GROUP ACTIONS

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1. G-Manifolds

Let G be a Lie group and M a smooth manifold. In particular this means G is discrete if and only if it is at most countable.

Definition 1. An **smooth action** of G on M is a group homomorphism $A \in \mathrm{Hom}(G, \mathrm{Diff}(M))$, denoted $g \mapsto A_g$, such that the **action map** $A(x, g) = A_g(x)$ is a smooth map $M \times G \to M$. We call M a G-**manifold**. Replacing $\mathrm{Diff}(M)$ with $\mathrm{Homeo}(M)$ defines a **continuous action** and P is called a G-space. For fixed $x \in M$ the map $\theta_x(g) = A_g(x)$ is called the **orbit map**. The image is of the orbit map is denoted Gx and called the **orbit** of x.

Remark 2. We also use the notation gx for $A_g(x)$.

Definition 3. Let $F\colon M \times G \to M \times M$ be the map $F(x, g) = (x, gx)$. If F is injective we say the action is **free**, and if F is proper then we say the action is **proper**. If F is an embedding then we call the action **weakly proper**. We call a G-space **free** if the action is free.

Remark 4. If F is injective and proper then it is automatically an embedding. Conversely, if F is an embedding then it is proper if and only if its image is closed.

Lemma 5. *For a G-manifold M the following are equivalent:*

(1) The action is proper.
(2) If x_n and g_n are sequences in M and G (respectively) such that x_n and $g_n x_n$ converge, there exists a convergent subsequence of g_n.
(3) For every compact $K \subset M$ the set $\{g \in G : gK \cap K \neq \emptyset\}$ is compact.

Furthermore, any of the above imply that the stabilizer subgroups are compact and moreover the orbits $\mathcal{O}$ of the action are closed embedded submanifolds of M with $T_x\mathcal{O} = \{\xi_M(x) : \xi \in \mathfrak{g}\}$.

Remark 6. In general the orbits of a smooth G-action are only immersed submanifolds.

Lemma 7. *If the action map is proper then the action is proper.*

Proof. Let $K \subset M$ be compact, define $G_K = \{g : gK \cap K \neq \emptyset\}$ and let (g_n) be a sequence in G_K. By definition of G_K, for each n there exists $x_n \in g_n K \cap K$. Since $x_n \in g_n K$ we can write $x_n = g_n y_n$ for $y_n \in K$. Since $x_n \in K$ we have $\mathcal{A}(g_n, y_n) = x_n \in K$ so $(g_n, y_n) \in \mathcal{A}^{-1}(K)$. The action map is assumed to be proper so $\mathcal{A}^{-1}(K)$ is compact hence there exists a convergent subsequence (g_{n_k}, y_{n_k}) of (g_n, y_n). In particular g_{n_k} is a convergent subsequence of g_n so G_K is compact, as desired. $\square$

Remark 8. The converse is not true. Non compact groups acting properly on compact manifolds give a large family of counterexamples. For an example where M is not compact consider the action of $\mathbb{Z}$ on $\mathbb{R}^2 \setminus 0$ given by:

$$n \cdot (x, y) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix}$$

The observation that compact groups have a proper action map leads gives:

Corollary 9. *The action of a compact group is proper.*

2. When is a G-space a principal bundle?

Definition 10. A **smooth principal G-bundle over** M is a smooth map $\pi\colon P \to M$ from a G-manifold P such that every point in M has a neighbourhood U and a G-equivariant diffeomorphism $\varphi_U\colon U \times G \to \pi^{-1}(U)$ making the following diagram commute:

$$\begin{tikzcd} U \times G \arrow[r, "\varphi_U"] \arrow[dr, "\mathrm{proj_U}"'] & \pi^{-1}(U) \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

where equivariance is taken with respect to the action $g(x,h) = (x, hg^{-1})$. The pair (U, φ_U) is called a **local trivialization** or a **trivialization over** U. We will often use the notation P_V or $P|_V$ to mean $\pi^{-1}(V)$ (even if V is not necessarily open).

For two smooth principal bundles $\pi_i\colon P_i \to M$, a **bundle isomorphism** is a G-equivariant diffeomorphism $f\colon P_1 \to P_2$ such that $\pi_1 = \pi_2 \circ f$. A bundle isomorphism $P \to P$ (also called a **bundle automorphism**) is called a **gauge transformation** and the notation $\mathrm{Gau}(P) \coloneqq \mathrm{Aut}(P)$ is often used and $\mathrm{Gau}(P)$ called the **gauge group**.

