Local Connectivity of the Mandelbrot Set.

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1 Introduction.

The celebrated Riemann Mapping Theorem states that every nonempty simply connected strict open subset U of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$. We call the biholomorphism $\varphi : \mathbb{D} \to U$ a Riemann mapping. Unfortunately however, it is not true in general that φ extends to a biholomorphism $\overline{\varphi} : \overline{\mathbb{D}} \to \overline{U}$ (where the bar is taken to mean the closure rather than the complex conjugate). To this end, there is another celebrated result (of Carathéodory) that states φ *is* extendable to $S^1 = \partial\mathbb{D}$ if and only if ∂U is locally connected. Then wee see that the issue of local connectedness of certain subsets of $\mathbb{C}$ is of the utmost importance.

The interior of the Mandelbrot set M has been shown to be simply connected, hence (by the Riemann Mapping theorem) is conformally equivalent to $\mathbb{D}$, but it is an open problem whether M is locally connected.

2 The Mandelbrot Set.

2.1 Definition.

For $w \in \mathbb{C}$ we define $f_w(z) = z^2 + w$. From this we define a sequence $w_1 = f_w(0)$, $w_2 = f_w(f_w(0))$, $w_n = (f_w \circ \cdots \circ f_w)(0)$ where there are n terms in the composition. We then define the *Mandelbrot set* M by:

$$M = \{w \in \mathbb{C} \;:\; |w_n| \leq C, \quad C \in \mathbb{R}\}$$

2.2 Basic Properties.

a) M is compact.

Proof. Suppose $|w| > 4$. We will show that $w \notin M$. First we claim that $|w_n| \geq 3^{n-1}|w|$. Proceeding by induction:

$$|w_1| = |w| = 3^0|w|.$$

Now assume the claim holds for $1 \leq k \leq n$. Then:

$$|w_{n+1}| = |w_n^2+w| \geq |w_n|^2-|w| \geq (3^{n-1}|w|)^2-|w| = |w|(3^{2n-2}|w|-1) \geq |w|(3^{2n-2}\cdot 4-1) \geq 3^n|w|$$

Then $M \subset 4\mathbb{D}$ hence M is bounded. Now suppose $|w| < 4$ but $|w_N| > 4$ for some N. Then:

$$|w_{N+1}| = |w_N^2 + w| \geq |w_N|^2 - |w| > 4|w_N| - 4$$

hence $\lim_{n\to\infty} |w_n| \to \infty$. Thus we see that $w \in m$ if and only if $|w_n|$ never exceeds 4 (in fact more is true, $|w_n|$ never exceeds 2, but we don't need this fact here). Let $C_n = \{z \in \mathbb{C} : |z_n| < 4\}$. Then by what was shown above we have that:

$$M = \bigcap_{n=1}^{\infty} C_n$$

Define a sequence of functions $f_1(w) = w$, $f_2(w) = w^2 + 2$, $f_n(w) = (f_{n-1}(w))^2 + w$. Then we see that:

$$C_n = f_n^{-1}([0,4])$$

so each C_n is closed. Finally we conclude that since W is the intersection of closed sets it is closed, hence M is compact.

□

b) (Douady and Hubbard) M is connected. We sketch a proof of this fact below.

Proof. Define $\varphi : \hat{\mathbb{C}} \setminus M \to \hat{\mathbb{C}}$ by:

$$\varphi(z) = \lim_{n\to\infty} (f_w^n(w))^{\frac{1}{2^n}}$$

First note by the Bottcher-Fatou lemma, φ is analytic on $\hat{\mathbb{C}} \setminus M$, hence it is an open map. Moreover φ is proper (this follows immediately from the fact that the Green functions $G_c(z)$ of the filled in Julia sets for f_c are continuous). Then since φ is proper and open it is also closed.

Now we see that φ is surjective onto $\hat{\mathbb{C}} \setminus \overline{D}$ because the interior of $\hat{\mathbb{C}} \setminus M$ gets mapped to an open subset of $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and the boundary of $\varphi(\hat{\mathbb{C}} \setminus \overline{D})$ coincides with the boundary of $\mathbb{D}$.

But φ is also injective. This follows from applying the argument principle:

$$k_z|\{\varphi^{-1}(z)\}| = \frac{1}{2\pi i}\int_\gamma \frac{\varphi'(\alpha)}{\varphi(\alpha) - z} d\alpha.$$

Choosing γ to contain all z with $|z| < M$, we see that since $\varphi^{-1}(\infty) = \infty$ and k_z is locally constant it must be that in fact k_z is identically 1 hence φ is injective.

Finally by Goursat's theorem since φ is injective analytic and open φ^{-1} is analytic. Then we have found a biholomorphism of $\hat{\mathbb{C}} \setminus M$ with $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ so $\hat{\mathbb{C}} \setminus M$ is simply connected in $\hat{\mathbb{C}}$ hence M is connected.

□

c) M is "simply connected". Simply connected is in quotation marks because simply connected is only defined on spaces that are path connected, which is currently still an open question for M. M is "simply connected" in that it is *full*.

d) (Shishikura) ∂M has Hausdorff dimension 2.

2.3 The MLC (Mandelbrot local connectivity) conjecture.

M is locally connected.

