SPIN MANIFOLDS OF POSITIVE SCALAR CURVATURE AND THE VANISHING OF THE $\hat{A}$-GENUS.

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1. Introduction.

Given a smooth manifold M, obstructions to the existence of Riemannian metrics on M having certain curvature properties is a classical subject with a wealth of interesting results. Given the curvature hierarchy:

$$\text{Curvature tensor} \Leftrightarrow \text{Sectional curvature} \Rightarrow \text{Ricci curvature} \Rightarrow \text{Scalar curvature}$$

obstructions to the existence of metrics with positive scalar curvature immediately gives an obstruction to the existence of metrics with certain Ricci or sectional curvature (notably those with positive Ricci curvature or positive sectional curvature). We analyze the case of spin manifolds with scalar sectional curvature here.

2. The Spinor bundle.

2.1. Principal bundles.

2.1.1. *Definition.* If $\pi : E \to B$ is a fiber bundle with fiber G where G is a topological group such that $G \circlearrowright E$ smoothly and freely, and the homeomorphism $h : \pi^{-1}(U) \to U \times G$ respects the action of G (ie. for all $x \in \pi^{-1}(U)$ we have $h(xg) = h(x)g$ where the action on $U \times G$ is the natural one $(x, g_1)g_2 = (x, g_1g_2)$) then we say that E is a *principal* G-bundle and write $B = E/G$.

Implicitly in this definition is that $\pi(x_1) = \pi(x_2)$ if and only if they are in the same G-orbit. To see this suppose $\pi(x_1) = \pi(x_2) = y$. Let U be a local trivialization near y and $\varphi : \pi^{-1}(U) \to U \times G$. Since $\pi : E \to B$ is a fiber bundle we have $\pi(x_1) = \pi(x_2) = \pi_U(\varphi(x_1)) = \pi_U(\varphi(x_2)) = y$ so $\varphi(x_1) = (y, g_1)$ and $\varphi(x_2) = (y, g_2)$. But $x_1 = \varphi^{-1}(y, g_1) = \varphi^{-1}(y, g_2g_2^{-1}g_1) = \left(\varphi^{-1}(y, g_2)\right) g_2^{-1}g_1 = x_2g_2^{-1}g_1$. This justifies the notation $B = E/G$.

2.1.2. *Examples.*

1. If G is a Lie group and $H \leq G$ is a closed subgroup then the coset action $H \circlearrowright G$ is free and smooth, and so defines a principal H bundle of G over G/H.
2. As a concrete example of 1. we have the action of $S^1 = U(1)$ on S^{2n+1} by scalar multiplication. This gives us a principal S^1-bundle of S^{2n+1} over $S^{2n+1}/S^1 = \mathbb{C}P(n)$.
3. Given an orientable Euclidean vector bundle $\pi : E \to B$ (ie. a vector bundle with an inner product space in each fiber) of rank n, $SO(n)$ acts smoothly and freely on the fibers E_x. This allows us to define the principal $SO(n)$-bundle $\pi : F_{SO}(E) \to B$, called the *orthonormal frame bundle.*

2.1.3. *Definition.* Given a principal G-bundle $\pi_G : P \to B$, if $G \circlearrowright F$ for some topological space F we define the *associated bundle* of P with respect to the G-action on F.

First note that $G \circlearrowright P \times F$ by the formula $(x, f)g = (xg, g^{-1}f)$. Define $P \times_G F = (P \times F)/G$, the base space of the principal G-bundle $\pi_1 : P \times F \to (P \times F)/G$. Given $v \in P \times_G F$ we can write $v = \pi_1(p, f)$ for some $p \in P$ and $f \in F$. Then we define $\pi : P \times_G F \to B$ to be $\pi(v) = \pi_G(p)$. As

$P \times_G F$ is a principal G-bundle this map is well defined and turns $\pi : P \times_G F \to B$ into a fiber bundle over B with fiber F.

