sum $\sum_i n_i f_i$ ($n_i \in \mathbb{R}$) called an *n-chain.* Define the *boundary map* $\partial_n : C_n(X) \to C_{n-1}(X)$ by:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{F_i}$$

where F_i denotes the i'th face of Δ^n (ie setting $x_i = 0$). Implicit is the canonical isomorphism $\Delta^{n-1} \cong F_i$ so that the above map ∂_n is well defined. One can show that $\partial_n \circ \partial_{n+1} : C_{n+1}(X) \to C_{n-1}(X)$ vanishes identically. This allows us to make the definition $H_n(X) = \ker(\partial_n)/\partial_{n+1}(C_{n+1})$, the *$n$'th (singular) homology group* of X.

Since $C_n(X)$ is a vector space and ∂_n are linear transformations, $H_n(X)$ is a (quotient) vector space. When only finitely many $H_n(X)$ are nonzero and each $H_n(X)$ is finite dimensional we say that X has *finitely generated homology.* In this case, we can write:

$$H_n(X) = \mathbb{R}^m$$

and define the *n'th Betti number* of X, $B_n(X)$, to be $m = \dim(H_n(X))$. We further define the *Poincare polynomial* of X, $p_X(t)$, to be the generating function for $f(n) = B_n(X)$:

$$p_X(t) = \sum_{n=0}^{\infty} B_n(X) t^n$$

Given two topologial spaces X and Y we say that X is *homotopy equivalent to* Y if there exists continuous maps $f : X \to Y$, $g : Y \to X$, $F : X \times [0, 1] \to X$, and $G : Y \times [0, 1] \to Y$, such that $F(x, 0) = (g \circ f)(x)$, $F(x, 1) = x$, $G(y, 0) = (f \circ g)(y)$ and $G(y, 1) = y$. That is to say that $(g \circ f)$ is *homotopic* to the identity on X with homotopy F, and $(f \circ g)$ is *homotopic* to the identity on Y with homotopy G. In this case we call both f and g *homotopy equivalences.*

An important fact is that a homotopy equivalence $f : X \to Y$ induces isomorphisms on the singular homology groups $f_* : H_n(X) \to H_n(Y)$ for all n. Then the singular homology groups give *homotopy invariants* that can be used to check that two spaces are not homotopy equivalent. Since a homeomorphism is a homotopy equivalence, homology can a fortiori be used to verify that two spaces aren't homeomorphic.

Because the coefficients on the n-chains came from $\mathbb{R}$ the above homology theory is sometimes called *homology with coefficients in* $\mathbb{R}$. If we replace $\mathbb{R}$ with another abelian group G we get *homology with coefficients in G.* However, when G is not a field $C_n(X)$ is free abelian rather than a vector space so $H_n(X)$ becomes an abelian group. There is a general algebraic formula for expressing homology with coefficients in G in terms of homology with coefficients in $\mathbb{Z}$ (called the Universal Coefficients Theorem) but we shall not need it here.

3 de Rham Cohomology of Manifolds

Suppose M is a smooth manifold. Let $\Omega^n(M)$ denote the space of smooth n-forms on M. Then the exterior derivative $d : \Omega^n(M) \to \Omega^{n+1}(M)$ satisfies $d \circ d \equiv 0$. We say $\omega \in \Omega^n(M)$ is *exact* if $\omega = d\eta$ for $\eta \in \Omega^{n-1}(M)$, and say it is *closed* if $d\omega = 0$. It is immediate from these definitions that if ω is exact it is closed so we are able to define the *n'th de Rham cohomology group of* M, denoted $H^n_{dr}(M)$, to be the closed n-forms modulo the exact n-forms.

The above definition was given for a *smooth* manifold M. Since the de Rham cohomology groups are defined by taking equivalence classes of *smooth* differential forms on M one would quite reasonably expect that $H^n_{dr}(M)$ depends on the smooth structure endowed on M. The very deep and surprising result of de Rham (not proved here) tells us otherwise.

3.1 Theorem (de Rham's theorem).

If M is an orientable smooth manifold then $H^n_{dr}(M) \cong H_n(M)$ for all n.

(In fact de Rham's theorem says something stronger than whats written here. We are combining it with Poincare duality for cohomology with coefficients in $\mathbb{R}$ for a result that applies to this paper, as Lie groups are always orientable).

The singular homology defined in section 2 was done for an arbitrary topological space. A given topological manifold M can be endowed with many non-equivalent smooth structures, but the theorem tells us that the de Rham cohomology is induced entirely by the topology on M.

As an immediate consequence of de Rham's theorem, the Betti number $B_p(M)$ is equal to the dimension of the vector space of closed p-forms mod exact p-forms. In fact, for a Lie group G more is true:

3.2 Lemma.

The Betti number $B_p(G) = \dim(H(\mathfrak{g})_p)$ where $H(\mathfrak{g})_p$ is the vector space of bi-invariant p-forms on G.

Proof. We show that every coset $[\omega]$ has a bi-invariant representative.

Let $[\omega]$ be the equivalence class (modulo exact forms) of a closed p-form ω. Then:

$$\omega' = \int_{G\times G} (L_h \circ R_g)^* \omega \; dh \; dg$$

is a closed, bi-invariant p-form and $[\omega'] = [\omega]$. Conversely suppose ω is a bi-invariant 1-form. Then $Ad(g)^*\omega = \omega$ so $Ad(\exp(tX))^*\omega = \omega$. From this we conclude that $\omega([X, Y]) = 0$ for all $X, Y \in \mathfrak{g}$.