2.4 Remark.

If the MLC were proved true, the theorem of Caratheodory would give us an extension of the Riemann map $\varphi : \mathbb{D} \to \text{Int}(M)$ to S^1, giving a conformal equivalence of M with $\overline{\mathbb{D}}$. Given the fractal nature of M, this would be a very surprising result.

3 The Density of Hyperbolicity Conjecture.

3.1 Definitions.

Let $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. We say w is a *periodic point for φ with period n* if $\varphi^n(w) = w$ (if $n = 1$ this is a fixed point). In this case we call $\{w, \varphi(w), ..., \varphi^{n-1}(w)\}$ a *cycle*.

If w is a periodic point for φ with period n, the *multiplier of w* is defined to be:

$$\lambda_w = (\varphi^n)'(w).$$

If $|\lambda_w| < 1$ then we say w is *attractive*. If $|\lambda_{\varphi^j(w)}| < 1$ for all $1 \leq j \leq n$, we call $\{w, \varphi(w), ..., \varphi^{n-1}(w)\}$ an *attractive cycle*.Finally we define:

$$\mathcal{H}_n = \{w \in \mathbb{C} \mid f_w \text{ has an attractive n-cycle}\}$$

3.2 Proposition.

$$\mathcal{H}_1 = \{w = \frac{1}{4}(1 - re^{i\theta}) \ : \ r = 2(1 + \cos\theta)\}$$

Proof. If $f_w(z) = z^2 + w$ has a 1-cycle (a fixed point), $z^2 + w = z$. Then:

$$z = \frac{1 \pm \sqrt{1 - 4w}}{2}$$

Let $1 - 4w = re^{i\theta}$ ($r > 0$, $\theta \in [0, 2\pi)$). Equivalently $w = \frac{1}{4}(1 - re^{i\theta})$. Then $z = 1/2(1 \pm \sqrt{r}e^{i\theta/2})$. If z is attractive then $1 > |f'_w(z)| = |2(1/2(1 \pm \sqrt{r}e^{i\theta/2})) = 1 \pm \sqrt{r}e^{i\theta/2}$. Hence:

$$1 > |(1 + \sqrt{r}\cos(\frac{\theta}{2}) \pm i\sqrt{r}\sin(\frac{\theta}{2})|$$

ie:

$$1 > (1 \pm \sqrt{r}\cos(\frac{\theta}{2}))^2 + r\sin^2(\frac{\theta}{2}) = 1 \pm 2\sqrt{r}\cos(\frac{\theta}{2}) + r$$

hence $0 < r < \pm 2\sqrt{r}\cos(\frac{\theta}{2})$. Thus $r^2 < 4r\cos^2(\frac{\theta}{2}) = 2r(1+\cos(\theta))$ so $r < 2(1+\cos(\theta))$ as claimed.

□

3.3 Proposition.

$$\mathcal{H}_2 = \{w \in \mathbb{C} \ : \ |w+1| < \frac{1}{4}\}$$

Proof. If f_w has a 2-cycle then:

$$f_w(f_w(z)) = z^4 + 2wz^2 + w^2 + w = z$$

so:

$$z^4 + 2wz^2 + z + w^2 + w = 0$$

ie:

$$(z^2 - z + w)(z^2 + z + w + 1) = 0.$$

But $z^2 - z + w = 0$ means $f_w(z) = z$ so these form the 1-cycles, so we conclude that $z^2 + z + w + 1 = 0$. Now:

$$(f_w \circ f_w)'(z) = 4z^3 + 4wz = 4z(z^2 + w) = 4zf_w(z)$$

and using that $z^2 + z + w + 1 = 0$ we get $f_w(z) = z^2 + w = -z - 1$ hence:

$$(f_w \circ f_w)'(z) = 4z(-z-1) = -4(z^2 + z) = 4(w+1)$$

Finally if z is an attractive point:

$$1 > |(f_w \circ f_w)'(z)| = |4(w+1)|$$

so $|w+1| < 1/4$ as claimed.

□

3.4 Definition.

We say that $f_w(z)$ is *renormalizable* if there exists an open $U \subset \mathbb{C}$ containing 0 and $n \in \mathbb{Z} \setminus \{0\}$ such that one connected component V of $f_w^{-n}(U)$ is relatively compact in U, and $f_w^n|_V$ is *polynomial-like of degree* 2, ie. for all $z \in U$, $\#\{(f_w^n)|_V^{-1}(z)\} = 2$, and:

$$\{z \in V \mid (f_w^n|_V)^k(z) \in V \text{ for all } k \geq 0\}$$

is connected. In the case that $f_w^{-n}|_V$ is renormalizable, we say that f_w twice or 2 times renormalizable. Then it makes sense to define in the natural way what it means to be m-times renormalizable. If f_w is m-times renormalizable for all $m > 0$ we say that f_w is *infinitely renormalizable*, else it is *finitely renormalizable*.

These lead us to an important conjecture in Complex dynamics:

3.5 Density of Hyperbolicity Conjecture (DHC).

$$\mathcal{H} = \{w \in \mathbb{C} \ : \ f_w \text{ has an attractive cycle}\} \qquad \text{is dense in M}$$

3.6 Remark.

To this end, there are two key results. The first is by Douady and Hubbard from 1981-1982, and the second complementary theorem is due to Yoccoz (1989):

3.7 Theorem.

If M is locally connected then $\mathcal{H}$ is dense in M.

3.8 Theorem.

M is locally connected at every $w \in M$ such that f_w is finitely renormalizable.