2.1.4. *Definition.* If $\pi : E \to B$ is a fiber bundle with fiber F and $f : X \to B$ is a continuous function then we can form the *pullback bundle* $f^* : E_f \to X$ with fiber F where $E_f = \{(x, y) \in X \times E \mid f(x) = \pi(y)\}$. The map is given by $f^*(x, y) = f(x) = \pi(y)$. If we apply the pullback construction to a principal G-bundle we get a new principal G-bundle.

2.1.5. *Definition.* Let $\pi_1 : E_1 \to B$, $\pi_2 : E_2 \to B$ be two bundles over B. Then E_1 is *isomorphic* to E_2 as bundles over B, denoted $E_1 \cong E_2$ if there exists a homeomorphism $f : E_1 \to E_2$ such that $\pi_1(x) = \pi_2(f(x))$ for all $x \in E_1$. For principal G-bundles we also require that $f(xg) = f(x)g$ for all $x \in E_1$, $g \in G$.

2.2. **Clifford Algebras.**

2.2.1. *Definition.* Let (V, g) be a real inner product space. The *real Clifford algebra* of V, denoted $Cl_g(V)$ is the quotient of the tensor algebra $T(V)$ by the two sided ideal:

$$I = \{v_1 \otimes v_2 + v_2 \otimes v_1 - 2g(v_1, v_2) \mid v_1, v_2 \in V\}$$

The map $j : V \to V$ given by $j(v) = -v$ extends to an algebra morhpism J in $Cl_g(V)$ such that $J^2 = Id$. The eigenspaces of J give a $\mathbb{Z}_2$ grading for $Cl_g(V)$:

$$Cl_g(V) = Cl^0(V) \oplus Cl^1(V)$$

into even and odd subspaces. If $\{e_1, ..., e_n\}$ is an orthonormal (with respect to g) basis of V then $\{e_{i_1} \cdots e_{i_k}\ k \geq 0\}$ is a basis of $Cl_g(V)$. We denote the *complex Clifford algebra* $Cl(V)^{\mathbb{C}} = Cl(V) \otimes_{\mathbb{R}} \mathbb{C} = Cl(V^{\mathbb{C}})$ (using the complexification of g) by $\mathbb{C}l(V)$.

The Clifford algebra comes with a second involution (besides J) called the *transpose*, given on 2-vectors by $(v_1 v_2)^t = v_2 v_1$ and extended naturally.

2.2.2. *Definition.* A *Clifford module over* $\mathbb{C}l(V)$ is a complex Z_2 graded inner product space $E = E^0 \oplus E^1$ with a $\mathbb{Z}_2$ graded algebra morphism $\rho : \mathbb{C}l(V) \to \mathrm{End}(E)$.

2.2.3. *Definition.* A *Clifford module bundle over* M is a vector bundle $\pi_S : S \to M$ such that each fiber S_x is a $\mathbb{C}l(V)$-module and the Clifford module structure is preserved in local trivializations.

If $\pi_E : E \to M$ is an oriented Euclidean vector bundle, the *Clifford bundle of* E is the associated vector bundle $\pi : Cl(E) = F_{SO}(E) \times_\sigma Cl(\mathbb{R}^n) \to M$ where $\sigma : SO(n) \to \mathrm{Aut}(Cl(\mathbb{R}^n))$ is the natural induced representation. Note that the fibers $Cl(E)_x$ are Clifford algebras so $Cl(E)$ is a bundle of Clifford algebras over M. Denote $Cl(TM)$ by $Cl(M)$.

2.3. **Spin structures.** Using the double cover $p : SU(2) \to SO(3)$ and the exact sequence of homotopy groups one can show that $\pi_1(SO(n)) = \mathbb{Z}_2$ for all $n \geq 3$. In particular this means that $SO(n)$ is not simply connected, but instead has a simply connected double cover called $\mathrm{Spin}(n)$. Since $SO(n)$ is a Lie group, $\mathrm{Spin}(n)$ is also a Lie group. However, there is an alternative realization of $\mathrm{Spin}(n)$ as a concrete subgroup of $Cl_g(V)$ (that we will not define here).