But now since $d\omega(X, Y) = \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega[X, Y])$, we use that ω is bi-invariant to have the first two terms vanish. The third was computed to be 0 in the previous paragraph. Then ω is closed.

Any p-form can be written as a wedge of 1-forms. But if every bi-invariant 1-form is closed this tells us that every bi-invariant p-form is closed.

Now suppose ω is bi-invariant and $\omega = d\eta$. Then $\omega = L_g^*\omega = L_g^* d\eta = dL_g^*\eta = d\eta$. Then $L_g^*\eta - \eta$ is closed for all g (and similar for R_g^*) hence η is closed so $\omega = d\eta = 0$.

$\square$

The space of all bi-invariant differential forms on G forms an associative algebra $H(\mathfrak{g})$ under the wedge product, graded by dimension. Then $H(\mathfrak{g})_p$ is the subspace of $H(\mathfrak{g})$ given by restricting our attention to those of dimension p.

4 Lie groups and Lie algebras.

Let G be a simple compact Lie group with Lie algebra $\mathfrak{g}$. Let $T \leq G$ denote a *maximal torus* (a maximal, compact, abelian subgroup of G). As T is a compact subspace of G it is closed, hence by Cartan's closed subgroup theorem it is a Lie group with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Call the dimension ℓ of $\mathfrak{t}$ the *rank* of $\mathfrak{g}$. The purpose of this section is to define the *roots* of $\mathfrak{g}$. The exposition follows closely that of [5], although the presentation of the root system is a bit different.

4.1 Lemma.

Let T be a torus. If $\pi : T \to GL(V)$ is an irreducible finite dimensional complex Lie group representation then **(1)** $\dim(V) = 1$ so $GL(V) \cong \mathbb{C}^\times$ and **(2)** $\pi(T) \leq S^1$.

Proof. **(1)** Let $t, s \in T$, $v \in V$. Then $\pi(t)\pi(s)(v) = \pi(ts)(v) = \pi(st)(v) = \pi(s)\pi(t)(v)$ (where the second equality comes from T being abelian). Then by Schur's lemma we see that $\pi(t) = cI$.

Then $\pi(t)(v) = cv$ so v is an eigenvector for $\pi(t)$. But t and v were arbitrary so every vector in V is an eigenvector for every $\pi(t)$ so we can write V as a direct sum over the one dimensional subspaces spanned by a basis. Finally since π is assumed to be irreducible we conclude that $\dim(V) = 1$.

(2) Now since $\dim(V) = 1$, $V \cong \mathbb{C}$ so $GL(V) = GL(\mathbb{C}) \cong \mathbb{C}^\times$ (the group of units under multiplication). $\pi : T \to GL(V)$ is continuous so $\pi(T)$ is a compact subspace of $GL(V)$. Moreover π is a homomorphism so $\pi(T)$ is a subgroup of $GL(V)$.

Now if $H \leq \mathbb{C}^\times$ is a subgroup and $z \in H$ then $z^n \in H$. But if $|z| \neq 1$ then either $|z|^2 > 1$ or $|\frac{1}{z}|^2 > 1$. In either case we see that H is unbounded hence not compact.

Finally we conclude that since $\pi(T)$ is a compact subgroup of $\mathbb{C}^\times$, $\pi(T) \subset S^1$.

□

4.2 Lemma.

Let G be a compact, connected Lie group and let $\pi : G \to GL(V)$ be a finite dimensional Lie group representation. Then π is completely reducible.

Proof. (Weyl's 'unitarian trick') Fix an inner product $\langle\rangle_0$ on V and define:

$$\langle u, v\rangle = \int_G \langle \pi(g)(u), \pi(g)(v)\rangle_0 \; |dg|.$$

That this defines an inner product on V follows from $\langle\rangle_0$ being one. Now since left multiplication is a diffeomorphism and $|dg|$ is an invariant measure:

$$\begin{aligned}\langle \pi(h)(u), \pi(h)(v)\rangle &= \int_G \langle \pi(g)(\pi(h)(u)), \pi(g)(\pi(h)(v))\rangle_0 \; |dg| \\ &= \int_G \langle \pi(gh)(u), \pi(gh)(v)\rangle_0 \; |dg| \\ &= \int_{R_{h^{-1}}(G)} R^*_{h^{-1}} \langle \pi(gh)(u), \pi(gh)(v)\rangle_0 \; |dg| \\ &= \int_G \langle \pi(g)(u), \pi(g)(v)\rangle_0 \; R^*_{h^{-1}}|dg| \\ &= \int_G \langle \pi(g)(u), \pi(g)(v)\rangle_0 |dg| \\ &= \langle u, v\rangle\end{aligned}$$

so π is unitary. Now assume that π is not irreducible (if it were then it would certainly be completely reducible), and let W denote a nontrivial invariant subspace. Let $g \in G$, $w \in W$ and $w^\perp \in W^\perp$ where $\perp$ is taken with respect to $\langle\rangle$. Then:

$$\langle \pi(g)(w^\perp), w\rangle = \langle \pi(g^{-1})\pi(g)(w^\perp), \pi(g^{-1})(w)\rangle = \langle w^\perp, \pi(g^{-1})(w)\rangle = 0$$

so $W^\perp$ is an invariant subspace. Finally by induction on the dimensions of W and $W^\perp$ we see that π is completely reducible.