Let $(U_i, \varphi_i) \subset M$ be two trivializations with $U_{12} \coloneqq U_1 \cap U_2 \neq \emptyset$. The map $\varphi_{12} \coloneqq \varphi_1 \circ \varphi_2^{-1}\colon U_{12} \times G \to U_{12} \times G$ is a diffeomorphism satisfying $\varphi_{12}(x,g) = (x, \psi_{12}(x,g))$ for some $\psi_{12}\colon U_{12} \times G \to G$ such that $\psi_{12}(x,gh) = \psi_{12}(x,g)h$ i.e. φ_{12} is a gauge transformation of $U_{12} \times G \to U_{12}$. We call the map $g_{12}\colon U_{12} \to G$ defined by $g_{12}(x) = \psi_{12}(x,e)$ the **transition function** of (U_1, φ_1) and (U_2, φ_2). One could equivalently view this as a map $U_{12} \to \mathrm{Aut}(G)$ by $x \mapsto (g \mapsto \psi_{12}(x,g))$.

Remark 11. In the category of topological spaces, a **principal G-bundle** is defined by replacing the "smooth" words with "continuous" (G becomes a topological group, P becomes a G-space, π and φ_U become continuous). That said, we will usually simply refer to a smooth principal bundle as simply a principal bundle.

Using equivariance of the local trivializations and the right action of G on itself is proper we see the action of G on P is free and proper and moreover it restricts to a transitive action on the fibers of π. The quotient map $P \to P/G$ induces a homeomorphism $M \to P/G$, providing a smooth structure on the orbit space making $P \to P/G$ a submersion. The converse also holds:

Theorem 12 (Quotient Manifold Theorem)**.** *Let P be a free G-manifold. Then P/G is a smooth manifold making $P \to P/G$ into a smooth principal G-bundle if and only if the action is proper. The smooth structure on P/G is unique (up to diffeomorphism).*

Remark 13. All of the assumptions in the quotient manifold theorem are necessary to ensure P/G is a smooth manifold and $P \to P/G$ is a submersion. It is straightforward enough to come up with actions that are not free or not proper so that the quotient won't even be a topological manifold. The assumption that G is acting by diffeomorphisms is necessary, even if G is discrete and finite and P is compact:

Proposition 14. *There exists a free action of $\mathbb{Z}_2$ on S^4 by homeomorphisms such that $S^4/\mathbb{Z}_2$ is not smoothable* [2].

The quotient manifold theorem therefore provides a correspondence between smooth principal G-bundles and smooth actions of Lie groups on manifolds that are free and proper. The situation in the category of topological spaces is more subtle. The existence of equivariant local trivializations on a principal bundle implies the action is free and proper, but the converse is not true even if P and G are both compact Hausdorff:

Proposition 15. *There exists a compact metrizable topological group G and compact Hausdorff free G-space P such that the quotient map $P \to P/G$ is not a fiber bundle* [1].

It is true, however, under mild assumptions:

Proposition 16. *If G is a Lie group and P is a locally compact Hausdorff free G-space then $P \to P/G$ is a principal bundle if and only if the action is proper.*

If we require that P be a manifold we have the Hilbert-Smith conjecture (verified for P a 3-manifold, as well as some specific types of actions):

Conjecture 17 (Hilbert-Smith)**.** *If G is a locally compact topological group acting continuously and faithfully on a connected topological manifold then G is a Lie group.*

Remark 18. Actually, the Hilbert-Smith conjecture is equivalent to:

Conjecture 19 (Hilbert-Smith)**.** *There does not exist a continuous and faithful action of a p-adic group on a connected topological manifold.*

We summarize these results in the following table for G a topological group acting continuously on P:

	G a Lie group	G not a Lie group
P a manifold	$P \to P/G$ is a principal bundle.	Hilbert-Smith.
P LCH	$P \to P/G$ is a principal bundle.	$P \to P/G$ not necessarily a principal bundle (even if P is compact).