2.3.1. *Definition.* Let $\rho : E \to M$ an oriented Euclidean $n \geq 3$-bundle and $\pi_F : F_{SO}(E) \to M$ the orthonormal frame bundle. Then a *spin structure* on E is a principal $\mathrm{Spin}(n)$-bundle $\pi_S : \mathrm{Spin}(E) \to M$ together with a double cover $P : \mathrm{Spin}(E) \to F_{SO}(E)$ such that $P(pg) = P(p)\rho_2(g)$ for all $p \in \mathrm{Spin}(E)$ and $g \in \mathrm{Spin}(n)$, where $\rho_2 : \mathrm{Spin}(n) \to SO(n)$ is the double covering.

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r] \arrow[d, "\rho_2"] & \mathrm{Spin}(E) \arrow[r, "P"] \arrow[d, "\pi_S"'] & F_{SO}(E) \arrow[dl, "\pi_F"] \\
SO(n) & M &
\end{tikzcd}$$

A *spin manifold* is an oriented Riemannian manifold with a spin structure on TM.

2.3.2. *Definition.* Let $\rho : \mathbb{C}l(\mathbb{R}^n) \to \mathrm{End}(M_{\mathbb{C}})$ be a Clifford module and $E \to M$ an oriented Euclidean vector bundle with spin structure $P : \mathrm{Spin}(E) \to F_{SO}(E)$. Let $\rho_n : \mathrm{Spin}(n) \to SO(M_{\mathbb{C}})$ denote the restriction of ρ to $\mathrm{Spin}(n) \leq \mathbb{C}l(\mathbb{R}^n)$. Then the associated Clifford module bundle $S_{\mathbb{C}}(E) = \mathrm{Spin}(E) \times_{\rho_n} M_{\mathbb{C}} \to M$ is a (complex) *spinor bundle* and sections of $S_{\mathbb{C}}(M)$ are called *spinors.*

If M is an even dimension spin manifold (ie. with a spin structure on TM) there is a unique irreducible spinor bundle which we denote by $\varpi_{\mathbb{C}}$.

3. The index of elliptic operators.

3.1. **Fredholm Theory.**

3.1.1. *Definition.* A bounded linear map $T : V \to W$ between Banach spaces is called *Fredholm* if $\ker(T)$ and $\mathrm{coker}(T)$ are both finite dimensional. If T is Fredholm we define the *index of* T by $\mathrm{Index}(T) = \ker(T) - \mathrm{coker}(T)$.

3.2. **Elliptic Operators.**

3.2.1. *Definition.* Let E, F be smooth vector bundles over M of ranks m, n (respectively). We say that $P : \Gamma(E) \to \Gamma(F)$ is a *differential operator of order* k and write $P \in DO^k(E, F)$ if

(1) For all $u \in \Gamma(E)$, $\mathrm{supp}Pu \subset \mathrm{supp}u$.
(2) For any $U \subset M$ and trivializations $E|_U$, $F|_U$, P is given by:

$$(Pu)(x) = \sum_{|\alpha| \leq k} A_\alpha(x)(D^\alpha u)(x)$$

where the sum is taken over multi indices α and $A_\alpha(x) : E|_U \to F|_U$ is an $n \times m$ matrix valued function of x.

Given $P \in DO^k(E, F)$ and $f_1, ..., f_k \in C^\infty(M)$, it is a consequence of polarization that $\mathrm{ad}(f_1) \cdots \mathrm{ad}(f_k)P \in \Gamma(E^* \otimes F)$ is symmetric in the f_i so $\mathrm{ad}(f_1) \cdots \mathrm{ad}(f_k)P = \mathrm{ad}(f)^k P$. Furthermore, by Taylor's theorem the value of $\mathrm{ad}(f)^k P$ at x depends only on $d_x f$. Thus, the following is well defined.

3.2.2. *Definition.* Let $\pi : T^*M \to M$ denote the bundle projection. If $P \in DO^k(E, F)$, the *principal symbol* $\sigma_P \in \Gamma(\pi^*(E^* \otimes F))$ is given by:

$$\sigma_P(\xi) = \frac{1}{k!}\mathrm{ad}(f)^k P$$

where f is any function satisfying $d_x f = \xi$.