□

Let $Ad : G \to GL(\mathfrak{g}^{\mathbb{C}})$ denote the adjoint representation. Then by lemma 4.2 if we restrict to a maximal torus T, Ad is completely reducible so we can write:

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$$

where Ad is irreducible on $\mathfrak{g}_i$. By lemma 4.1 we see that for $v_i \in \mathfrak{g}_i$ we have $Ad_t(v_i) = \alpha_t v_i$ where $\alpha_t \in S^1$. We then call the map $\alpha : T \to S^1$ sending $t \mapsto \alpha_t$ a *weight* of $\mathfrak{g}$, and if moreover α is not the trivial homomorphism we call it a *root* of $\mathfrak{g}$.

Phrased differently, there exists a basis $\beta = \{X_1, ..., X_n\}$ (called *weight vectors*) for $\mathfrak{g}^{\mathbb{C}}$ where $Ad_t(X_i) = \alpha^i_t X_i$, the maps $t \to \alpha^i_t$ are the weights, and the nontrivial weights are the roots (with corresponding *root vectors* X_α). Denote by $\mathfrak{R}$ the set of roots.

Since $T \cong T^\ell$ we can write $t \in T$ as $t = (t_1, ..., t_\ell)$ with $t_i \in S^1$. Then if $\alpha : T \to S^1$ is a root we see that $\alpha(t) = \alpha(t_1, ..., t_\ell) = t_1^{a_1} \cdots t_1^{a_\ell}$ where a_i are the *winding numbers*. In this case we have that since $d_e\alpha : \mathfrak{t} \to i\mathbb{R}$, then $d_e\alpha \in \mathfrak{t}^*$. Let $\{X_1, ..., X_\ell\}$ be a basis for $\mathfrak{t}$ and let $\{X_1^*, ..., X_\ell^*\}$ be the dual basis. Then $\alpha \mapsto (a_1, ..., a_\ell)$ gives a correspondence between $\mathfrak{R}$ and a discrete subset of $\mathfrak{g}^*$ written as integer tuples in $\{X_i^*\}$ coordinates.

Then we may view the roots as elements of $\mathbb{R}^\ell \cong \mathfrak{g}^*$. One can show that there is an inner product $B(\cdot, \cdot)$ on $\mathbb{R}^\ell$ such that $\mathfrak{R}$ forms a *root system*. Let σ_α denote the hyperplane orthogonal to α with respect to B. Then $\mathbb{R}^\ell \setminus \bigcup_{\alpha \in \mathfrak{R}} \sigma_\alpha$ is a disconnected space. We call each connected component a *Weyl chamber*. For a fixed Weyl chamber C, call $\alpha \in \mathfrak{R}$ a *positive root* if $B(\alpha, v) > 0$ for all $v \in C$ and denote the set of positive roots (relative to C) by $\mathfrak{R}^+$. We call $\alpha \in \mathfrak{R}^+$ a *simple root* (relative to C) if it cannot be written as a nontrivial sum of elements of $\mathfrak{R}^+$. Finally we define a partial order $\succ$ on $\mathfrak{R}$. We will say that $\alpha \succ \beta$ if $\alpha - \beta \in \text{span}_{\mathbb{R}_{\geq 0}} \mathfrak{R}^+$. One can show that there is a unique $\alpha \in \mathfrak{R}$ such that for all $\beta \in \mathfrak{R}$, $\alpha \succ \beta$. We call this the *highest root* or *highest weight.*

Let $R_\alpha : \mathbb{R}^\ell \to \mathbb{R}^\ell$ denote reflection across σ_α, ie. $R_\alpha(v) = v - 2\frac{B(\alpha,v)}{B(\alpha,\alpha)}\alpha$. We define the *Weyl group* W of G relative to the Weyl chamber C to be the group generated by all such R_α. One can show that $W \cong N_G(T)/T$. We call $\gamma \in W$ a *Coxeter-Killing transformation* or a *Coxeter-Killing element* if $\gamma = R_{\alpha_1} \cdots R_{\alpha_n}$ where $\{\alpha_1, ..., \alpha_n\}$ is the set of simple roots in $\mathfrak{R}$, and write $h = |\gamma|$. To each Coxeter element γ there exists a unique plane $P \subset \mathbb{R}^\ell$ such that γ acts on P by a rotation, called the *Coxeter plane.*

4.3 Example.

We compute the root system for $SU(3)$, the group of 3×3 unitary matrices with determinant 1. Let T be the maximal torus given by diagonal matrices. Let $X_{i,j} \in \mathfrak{su}(3)$ $(i \neq j)$ be the matrix that has a 0 in every entry except a 1 in the (i,j) position. Then if:

$$t = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_1^{-1}z_2^{-1} \end{pmatrix}$$

we see that:

$$\begin{aligned} Ad_t(X_{1,2}) &= z_1 z_2^{-1} X_{1,2} \\ Ad_t(X_{1,3}) &= z_1^2 z_2 X_{1,3} \\ Ad_t(X_{2,1}) &= z_1^{-1} z_2 X_{2,1} \\ Ad_t(X_{2,3}) &= z_1 z_2^2 X_{2,3} \\ Ad_t(X_{3,1}) &= z_1^{-2} z_2^{-1} X_{3,1} \\ Ad_t(X_{3,2}) &= z_1^{-1} z_2^{-2} X_{3,2} \end{aligned}$$

and $Ad_t(X) = X$ for all diagonal matrices X. The corresponding integer tuples of the winding numbers are $(1,-1), (2,1), (-1,1), (1,2), (-2,-1), (-1,-2)$. Take:

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

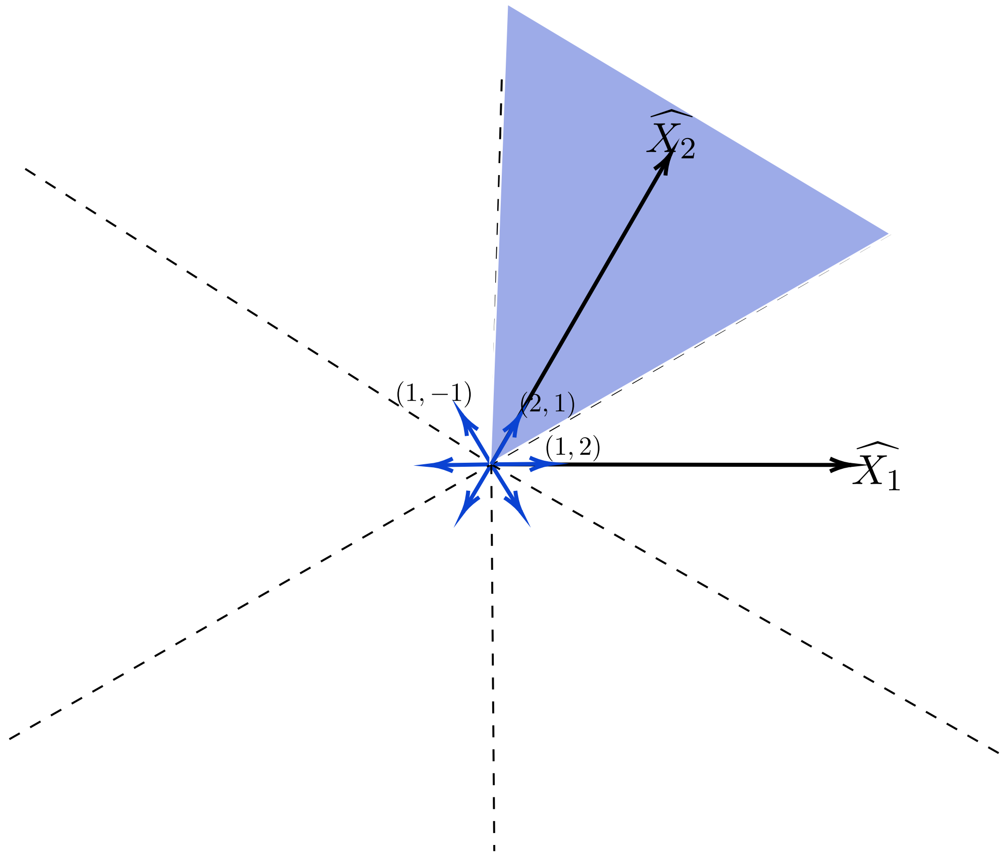
as a basis for $\mathfrak{t}$. Let $\widehat{X_1}, \widehat{X_2}$ be the dual basis of $\mathfrak{t}^*$ under the Killing form K. Then since $\widehat{X_1}(X_1) = \widehat{X_2}(X_2) = 12$ and $\widehat{X_1}(X_2) = \widehat{X_2}(X_1) = 6$ we see in the $\{X_i^*\}$ coordinates that $\widehat{X_1} = (12, 6)$ and $\widehat{X_2} = (6, 12)$. It follows that $(2,1) = \widehat{X_1}/6$, $(1,2) = \widehat{X_2}/6$, and $(1,-1) = \widehat{X_1}/6 - \widehat{X_2}/6$.Let $B(\cdot,\cdot)$ be the inner product on $\mathfrak{t}^*$ given by $B(\widehat{X_i}, \widehat{X_j}) = K(X_i, X_j)$. Then with respect to B we see that:

$$|(1,-1)| = |(1,2)| = |(2,1)| = 1/3$$

Moreover:

$$\langle (2,1),(1,2)\rangle = \frac{1}{6} = \cos(\theta_1)/3,$$
$$\langle (2,1),(1,-1)\rangle = \frac{1}{6} = \cos(\theta_2)/3,$$
$$\langle (1,2),(1,-1)\rangle = -\frac{1}{6} = \cos(\theta_3)/3$$

Then with respect to B, $(1,-1)$, $(1,2)$ and $(2,1)$ are all vectors with length $1/3$, ,the angle between the first two is $\theta_1 = \pi/6$, the angle between the second two is $\theta_2 = \pi/6$, and the angle between the first and third is $\theta_3 = \pi/3$. Thus we see that the root system look as in the following image:



The blue arrows indicate the roots of $SU(3)$, the dashed lines indicate the orthogonal hyperplanes σ_α, the blue shaded region is a choice of Weyl chamber C, the labelled roots are the positive roots with respect to C, the two positive roots not in C are the simple roots (with respect to C) and the single root inside of the Weyl chamber is the highest root.