Definition 20. Let $\pi\colon P \to M$ be a principal G-bundle. A **(prinicpal) connection** is a G-equivariant $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P, \mathfrak{g})$ such that $\theta(X^*) = X$ for every $X \in \mathfrak{g}$ (where X^* is the generating vector field for X associated to the action of G on P). By G-equivariance we mean with respect to the adjoint action on $\mathfrak{g}$, i.e. $A_g^*\theta = \mathrm{Ad}_g(\theta)$ where by the right hand side we mean the composition of θ with Ad_g when the $\mathfrak{g}$-valued 1-form as a map $\theta\colon TP \to \mathfrak{g}$ (recall that at $x \in P$, θ_x is a map $\theta_x\colon T_xP \to \mathfrak{g}$). We further define the **horizontal subbundle** $H_\theta = \ker(\theta)$. Note that $TP = \ker(\pi_*) \oplus H_\theta$.

Since $\theta(X^*) = X$, for any two connections θ_1, θ_2 we have $(\theta_1 - \theta_2)(X^*) = 0$ so $\theta_1 - \theta_2$ corresponds to some $A \in \Omega^1(M, \mathrm{Ad}\, P)$, i.e. the space of connections $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(M, \mathrm{Ad}\, P)$

Let $P = M \times G$ be the trivial principal G-bundle and $\theta \in \Omega^1(P, \mathfrak{g}) = \Omega^1(M \times G, \mathfrak{g})$ a principal connection. At $(x, g) \in P$ we have $\theta_{(x,g)}\colon T_{x,p}(M \times G) = T_xM \oplus T_pG \to \mathfrak{g}$. Writing $\theta_{x,g} = \theta_x + \theta_g$ for $\theta_x = \theta_{x,g}|_{T_xM}$ and $\theta_g = \theta_{x,g}|_{T_gG}$, it follows from the definition that $\theta_x = \mathrm{Ad}_{g^{-1}}(A)$ for $A \in \Omega^1(M, \mathfrak{g})$ and $\theta_g = \theta^L$ is the left-invariant Maurer-Cartan form. We call A the **connection 1-form**. Viewing $\mathfrak{g}$ as a matrix algebra, A is a matrix whose entries are elements of $\Omega^1(M)$, i.e. A is a **matrix-valued 1-form**.

The gauge group acts on $\mathcal{A}(P)$ by **gauge transformations** $\varphi \cdot \theta = (\varphi^{-1})^*\theta = \varphi_*(\theta)$ and we call θ and $\varphi^*\theta$ **gauge equivalent**. Suppose $\varphi \in \mathrm{Gau}(P)$ and (U, ψ_U) is a local trivialization so $\varphi(P_U) = P_U$. Then $\Phi = \psi_U^{-1} \circ \varphi \circ \psi_U$ is a gauge transformation $U \times G \to U \times G$ i.e. $\Phi(x, g) = (x, \phi(x, g))$ for some $\phi\colon U \times G \to G$ with $\phi(x, gh) = \phi(x, g)h$. Write $\phi(x) = \phi(x, e)$. In this case $\Phi^{-1}(x, g) = (x, \phi(x)^{-1}g)$. Observe that the gauge action on θ becomes:

$$\begin{aligned}
\Phi \cdot \theta = (\Phi^{-1})^*\theta = (\Phi^{-1})^*(g^{-1}Ag + \theta^L) &= (\Phi^{-1})^*(g^{-1}Ag) + (\Phi^{-1})^*\theta^L \\
&= g^{-1}\phi A\phi^{-1}g + g^{-1}\phi d(\phi^{-1}g) \\
&= g^{-1}\phi A\phi^{-1}g + g^{-1}\phi(-\phi^{-1}d\phi\phi^{-1}g + \phi^{-1}dg) \\
&= g^{-1}\phi A\phi^{-1}g - g^{-1}d\phi\phi^{-1}g + g^{-1}dg \\
&= g^{-1}(\phi A\phi^{-1} - \phi^*\theta^R)g + \theta^L \\
&= g^{-1}Bg + \theta^L
\end{aligned}$$

We see then that locally the gauge action on $\mathcal{A}(P)$ reduces to the **gauge action** of $C^\infty(M, G)$ on $\Omega^1(M, \mathfrak{g})$ as in $\phi \cdot A = \mathrm{Ad}_\phi(A) - \phi^*\theta^R$. In particular if (U_1, φ_1), (U_2, φ_2) are local trivializations with transition function g_{12}, writing $\varphi_i^*\theta = g^{-1}A_ig + \theta^L$ we see:

$$g^{-1}A_2g + \theta^L = \varphi_{12}^*(g^{-1}A_1g + \theta^L) = \varphi_{21} \cdot (g^{-1}A_1g + \theta^L) = g^{-1}(g_{12}^{-1}A_1g_{12} + g_{12}^*\theta^L)g + \theta^L$$

thus:

$$A_2 = g_{12}^{-1}A_1g_{12} + g_{12}^*\theta^L$$

is how the connection 1-form transforms under change of trivializations.