3.2.3. *Remark.* The above definition involves two standard identifications - the first being $E^* \otimes F \cong \mathrm{Hom}(E, F)$ and the the second being that sections of a pullback bundle $f^*s \in \Gamma(f^*E)$ are given by composing f with section s of E: $f^*s = s \circ f$.

3.2.4. *Example.* In local trivializations, $Pu = \sum_{|\alpha|\leq k} A_\alpha(x)(D^\alpha u)$. Then $\mathrm{ad}(f)^k P = k! \sum_{|\alpha|=k} A_\alpha(x)(D^\alpha f)$ so σ_P is given at $x \in U$ by

$$\sigma_P(x,\xi) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha$$

3.2.5. *Definition.* A differential operator $P : \Gamma(E) \to \Gamma(F)$ is called *elliptic* if $\sigma_P(\xi, x) : E_x \to F_x$ is invertible for all $\xi \in T^*M \setminus M$.

3.2.6. *Theorem.* If E, F are vector bundles over a compact manifold M, $s \in \mathbb{R}$, then any elliptic differential operator $P : \Gamma(E) \to \Gamma(F)$ extends to a Fredholm operator $\tilde{P} : H^s(M,E) \to H^{s-k}(M,F)$ where $H^j(M,S)$ denotes the Sobolev space $W^{j,2}(\Gamma(S))$ and the index of $\tilde{P}$ is independent of s.

3.2.7. *Definition.* The *analytic index* of an elliptic differential operator P is defined to be $\mathrm{Index}(P) = \dim(\ker P) - \dim(\ker P^*)$ where P^* is a formal adjoint to P under the chirality element ω. By theorem 3.2.6 it is the case that $\mathrm{Index}(P) = \mathrm{Index}(\tilde{P})$.

4. The $\hat{A}$-genus.

4.1. **Characteristic classes.**

4.1.1. *Theorem. (Stiefel-Whitney classes).* Let M be a smooth manifold and $Vect(n)$ denote the isomorphism classes of real n-bundles over M. Then there exists a unique map $w : Vect(n) \to H^*(M;\mathbb{Z})$ such that:

(1) $w_i(f^*E) = f^*(w_i(E))$
(2) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$
(3) $w_i(E) = 0$ for $i > n$
(4) If $E \to \mathbb{R}P(\infty)$ is the canonical line bundle, $H^1(\mathbb{R}P(\infty),\mathbb{Z}) = \langle w_1(E)\rangle$

where $w = (w_1, ..., w_n)$ in coordinates (ie. $w_i(E) \in H^i(M;\mathbb{Z})$). We call w_i the i'th *Stiefel-Whitney class* and the map w the *total Stiefel-Whitney class.*

4.1.2. *Proposition.* An orientable Riemannian manifold M has a spin structure if and only if $w_2(TM) = 0$.

4.1.3. *Theorem (Chern classes).* Let M be a smooth manifold and $Vect(n)$ denote the isomorphism classes of complex n-bundles over M. Then there exists a unique map $c : Vect(n) \to H^{ev}(M;\mathbb{Z})$ such that:

(1) $c_i(f^*E) = f^*(c_i(E))$
(2) $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$
(3) $c_i(E) = 0$ for $i > n$
(4) If $E \to \mathbb{C}P(\infty)$ is the canonical line bundle, $H^2(\mathbb{C}P(\infty),\mathbb{Z}) = \langle c_1(E)\rangle$

where $c = (c_1, ..., c_n)$ in coordinates (ie. $c_i(E) \in H^{2i}(M;\mathbb{Z})$). We call c_i the i'th *Chern class* and the map c the *total Chern class.*

4.1.4. *Definition.* Let $E \to M$ a real vector bundle of rank n and $E^{\mathbb{C}}$ its complexification. We define the i'th *Pontryagin class* of E by $p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}})$, and the *total Potryagin class*:

$$p(E) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} p_i$$

We define the Potryagin classes of a manifold M to be $p_i(M) = p_i(TM)$.