5 The Coxeter element

The principle result of this section is to show that $h\ell = 2r$ where $h = |\gamma|$, ℓ is the rank of $\mathfrak{g}$, and $r = |\mathfrak{R}^+|$. This was the crucial step needed to conclude the proof that $p_G(t)$ can be computed explicitly by the eigenvalues of γ, and was originally show by Kostant in 1959 [4].

5.1 Theorem.

Let γ be a Coxeter-Killing element. Then the number $N(\gamma)$ of roots $\alpha \in \mathfrak{R}^+$ such that $\gamma(\alpha) \notin \mathfrak{R}^+$ is ℓ.

Proof. Define $\phi_i = R_{\alpha_\ell} R_{\alpha_{\ell-1}} \cdots R_{\alpha_{i+1}} \alpha_i$. Then $\phi_i = \alpha_i + \sum_{j>i} c_j \alpha_j$ so $\phi_i \in \mathfrak{R}^+$. It will be shown in lemma 5.5 that the ϕ_i are linearly independent.

Now $\gamma(\phi_i) = R_{\alpha_1} \cdots R_{\alpha_{i-1}}(-\alpha_i)$ so $\gamma(\phi_i) = -\alpha_i + \sum_{j<i} m_j \alpha_j$. Thus $\phi_i \in \mathfrak{R}^+$ and $\gamma(\phi_i) \notin \mathfrak{R}^+$. But ϕ_i are linearly independent so $N(\gamma) \geq \ell$.

Now suppose $\phi \in \mathfrak{R}^+$ and $\gamma(\phi) \notin \mathfrak{R}^+$. Then since:

$$R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_\ell} \phi \notin \mathfrak{R}^+$$

but:

$$\phi \in \mathfrak{R}^+$$

there must exist a largest i such that:

$$R_{\alpha_i} R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \phi \notin \mathfrak{R}^+$$

ie. :

$$R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \phi \in \mathfrak{R}^+$$

This means that $R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \phi$ is a positive root that changed sign under R_{α_i}. But α_i is the only possible such element, so $R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \phi = \alpha_i$ hence $\phi = \phi_i$ so $N(\gamma) \leq \ell$.

Since $N(\gamma) \leq \ell$ and $N(\gamma) \geq \ell$, $N(\gamma) = \ell$ as claimed.

□

The following result will be used several times, although we do not prove it here. For a proof see [2].

5.2 Theorem (Coleman).

There exists an eigenvector v for γ with eigenvalue ζ such that $v \notin \sigma_\alpha$ for all $\alpha \in \mathfrak{R}$. Such an eigenvector is called *regular*.

5.3 Lemma.

If λ is an eigenvalue for γ then $\lambda = \zeta^{m_i}$ for some m_i.

Proof. Suppose $\gamma(v) = \lambda v$. Then since γ has order h, $v = \gamma^h(v) = \lambda^h v$ ie. $\lambda^h = 1$. Since the primitive h'th roots of unity generate the h'th roots of unity, $\lambda = \zeta^{m_i}$ for some m_i.

□

5.4 Remark.

In fact more is true - every primitive h'th root of unity occurs as an eigenvector for γ. To see this, notice that since ζ is an eigenvalue, it must be a root of the characteristic polynomial p_γ. But then the minimial polynomial Φ (the h'th cyclotomic polynomial) for ζ must divide p_γ. Now since the other primitive h'th roots of unity are roots of Φ we see that they too must be roots of p_γ.

5.5 Lemma.

If γ is a Coxeter-Killing element then γ has no non-zero fixed points (ie. 1 is not an eigenvalue).

Proof. Let $\{e_i\}$ be a basis for $\mathfrak{t}$ dual under the Killing form to the simple roots of $\mathfrak{g}$. Define:

$$e_i' = \frac{2}{\langle \alpha_i, \alpha_i \rangle} e_i.$$

Then:

$$\gamma(e_i') = R_{\alpha_1} \cdots R_{\alpha_\ell} e_i' = e_i' + R_{\alpha_1} \cdots R_{\alpha_i}(\alpha_i) = e_i' + R_{\alpha_1} \cdots R_{\alpha_{i-1}}(-\alpha_i)$$

Define $\phi_i = R_{\alpha_\ell} R_{\alpha_{\ell-1}} \cdots R_{\alpha_{i+1}}(\alpha_i)$. Then $\gamma(e_i') = e_i' - \gamma(\phi_i)$. Let $X = \sum_{i=1}^{\ell} a_i e_i'$ and suppose $\gamma(X) = X$. Then:

$$X = \gamma(X) = \sum_{i=1}^{\ell} \alpha_i (e_i' + \gamma(\phi_i)) = X + \sum_{i=1}^{\ell} \alpha_i \gamma(\phi_i)$$

so:

$$\sum_{i=1}^{\ell} a_i \gamma(\phi_i) = 0$$

We see that:

$$\phi_i = \alpha_i + \sum_{j>i} c_j \alpha_j$$

hence the ϕ_i are linearly independent. But this means that $a_i = 0$ for all i, so $X = 0$ as required.

□

5.6 Lemma.

Let $\mathcal{O}_i$ denote the orbits in $\mathfrak{R}$ under the action of $< \gamma >$. Then:

$$\mathfrak{h}_i = \bigoplus_{\alpha \in \mathcal{O}_i} X_\alpha$$

is a Lie subalgebra of $\mathfrak{g}$ of dimension h.