Definition 21. Let $\pi\colon P \to M$ be a principal bundle and θ a connection. A path $\gamma\colon I \to P$ is called **horizontal** if $\dot\gamma(t) \in H_\theta$.

Proposition 22. *For every smooth $\gamma\colon I \to M$ with $\gamma(0) = x$ and $p \in \pi^{-1}(x)$, there exists a* ***horizontal lift*** *$\widetilde{\gamma}_p\colon I \to P$ with $\widetilde{\gamma}_p(0) = p$, i.e. $\pi \circ \widetilde{\gamma}_p = \gamma$ and $\widetilde{\gamma}_p$ is horizontal. The map $\tau_\gamma\colon \pi^{-1}(x) \to \pi^{-1}(\gamma(1))$ given by $\tau_\gamma(p) = \widetilde{\gamma}_p(1)$ is called the* ***parallel transport along*** *γ and it is an isomorphism.*

Definition 23. Let $\gamma\colon I \to M$ be a loop based at $x \in M$. The **holonomy** $\mathrm{Hol}_{x,\gamma}\colon P_x \to P_x$ **of** γ **at** x is defined by $\mathrm{Hol}_{x,\gamma}(p) = \widetilde{\gamma}_p(1)$. This defines a map $\mathrm{Hol}_x\colon C(x) \to \mathrm{Homeo}(P_x)$, $\mathrm{Hol}_x(\gamma) = \mathrm{Hol}_{x,\gamma}$.

A useful reinterpretation of the holonomy group as a subgroup of G is as follows.

Definition 24. Fix $p \in P_x$. For $\gamma \in C(x)$ there exists a unique $g_\gamma \in G$ such that $\mathrm{Hol}_{x,\gamma}(p) = g_\gamma \cdot p$ defining a map $\mathrm{Hol}_p\colon C(x) \to G$ by $\gamma \mapsto g_\gamma$. The image $\Phi_p \leq G$ of Hol_p is called the **holonomy subgroup of** G (with **reference point p**) and there exists a unique isomorphism $\Phi_x \to \Phi_p$ such that the following diagram commutes:

$$\begin{tikzcd} C(x) \arrow[d] \arrow[dr] & \\ \Phi_x \arrow[r] & \Phi_p \end{tikzcd}$$

Finally, we define two notions of reducibility. If Φ_p is not equal to G then we say the connection is **reducible to** Φ_p. Alternatively (but *not* equivalently), if H is a proper subgroup of G then we say A is **reducible** if there exists a reduction of the structure group $\iota\colon Q \to P$ to a principal H-bundle Q such that ι^*A is a principal connection for Q. The second definition of reducible is equivalent to ι^*A taking values in $\mathfrak{h}$.

Definition 25. If P is a principal G-bundle with θ a principal connection, the **curvature** $\mathrm{Curv}(\theta) = \mathrm{Curv}_\theta \in \Omega^2(P, \mathfrak{g})$ of θ is:

$$\mathrm{Curv}(\theta) = d\theta + [\theta, \theta]$$

where $d\theta$ is the exterior derivative on $\Omega^k(P, \mathfrak{g})$ and:

$$[\eta_1, \eta_1](u, v) = [\eta_1(u), \eta_1(v)].$$

where the right hand side is the Lie bracket on $\mathfrak{g}$. If $\mathrm{Curv}(\theta) \equiv 0$ we say θ is **flat** and denote the space of flat connections by $\mathcal{A}_{\mathrm{flat}}(P)$. When $P = M \times G$, $\mathrm{Curv}(\theta^L) = 0$ so we call θ^L the **canonical flat connection** on $M \times G$.

Remark 26. The space of flat connections is a gauge invariant subspace of $\mathcal{A}(P)$.

Definition 27. Let M be a manifold, $p \in M$, and fix $\phi \in \mathrm{Hom}(\pi(M, p), G)$ for a Lie group G. For $\psi \in \pi(M, p)$ let $F^\phi_\psi\colon \widetilde{M} \times G \to \widetilde{M} \times G$ be the diffeomorphism $F^\phi_\psi(x, g) = (\psi(x), \phi(\psi)g)$. The canonical flat connection θ^L on the trivial bundle $\widetilde{M} \times G$ satisfies $(F^\phi_\psi)^*\theta^L = \theta^L$ for every $\psi \in \pi(M, p)$ so it induces a flat connection θ_ϕ on the associated bundle $\widetilde{M} \times_\phi G$ that we call the **canonical flat connection**.