4.1.5. *Definition.* Let $f(x)$ be a formal power series in x with rational coefficients such that $f(0) = 1$. Writing:

$$\prod_{i=1}^{n} f(x_i) = 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + \cdots$$

where:

$$\sigma_k(x_1, ..., x_n) = \sum_{i_1 < \cdots < i_k} \prod_{j=1}^{k} x_{i_j}$$

is the k'th *elementary symmetric function*, and F_k is weighted homogeneous of degree k, ie. $F_k(t\sigma_1, ..., t^k\sigma_k) = t^k F_k(\sigma_1, ..., \sigma_k)$. The sequence $(F_k) = (F_k(\sigma_1, ..., \sigma_k))$ is called the *multiplicative sequence* associated to f.

4.1.6. *Examples.*

(1) Let $f_a(x)$ be the power series given by the Taylor series of:

$$f_a(x) = \frac{\sqrt{x}}{2\sinh(\frac{\sqrt{x}}{2})}$$

ie:

$$f_a(x) = 1 - \frac{1}{24}x + \frac{7}{5760}x^2 \pm \cdots$$

The multiplicative sequence associated to f_a, denoted $(\hat{A}_i)$, is called the $\hat{A}$-sequence. The first term is given by $\hat{A}_1(\sigma_1(x_1)) = -\frac{1}{24}x_1$.

(2) Let $f_\ell(x)$ be the power series given by the Taylor series of:

$$f_\ell(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})}$$

ie.

$$f_\ell(x) = 1 + \frac{1}{3}x - \frac{1}{45}x^2 \pm \cdots$$

The multiplicative sequence in this case, denoted (L_i), is called the *Hirzebruch L-sequence*. The first term is given by $L_1(\sigma_1(x_1)) = \frac{1}{3}x_1$.

4.1.7. *Definition.* Let M be a compact, oriented n-dimension manifold and (F_k) a multiplicative sequence associated to some power series $f(x)$. We define the *total F-class* associated to f by:

$$\mathbf{F}(M) = \sum_{k=1}^{n} F_k(p_k(M))$$

and define the *F-genus* of M, denoted $F(M)$, by $\langle \mathbf{F}(M), [M] \rangle$ where $[M]$ denotes the fundamental class of M and $\langle \cdot, \cdot \rangle$ denotes the pairing between $H^n(M; \mathbb{Q})$ and $H_n(M; \mathbb{Q})$. Equivalently, $F(M)$ is 0 if n is not a multiple of 4 and otherwise:

$$F(M) = \langle F_k(p_1(M), ..., p_k(M)), [M] \rangle$$

where $n = 4k$.

4.1.8. *Definition.* Choosing $f = f_a$ from above, we have the **$\hat{A}$-genus** of M denoted $\hat{A}(M)$. If M has dimension 4 we see by the formulas for $\hat{A}_1$ and L_1 given above:

$$\hat{A}(M) = -\frac{1}{24}\langle p_1(M), [M]\rangle = -\frac{1}{8}L(M)$$

4.1.9. *Examples.*

(1) If $M = \mathbb{C}P(2)$, $\langle p_1(M), [M]\rangle = 3$ so $\hat{A}(M) = -1/8$.
(2) If $M = S^4$, $p_1(M) = 0$ so $\hat{A}(M) = 0$.

4.2. **Dirac operators.**

4.2.1. *Definition.* Let M a Riemannian manifold with Clifford bundle $C\ell(M)$ and let $S \to M$ a Euclidean vector bundle of $C\ell(M)$-modules (ie. the fibers S_x are $C\ell(M)_x$-modules) with a Riemannian connection ∇. Then the map $\Delta : \Gamma(S) \to \Gamma(S)$ given by:

$$\Delta(\sigma) = \sum_{i=1}^{n} e_i \times \nabla_{e_i}\sigma$$

is called the *Dirac operator* of S, where $\times$ denotes Clifford multiplication.