Proof. It suffices to prove that $|\mathcal{O}_i| = h$. Let $\alpha \in \mathfrak{R}$ and assume $\gamma^n(\alpha) = \alpha$. Let ζ be a primitive h'th root of unity. By theorem 5.2 there exists a regular eigenvector v of γ with eigenvalue ζ. Hence:

$$\langle v, \alpha \rangle = \langle v, \gamma^n \alpha \rangle = \langle \gamma^{-n}(v), \alpha \rangle = \zeta^{-n} \langle v, \alpha \rangle.$$

But v is regular so $\langle v, \alpha \rangle \neq 0$ thus $\zeta^{-n} = 1$, ie. $\exp(\frac{-2\pi n i}{h}) = 1$. This means that $\frac{n}{h} \in \mathbb{Z}$ so h divides n.

Now since $\mathcal{O}_i = \{\gamma^k(\alpha)\} \subset \{\alpha, \gamma(\alpha), \gamma^2(\alpha), ..., \gamma^{h-1}(\alpha)\}$ and h divides n whenever $\gamma^n(\alpha) = \alpha$ we conclude that $|\mathcal{O}_i| = h$.

□

5.7 Theorem.

Let γ be a Coxeter-Killing element and suppose $|\gamma| = h$. Then $h\ell = 2r$ where $r = |\mathfrak{R}^+|$.

Proof. Define $\mathfrak{h}_i$ as in the previous lemma. Since $\mathfrak{R} = \bigcup_i \mathcal{O}_i$ we have that:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{i=1}^{L} \mathfrak{h}_i$$

where L is the number of distinct orbits $\mathcal{O}_i$. Since $\mathfrak{g}$ has dimension $\ell + 2r$, the above direct sum decomposition shows that $\ell + 2r = \ell + hL$ ie. $2r = hL$. We claim that $L = \ell$.

Let $M \leq L$ denote the number of orbits where α acts as the identity on $\mathfrak{h}_i$ for every $\alpha \in \mathcal{O}_i$. Write:

$$\mathfrak{g}^\gamma = \bigoplus_{i=1}^{L} \mathfrak{h}_i \cap \mathfrak{g}^\gamma.$$

Each $\mathfrak{h}_i \cap \mathfrak{g}^\gamma$ has dimension 0 or 1 depending on whether every $\alpha \in \mathcal{O}_i$ acts as the identity on $\mathfrak{h}_i$ or not. Then we see that M is the dimension of $\mathfrak{g}^\gamma$. But $\mathfrak{t} \subset \mathfrak{g}^\gamma$ so we conclude that $\ell \leq M \leq L$.

Each $\mathcal{O}_i$ must contain at least one positive root and at least one negative root. For contradiction, suppose not. With no loss in generality assume it contains only positive roots. Then $\sum_{\alpha \in \mathcal{O}_i} \alpha \neq 0$. Since γ permutes the elements of $\mathcal{O}_i$, $\gamma(\sum_{\alpha \in \mathcal{O}_i} \alpha) = \sum_{\alpha \in \mathcal{O}_i} \alpha$. But by lemma

5.5 γ doesn't have any fixed points, a contradiction.

But now since $\mathcal{O}_i$ is an orbit, there must exist a positive root $\alpha \in \mathcal{O}_i$ such that $\gamma(\alpha)$ is a negative root, ie. that α changes sign under the action of γ. Hence there is at least one element in each $\mathcal{O}_i$ that changes sign under the action of γ. But by theorem 5.1 there are ℓ such elements so $L \leq \ell$.

Finally since $\ell \leq L$ and $L \leq \ell$ we conclude that $\ell = L$. So $h\ell = 2r$.

□

6 The Betti numbers of G

Write $\{\zeta^{m_i}\}$ for the eigenvalues of γ. The goal of this section is to show:

$$p_G(t) = \prod_{i=1}^{\ell}(1 + t^{d_i}) \qquad (*)$$

where $d_i = 2m_i + 1$. The numbers m_i are called the *exponents* of $\mathfrak{g}$. Hence the Poincare polynomial for G can be computed entirely from knowledge of a Coxeter-Killing element.

It was shown by Hopf [3] in 1941 that $p_G(t)$ is given by $(*)$. His original derivation of the d_i's is given by lemma 6.1. Nearly ten years later, Chevalley [1] proved (see theorem 6.3 below) that the d_i arise as the degrees of W-invariant polynomials on $\mathfrak{t}$, and he used explicit calculations of these polynomials to find exact values of the d_i.

It was noted by Coxeter that the explicit values for the d_i by Chevalley were related to the eigenvalues of the Coxeter-Killing element γ described in the previous section. In 1958, Coleman [2] showed that this was more than mere coincidence. He proved (theorem 6.5 below) that $d_i = 2m_i + 1$ where the eigenvalues of γ are given by lemma 5.3. However, his proof of $d_i = 2m_i + 1$ relied on the assumption that $h\ell = 2r$ where h is the order of γ, ℓ is the rank of $\mathfrak{g}$, and r is the number of positive roots. This equality was shown to hold for all of the examples previously computed by Chevalley, but there was no general proof of it at the time.

Finally, Kostant showed [4] (theorem 5.7 in the previous section) in 1959 that in general $h\ell = 2r$.