One computes that when $\theta = g^{-1}Ag + \theta^L$ we have $\mathrm{Curv}(\theta) = \mathrm{Ad}_{g^{-1}}(\mathrm{Curv}(A))$ It follows that in the overlap:

$$\mathrm{Curv}(A_\alpha) = g_{\alpha\beta}\mathrm{Curv}(A_\beta)g^{-1}_{\alpha\beta}$$

so the $\mathfrak{g}$-valued 2-forms $\mathrm{Curv}(A_\alpha) \in \Omega^2(M, \mathfrak{g})$ glue together to form $F_A \in \Omega^2(M, \mathrm{ad}P)$. Moreover, under bundle projection $\pi\colon P \to M$ we have $\mathrm{Curv}_\theta = \pi^*F_A \in \Omega^2(P, \pi^*\mathrm{ad}P) = \Omega^2(P, \mathfrak{g})$.

Another way to define F_A is as follows. For $X, Y \in \mathfrak{X}(M)$ let $X_H, Y_H \in \mathfrak{X}(P)$ denote their horizontal lifts (horizontal with respect to θ). View the connection as an equivariant splitting $\theta\colon TP \to V \cong P \times \mathfrak{g}$, let $q\colon P \times \mathfrak{g} \to \mathrm{ad}P$ denote the quotient map and define:

$$F_A(X, Y) = q(\theta[X_H, Y_H])$$

Observe that the H_θ is integrable if and only if θ is flat - definition, a vector $X \in T_pP$ is horizontal if $\theta(X) = 0$. Then for horizontal vector fields X, Y on P it follows easily that $\theta([X, Y]) = -2\mathrm{Curv}_\theta(X, Y)$ so the Lie bracket of horizontal vector fields is again horizontal if and only if the curvature vanishes.

Proposition 28. *If $\pi\colon (P, \theta) \to M$ is a flat G-bundle and $\Gamma\colon [0,1]^2 \to M$ is a homotopy then the holonomy of $\gamma_s(t) = \Gamma(s, t)$ is the same for all $s \in [0, 1]$.*

Proof. Let $\widetilde{\Gamma}\colon [0,1]^2 \to P$ denote the horizontal lift of Γ starting at $x \coloneqq \widetilde{\Gamma}(s, 0)$. Since $\widetilde{\gamma}_s(t) = \widetilde{\Gamma}(s, t)$ is horizontal, $\widetilde{\gamma}_s(t) \in P_x$ for all s and t where P_x is the leaf of the horizontal foliation through x. But $\dim(P_x) = \dim(M)$ so $\pi|_{P_x}$ is a covering map hence $\widetilde{\Gamma}(s, 1)$ is constant in s. □

From this we see that the holonomy of a flat connection is homotopy invariant so Hol_p descends to a homomorphism $\phi_\theta\colon \pi_1(M) \to G$ that we call the **holonomy homomorphism**.

Let $\mathcal{A}(M, G) = \{(P, \theta) : \text{P a principal G-bundle, } \theta \text{ a connection}\}$ and $\mathcal{A}_{\text{flat}}(M, G)$ the subspace where θ is taken to be flat. Put an equivalence relation $\sim$ on $\mathcal{A}(M, G)$ by declaring $(P_1, \theta_1) \sim (P_2, \theta_2)$ if there exists a bundle isomorphism $f\colon P_1 \to P_2$ such that $f^*\theta_2 = \theta_1$.