4.2.2. *Definition.* Let M be a spin manifold and S any spinor bundle associated to TM. Then S is a $C\ell(M)$-module bundle with a canonical connection induced by the Levi-Civita connection on TM. The Dirac operator in this case is called the *Atiyah-Singer operator*. If $S = \varpi_{\mathbb{C}}$ we denote this operator by $\not{D}$.

4.2.3. *Theorem (Lichnerowicz).* Let M,$\not{D}$ as in definition 4.2.2 and let κ denote the scalar curvature of M. Then:

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\kappa$$

4.2.4. *Corollary.* Let $M, \not{D}, \kappa$ as in theorem 4.3.2. If $\kappa \geq 0$ and there exists $p \in M$ with $\kappa(p) > 0$ then $\ker \not{D} = 0$.

Proof. Let $\sigma \in \ker(\not{D})$. By theorem 4.2.3 $\nabla^*\nabla\sigma + \frac{1}{4}\kappa\sigma = 0$ so $-\nabla^*\nabla\sigma = \frac{1}{4}\kappa\sigma$ hence $g(\frac{1}{4}\kappa\sigma, \sigma) = -g(\nabla^*\nabla\sigma, \sigma)$. Equivalently:

$$\frac{1}{4}\kappa||\sigma||^2 = \frac{1}{4}\kappa g(\sigma, \sigma) = -g(\nabla\sigma, \nabla\sigma) = -||\nabla\sigma||^2$$

thus $\sigma \equiv 0$.

□

5. The (cohomological) Atiyah-Singer Index theorem.

5.1. **The Atiyah-Singer Index theorem.**

5.1.1. *Theorem (Atiyah-Singer).* Let M a compact oriented n manifold and $E, F \to M$ vector bundles with an elliptic differential operator $P : \Gamma(E) \to \Gamma(F)$. Let $[\sigma_P] \in K(M)$ denote the K-theory class of the principal symbol of P. Then we have:

$$\mathrm{Index}(P) = (-1)^{\frac{n(n+1)}{2}}\langle \pi_{\mathrm{i}}\mathrm{ch}[\sigma_P]\hat{\mathbf{A}}(M)^2, [M]\rangle$$

where $\pi : TM \to M$ is the canonical projection map.

5.1.2. *Theorem.* Let M be a compact spin manifold of dimension $n = 2k$ and $\not{D}$ the Atiyah-Singer operator. Then:

$$\text{Index}(\not{D}^+) = \hat{A}(M)$$

Proof. Since $\pi_{!}\text{ch}[\sigma_{\not{D}^+}] = (-1)^k \hat{\mathbf{A}}^{-1}$, by theorem 5.1.1 we see:

$$\text{Index}(\not{D}^+) = (-1)^k \langle (-1)^k \hat{\mathbf{A}}^{-1} \hat{\mathbf{A}}^2, [M] \rangle = \langle \hat{\mathbf{A}}, [M] \rangle = \hat{A}(M)$$

□

5.1.3. *Corollary.* If M is as in the previous theorem $\hat{A}(M)$ is an integer.

5.1.4. *Corollary.* If M is a compact Riemannian spin manifold with positive scalar curvature then $\hat{A}(M) = 0$.

Proof. By theorem 5.1.2, $\hat{A}(M) = \text{Index}(\not{D}^+) = \dim(\ker(\not{D})) - \dim(\text{coker}(\not{D}^+)) = \dim(\ker(\not{D}))$. But by corollary 4.2.4 if $\kappa > 0$ then $\dim(\ker D^+) = 0$ so $\hat{A}(M)$=0. □

5.2. **Remark.** If the dimension of M is not a multiple of 4 then $\hat{A}(M) = 0$, so the previous corollary only gives a new obstruction to positive scalar curvature in the case $n = 4k$. But there are plenty of non equivalent compact spin manifolds of dimension $n = 4k$, some with $\hat{A}(M) = 0$ and some with $\hat{A}(M) \neq 0$.