6.1 Lemma.

The Poincare polynomial $p_G(t)$ is given by:

$$p_G(t) = \prod_{i=1}^{\ell}(1 + t^{d_i})$$

where d_i are some odd integers.

Proof. Let:

$$p_G(t) = \sum_p B_p t^p$$

denote the Poincare polynomial for G. As was stated in section 4, B_p is given by the dimension of $H(\mathfrak{g})_p$. It was shown by Hopf that $H(\mathfrak{g})$ is isomorphic (as a graded algebra) to the exterior algebra of a subspace $P(\mathfrak{g}) \subset H(\mathfrak{g})$ of dimension ℓ with a basis given by forms with odd degrees $\{d_1, ..., d_\ell\}$.

Then any bi-invariant p-form corresponds uniquely to the choice of k odd dimension forms of degrees d_i in $P(\mathfrak{g})$ such that $\sum_{i=1}^{k} d_i = p$. So if we write:

$$f(t) = \prod_{i=1}^{\ell}(1 + t^{d_i})$$

we see that the coefficient on t^p is given by the number of ways to write p as a sum of d_i's, ie. is exactly B_p. Thus:

$$p_G(t) = \prod_{i=1}^{\ell}(1 + t^{d_i})$$

□

6.2 Remark.

Examination of the above proof shows that in order to compute the d_i we need only to find a basis for $P(\mathfrak{g})$. Below, we construct a surjective linear map onto $P(\mathfrak{g})$ such that the preimage of a $2k - 1$ form is a polynomial of degree k.

6.3 Theorem.

Write $d_i = 2k_i - 1$. Then k_i is the degree of a W-invariant polynomial on $\mathfrak{t}$.

Proof. Let $S(\mathfrak{g})$ denote the symmetric algebra over $\mathfrak{g}$ and let $I(\mathfrak{g}) \subset S(\mathfrak{g})$ be the polynomials that are both left and right Ad-invariant.

There is a surjective linear map $T : I(\mathfrak{g}) \to P(\mathfrak{g})$ given by:

$$P(X_1, ..., X_n) \mapsto \frac{1}{n}\sum_{i=1}^{n}(\frac{\partial P}{\partial x_i})(dX_1, ..., dX_n)X_i$$

whose kernel is generated by 1 and products of homogeneous polynomials. If $P \in I(\mathfrak{g})/\ker(T)$ has degree n, $T(P)$ is a $2n-1$ form. Then since $P(\mathfrak{g})$ is generated by the the $d_i = 2k_i - 1$ forms $\{\omega_1, ..., \omega_\ell\}$, we have that $I(\mathfrak{g})/\ker(T)$ (and hence $I(\mathfrak{g})$) is generated by deree k_i polynomials $\{P_1, ..., P_\ell\}$.

Since every element of G is conjugate to an element in T, if $P \in I(\mathfrak{g})$ then $P(Ad_g(X)) = P(X)$. Thus if P has a root in $\mathfrak{t}$ then $P \equiv 0$ on $\mathfrak{t}$. So $I(\mathfrak{g})$ is isomorphic to the restriction of $I(\mathfrak{g})$ to $\mathfrak{t}$.

But P is the restriction to $\mathfrak{t}$ of $P' \in I(\mathfrak{g})$ if and only if P is W-invariant. Thus the generators for the Ad-invariant polynomials $I(\mathfrak{g})$ correspond exactly to W-invariant polynomials on $\mathfrak{t}$.

□

Given theorem 6.3 what remains to be found are the values k_i given by the degree of the W-invariant polynomials in $I(\mathfrak{g})$. Theorem 6.5 shows that $k_i = m_i + 1$, concluding the calculation of $p_G(t)$.

6.4 Lemma (Coleman)

$|\mathfrak{R}^+| = \sum_i (k_i - 1)$.

Proof. Let $w = |W|$. By a theorem of Molien we have:

$$w \prod_{i=1}^{\ell} \frac{1}{1-t^{k_i}} = \sum_{\alpha \in W} \prod_{i=1}^{\ell} \frac{1}{1-\lambda_i^{\alpha} t}$$

where λ_i^{α} denotes the i'th eigenvalue of α. Multiply both sides by $(1-t)^{\ell}$:

$$w \prod_{i=1}^{\ell} \frac{1-t}{1-t^{k_i}} = \sum_{\alpha \in W} \prod_{i=1}^{\ell} \frac{1-t}{1-\lambda_i^{\alpha} t}$$

Then using L'Hospital's rule on the left hand side we get:

$$\lim_{t \to 1} w \prod_{i=1}^{\ell} \frac{1-t}{1-t^{k_i}} = \frac{w}{\prod_{i=1}^{\ell} k_i}$$

and since:

$$\lim_{t \to 1} \prod_{i=1}^{\ell} \frac{1-t}{(1-a_n t)} = 0$$

if any of the $a_n \neq 1$, then all the summands on the right hand side vanish except for the identity (which has all eigenvalues equal to 1). Hence the right hand side reduces to 1 so:

$$w = \prod_{i=1}^{\ell} k_i$$

Putting this back into the expression of Molien, subtract $\frac{1}{(1-t)^{\ell}}$ from both sides and multiply through by $(1-t)^{\ell-1}$:

$$\frac{(1-t)^{\ell-1} \prod_{j=1}^{\ell} d_j}{\prod_{i=1}^{\ell}(1-t^{k_i})} - \frac{1}{1-t} = -\frac{1}{1-t} + \sum_{\alpha \in W} \frac{(1-t)^{\ell-1}}{\prod_{i=1}^{\ell}(1-\lambda_i^{\alpha} t)}$$

we repeat a similar process to above (using L'Hospital's rule and take the limit as $t \to 1$) and what remains on the left hand side is:

$$\frac{\sum_i (k_i - 1)}{2}.$$

On the right hand side, the $-\frac{1}{1-t}$ term cancels the summand corresponding to the identity element. Finally we note that if a summand corresponds to a reflection one eigenvalue is -1 and the rest are 1 hence as $t \to 1$ that summand gives $\frac{1}{2}$, and if it does not correspond to a reflection then as $t \to 1$ that summand vanishes. Hence what we are left with is:

$$\frac{\sum_i (k_i - 1)}{2} = \frac{|\mathfrak{R}^+|}{2}$$

and the result follows.

□

6.5 Theorem.

$$k_i = m_i + 1$$

Proof. Let Λ denote the eigenvalues for γ. Since γ is a linear transformation on the *real* vector space $\mathfrak{t}$, if $\lambda \in \Lambda$ then $\bar{\lambda} \in \Lambda$. Hence if m_i is an exponent of $\mathfrak{g}$, ζ^{m_i} is an eigenvalue of γ so ζ^{h-m_i} is also an eigenvalue for γ. Then $h-m_i$ is an exponent whenever m_i is an exponent. Hence:

$$\sum_i m_i = \sum_i (h - m_i) = \sum_i h - \sum_i m_i = \ell h - \sum_i m_i$$

Then we see that:

$$\sum_i m_i = \frac{\ell h}{2}$$

But by theorem 5.7 we know that $\ell h/2 = r$ and by lemma 6.4, $r = \sum_i (k_i - 1)$. Thus:

$$\sum_i m_i = \frac{\ell h}{2} = r = \sum_i (k_i - 1).$$

Now we show that for each m_i there exists a j such that $m_i \equiv k_j - 1 \mod h$.

Let $\{v_1, ..., v_\ell\}$ denote a basis of eigenvectors for γ with eigenvalues $\{\zeta^{m_1}, ..., \zeta^{m_\ell}\}$ where $m_1 = 1$. Let $\{P_1, ..., P_\ell\}$ denote the generators of $I(\mathfrak{g})$ with degrees $\{k_1, ..., k_\ell\}$, and let $P = (P_1, ..., P_\ell)$. Then the Jacobian $d_v P$ of P at v is an $\ell \times \ell$ matrix with determinant equal to 0 if and only if v is not regular.

By theorem 5.2 v_1 is regular so $dP(v_1)$ has nonzero determinant. Then for all $1 \leq i \leq \ell$ there exists $1 \leq j \leq \ell$ such that the (i, j) entry is nonzero. But the (i, j) entry of d_P is $\frac{\partial P_i}{\partial x_j}$, and:

$$\frac{\partial P_i}{\partial x_j}(x_1, 0, ..., 0) = \frac{\partial}{\partial x_j}\bigg|_{(x_1,0,...,0)} \sum_p c_{i_p} x_1^{n_{p1}} x_2^{n_{p2}} \cdots x_\ell^{n_{p\ell}}$$

is 0 unless one of the summands is $x_1^{k_i-1}x_j$. But $P_i \in I(\mathfrak{g})$ so $\gamma(x_1^{k_i-1}x_j) = x_1^{k_i-1}x_j$. However:

$$\gamma(x_1^{k_i-1}x_j) = \zeta^{k_i-1}\zeta^{m_j}x_1^{k_i-1}x_j = \zeta^{k_i-1+m_j}x_1^{k_i-1}x_j$$

so $\zeta^{k_i-1+m_j} = 1$ ie. $k_i - 1 + m_j \equiv 0 \mod h$ as claimed.

Finally since:

$$\sum_i m_i = \frac{\ell h}{2} = \sum_j (k_j - 1)$$

and for each i there exists a j such that $m_i \equiv k_j - 1 \mod h$ we can change the indices so that $m_i = k_i - 1$ ie. $k_i = m_i + 1$ as claimed.

□

7 Corollaries of the Poincare polynomial formula.

Let G be a simple compact connected Lie group.

7.1 Corollary 1.

$\pi_1(G)$ is finite.

Proof. It was shown in lemma 5.5 that 0 is not an eigenvalue of γ so $m_i \neq 0$ for any i ie. $2m_i + 1 \neq 1$. Then $B_i(G) = 0$ for $i \in \{1, 2\}$. In particular $B_1(G) = 0$ so $H_1(G)$ is torsion free. But $\pi_1(G)$ is abelian (see eg. [5]) so $\pi_1(G) = H_1(G)$ is torsion free hence finite.

□

References

[1] C. Chevalley, *The Betti numbers of the exceptional simple Lie groups*, Proc. Int. Math. Cong. II (1950), 21-24.

[2] A. J. Coleman *The Betti numbers of the simple Lie groups*, Canadian Journal of Mathematics, vol. 10 (1958), 349-356.

[3] H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math, vol. 42 (1941).

[4] B. Kostant, *The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie Group*, American Journal of Mathematics, vol. 81, No 4 (1959), 973-1032.

[5] E. Meinrenken, *Lie groups and Lie algebras*, Course Notes (2010).