Theorem 29. *The map $\mathcal{A}_{\mathit{flat}}(M, G) \to \mathrm{Hom}(\pi_1(M), G)$ defined by $(P, \theta) \mapsto \phi_\theta$ descends to a bijection $\mathcal{A}_{\mathit{flat}}(P)/\sim\, \to \mathrm{Hom}(\pi_1(M), G)/G$. The map $\mathrm{Hom}(\pi_1(M), G) \to \mathcal{A}_{\mathit{flat}}(M, G)$ given by $\phi \mapsto (\widetilde{M} \times_\phi G, \theta_\phi)$ descends to its inverse.*

3. Some calculations

Let $G = \mathrm{SL}_2(\mathbb{R})$ with $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ having basis $e_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $e_h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that for $c \in \mathbb{R}$ we have:

$$\exp ce_a = \begin{pmatrix} e^c & 0 \\ 0 & \frac{1}{e^c} \end{pmatrix}, \quad \exp ce_k = \begin{pmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{pmatrix}, \quad \exp ce_h = \begin{pmatrix} \cosh c & \sinh c \\ \sinh c & \cosh c \end{pmatrix}$$

Let $B(X,Y) = 4\operatorname{tr}(XY)$ denote the Killing form on $\mathfrak{sl}_2(\mathbb{R})$. If e_a^*, e_k^*, e_h^* denote the dual basis to e_a, e_k, e_h then $f_a = 8e_a^*$, $f_k = -8e_k^8$, and $f_h = 8e_h^*$ is the B-dual basis.

For $t > 1$ and $z \in \mathbb{R}$ let $m = \sqrt{\frac{\sqrt{t^4z^2+1}+1}{2}}$, $n = \sqrt{\frac{\sqrt{t^4z^2+1}-1}{2}}$, and $s = \frac{t}{(t^4z^2+1)^{\frac{1}{4}}}$. Then:

$$\begin{pmatrix} \frac{ms}{t} & \frac{ns}{t} \\ \frac{-ns}{t} & \frac{ms}{t} \end{pmatrix} \begin{pmatrix} m & n \\ n & m \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix} = \begin{pmatrix} t & tz \\ 0 & \frac{1}{t} \end{pmatrix}$$

and since $\begin{pmatrix} \frac{ms}{t} & \frac{ns}{t} \\ \frac{-ns}{t} & \frac{ms}{t} \end{pmatrix} \in SO(2)$ we have a decomposition of G of the form $g = kha$ where $k \in SO(2)$, $a \in A$ is diagonal and h is of the form $\begin{pmatrix} m & n \\ n & m \end{pmatrix}$ with $m \geq 1 > 0$ and this decomposition is unique. Note that such a matrix can be written $h = \begin{pmatrix} \cosh(w) & \sinh(w) \\ \sinh(w) & \cosh(w) \end{pmatrix}$ for $w = \sinh^{-1}(n)$.

In Cartesian coordinates, the left-invariant vector fields on G associated to our basis for $\mathfrak{g}$ can be computed:

$$ge_a = x\partial_x - y\partial_y + z\partial_z - w\partial_w, \quad ge_k = -y\partial_x + x\partial_y - w\partial_z + z\partial_w, \quad ge_h = y\partial_x + x\partial_y + w\partial_z + z\partial_w$$

In Cartesian coordinates, the[1] dual basis for $\mathfrak{g}^*$ is:

$$e_a^* = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_k^* = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_h^* = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so the left-invariant 1-forms dual to the above left invariant vector fields are given in Cartesian coordinate by:

$$2g^{-t}e_a^* = wdx + zdy - ydz - xdw \quad 2g^{-t}e_k^* = zdx + wdy - xdz - ydw, \quad 2g^{-t}e_h^* = -zdx + wdy - xdz - ydw$$

and we denote them (respectively) by ω_a, ω_k, and ω_h. With respect to Iwasawa coordinates $G = KAN = S^1 \times \mathbb{R}_{>0} \times \mathbb{R}$ one computes that:

$$2\omega_k = (r^2 + \frac{1}{r^2} + r^2z^2)d\theta - dz - \frac{2z}{r}dr$$

and under the diffeomorphism $S^1 \times \mathbb{R}^2 \to KAN$ given by $(\theta, s, z) \mapsto (\theta, e^s, z)$ we have:

$$\omega_k = (2\cosh(2s) + e^{2s}z)d\theta - dz - 2zds$$

Conversely one can write the Cartesian 1-forms in terms of the left-invariant ones, e.g.:

$$d\theta = \frac{\omega_h - \omega_k}{x^2 + z^2} = \frac{1}{x^2 + z^2}g^{-t}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{xdz - zdx}{x^2 + z^2}$$

Alternatively, using Cartan decomposition $G = KP \cong S^1 \times \mathbb{R}^2$ it can be shown that:

$$d\theta_g = \frac{1}{|g|^2}((z - y)(dx + dw) + (x + w)(dy - dz)) = \frac{1}{|g|^2}((z - y)d(x + w) + (x + w)d(y - z))$$

where $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $|g|^2 = \mathrm{tr}(gg^t + I)$.

Let Σ be a closed oriented genus $g \geq 2$ surface. A pair (S, f) representing an element of Teichmuller space $\mathcal{T}_g$ is by definition an orientation preserving diffeomorphism $f\colon \Sigma \to S$ where S is a Riemann surface. The Riemann surface S has Fuchsian representation $S = \mathbb{H}/\Gamma_S$ for $\Gamma_S \leq G$, unique up to conjugation. The map f induces a group isomomorphism $f_*\colon \pi_1(\Sigma) \to \pi_1(S) = \Gamma_S \leq G$ hence $\iota \circ f_* \in \mathrm{Hom}(\pi_1(\Sigma), G)$ is injective with discrete image. The map $(S, f) \mapsto \iota \circ f_*$ therefore descends to an injective map $\mathcal{T}_g \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$. Conversely, if $\phi \in \mathrm{Hom}(\pi_1(\Sigma), G)$ is injective with discrete image, $\Gamma_\phi := \mathrm{Im}(\phi) \cong \pi_1(\Sigma)$ is Fuchsian so there exists a diffeomorphism $f_\phi\colon \Sigma \to S_\phi := \mathbb{H}/\Gamma_\phi$ such that $(f_\phi)_* = \phi$. The map f_ϕ pulls back the metric on S_ϕ to a metric on Σ so if f_ϕ is orientation preserving (there is a canonical orientation on S_ϕ) the pair (S_ϕ, f_ϕ) represents an element of $\mathcal{T}_g$.

Fix $P \to \Sigma$ a principal G-bundle with a flat connection $\theta \in \Omega^1(P, \mathfrak{g})$ and fix $p \in \Sigma$. Since θ is flat there exists a trivializing neighbourhood U_1 of p such that $\theta|_{P_{U_1}} = \theta^L$ and with no loss in generality we may take U_1 to be a disk. Let $U_2 = \Sigma \setminus p$ and fix a trivialization of $P|_{U_2}$. Writing $\theta = g^{-1}Ag + \theta^L$ in U_2 for $A = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & -\omega_1 \end{pmatrix} \in \Omega^1(U_2, \mathfrak{g})$, in U_{12} we have $A = g_{12}^{-1} dg_{12} = g_{12}^* \theta^L$. Flatness of θ is therefore equivalent to a solution to the system:

$$\begin{aligned} 2d\omega_1 &= \omega_3 \wedge \omega_2 \\ d\omega_2 &= \omega_2 \wedge \omega_1 \\ d\omega_3 &= \omega_1 \wedge \omega_3. \end{aligned}$$

Let $\alpha_1 = \omega_2 + \omega_3$, $\alpha_2 = 2\omega_1$ and $\omega = \omega_3 - \omega_2$. Flatness of θ is therefore equivalent to a solution to the **Cartan equations**:

$$\begin{aligned} d\alpha_1 &= \alpha_2 \wedge \omega \\ d\alpha_2 &= \omega \wedge \alpha_1 \\ d\omega &= \alpha_2 \wedge \alpha_1 \end{aligned}$$

We note here that ω (not only $d\omega$) is determined uniquely by α_i whenever $\alpha_1 \wedge \alpha_2 \neq 0$ - writing $d\alpha_i = -f_i \alpha_1 \wedge \alpha_2$ then $\omega = f_1\alpha_1 + f_2\alpha_2$.

Define $F\colon G \times \mathbb{H} \to \mathbb{H}$ by $F(g, x) = g(x)$. Then:

$$D_{(g,x)}F = \frac{1}{(d + cx)^2} \left(2x, -(x^2+1), 1, 1, i\right)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence the map $(g, x) \mapsto g^{-1}\phi(x)$ has derivative:

$$\begin{pmatrix} 2(\phi(x)(ad+bc) - cd\phi(x)^2 - ab) & 2(ac+bd)\phi(x) - (c^2+d^2)\phi(x)^2 - (a^2+b^2) & 2ac\phi(x) + c^2\phi(x)^2 + a^2 \end{pmatrix}$$

together with $d_x\phi$.

References

[1] R.F. Williams, *A useful functor and three famous examples in topology*, Trans. Amer. Math. Soc. 106 (1963) 319–329.

[2] D. Ruberman, *Invariant knots of free involutions of* S^4, Topology Appl. 18, 217-224 (1984). ZBL0559.57016.