KIRILLOV'S ORBIT METHOD

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1. Preliminaries.

1.1. **Definition.** Let G be a Lie group. Two unitary representations $\rho_1 : G \to u(\mathcal{H}_1)$, $\rho_2 : G \to u(\mathcal{H}_2)$ are **unitary equivalent** if there exists a unitary isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that $T \circ \rho_1(g) = \rho_2(g) \circ T$ for all $g \in G$. That this is an equivalence relation allows us to define the **unitary dual** $\widehat{G}$ to be the set of equivalence classes of irreducible unitary representations of G.

1.2. **Remark.**

(1) If G is a Lie group and $\pi : G \to U(\mathcal{H})$ is an irreducible unitary representation then $\mathcal{H}$ is necessarily separable.
(2) If G is a Lie group then $L^2(G)$ is separable (the measure on G being Haar measure).
(3) If $\mathcal{H}$ is separable with an at most countable orthonormal basis β, then $\mathcal{H} \cong \ell^2(\beta)$ as Hilbert spaces so any two separable Hilbert spaces of the same dimension are (unitarily) isomorphic.

From the above we see that if $\pi : G \to U(\mathcal{H})$ is an irreducible unitary representation of the Lie group G there is no loss in generality in assuming $\mathcal{H} \subseteq L^2(\mathbb{R})$.

Let G be a nilpotent Lie group and let $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ be the coadjoint representation. For $F \in \mathfrak{g}^*$ let $\mathcal{O}_F$ denote the orbit of F under the coadjoint action and let $\mathfrak{g}^*/G$ be the space of orbits endowed with the quotient topology.

1.3. **Kirillov's User Guide.** Let G be a nilpotent Lie group. We highlight some important pieces of Kirillov's *orbit method* - a tool for identifying elements of $\widehat{G}$ with coadjoint orbits.

(a) $\mathfrak{g}^*/G \cong \widehat{G}$ as topological spaces (the topology on $\widehat{G}$ under consideration here will be a black box).
(b) The character of π_F is given by:

$$\langle \chi, \varphi \rangle = \int_F \varphi^* F e^{\omega}$$

(c) For $A \in Z(\mathfrak{g})$ (the center of $U(\mathfrak{g})$), let $P_A \in \mathrm{Sym}(\mathfrak{g}^*)$ be the G-invariant polynomial on $\mathfrak{g}^*$ corresponding to A under the Harish-Chandra homomorphism. Then for $\mathcal{O}_F \in \mathfrak{g}^*/G$, the infinitesimal character of π_F on A is given by $P_A(F)$.
(d) For $\mathcal{O}_F \in \mathfrak{g}^*/G$, π_F and π_{-F} are dual representations.
(e) The Plancherel measure μ on $\widehat{G}$ is given by the decompostion of Lebesgue measure on $\mathfrak{g}^*$ canonically on the coadjoint orbits.

1.4. **The Correspondence.** If G is a Lie group, $H \leq G$ a closed subgroup and $\rho : H \to U(1)$ a 1-dimension unitary representation of H then there exists an irreducible unitary representation ρ' of G called the **induced representation of** ρ. To construct ρ' we let $M = H/G$ denote the coset space, let $L = |\Lambda_M|^{\frac{1}{2}}$ be the $\frac{1}{2}$-density line bundle over M and define $\mathcal{H} = L^2(|\Lambda_M|^{\frac{1}{2}})$ (the L^2-sections of L). Next we let $(U, (x_1, ..., x_n))$ be a coordinate chart covering a sufficient amount of M (sufficient for the purpose of integration) and let $s : U \to G$ be a section of $G \to M$ so we may identify $G \cong U \times H$ and define a function $h_s : U \times G \to H$ by the *master equation*:

$$s(m)g = h_s(m, g)s(y) \quad (*)$$

where $y = mg$. Since $m \in M = H/G$, $m = Hg_1$ and so $y = mg = Hg_1 g$. For Haar measures μ_G, μ_H on G, H, in the identification $G \cong U \times H$ we have:

$$d\mu_G(x,h) = r(x) d\mu_H(h) dx(x)$$

for some $r \in C^\infty(U)$. With this we define a measure $\mu_s = r(x)\Delta_G(s(x))dx(x)$ on U where Δ is the Haar modulus. In U, a $\frac{1}{2}$-density ω can be written $\omega = f(x)\sqrt{dx_1 \wedge \cdots \wedge dx_n}$ for some $f \in L^2(U, \mu_s)$ giving an identification $\mathcal{H} \cong L^2(U, \mu_s)$ and so we (finally) define the representation $\rho' : G \to L^2(U, d\mu_s)$ by:

$$[\rho'(g)f](m) = \left(\frac{\Delta_H(h_s(m,g))}{\Delta_G(h_s(m,g))}\right) \rho(h_s(m,g)) f(mg)$$

In particular if G is unimodular we have:

$$[\rho'(g)f](m) = \rho(h_s(x,g)) f(mg)$$

To give the correspondence between $\mathfrak{g}^*/G$ and $\widehat{G}$ let $\Omega_F \subseteq \mathfrak{g}^*$ be a coadjoint orbit and let $\mathfrak{h}_F \subseteq \mathfrak{g}$ be a maximal subalgebra satisfying $F|_{[\mathfrak{h},\mathfrak{h}]} = 0$. We define a 1-dimensional unitary representation $\pi_{F,H} : \mathfrak{h} \to \mathfrak{u}(1)$ by $\pi_{F,H}(X) = \langle F, X \rangle$ (that this is a representation of $\mathfrak{h}$ is equivalent to the condition $F|_{[\mathfrak{h},\mathfrak{h}]} = 0$) and exponentiate to get a 1-dimensional unitary representation $\rho_{F,H}(\exp(X)) = e^{2\pi i \langle F, X \rangle}$ (by a theorem of Chevalley there exists a unique closed connected $\iota : H \to G$ such that $d\iota_e(H) = \mathfrak{h}$ and moreover H is generated by $\exp(\mathfrak{h})$). With this in hand the correspondence $\mathfrak{g}^*/G \cong \widehat{G}$ is given by:

$$\Omega_F \mapsto \rho'_{F,H}$$

2. A worked example.

We consider the Heisenberg group H given by:

$$H = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \ x, y, z \in \mathbb{R} \}$$

with Lie algebra:

$$\mathfrak{h} = \{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \}$$

having as a basis:

$$\{ X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \}$$

and bracket relations:

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0$$

We also note here that:

$$Z(H) = \exp(zZ) = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H/Z(H) \cong \{ \begin{pmatrix} 1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \ x, y \in \mathbb{R} \}$$

and since:

$$\begin{pmatrix} 1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+a & * \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{pmatrix}$$

we see $H/Z(H) \cong \mathbb{R}^2$.

2.1. **Representations of H..** Let $\pi : H \to U(L^2(\mathbb{R}))$ be a complex irreducible unitary representation. For any complex irreducible representation it is a consequence of Schur's lemma that elements of the center act by scalar multiplication so $\pi(\exp(zZ)) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by $f \mapsto e^{2\pi i z\lambda} f$ for some $\lambda \in \mathbb{R}$.

(1) If $\lambda \neq 0$ it is a consequence of the Stone-von Neumann theorem that there exists a unique representation ρ_λ on $L^2(\mathbb{R})$ given by:

$$\rho_\lambda(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix})(f)(s) = e^{2\pi i(sy+\lambda z)} f(s+\lambda x)$$

that is unitarily equivalent to π.

(2) If $\lambda = 0$ then $\pi|_{Z(H)}$ is the trivial representation so π is equivalent to the induced representation $\tilde{\pi}$ on $H/Z(H) \cong \mathbb{R}^2$. Since $H/Z(H)$ is abelian we get (by Schur's lemma) that $\tilde{\pi}$ is 1-dimensional so there exists (α, β) such that:

$$\pi(\begin{pmatrix} 1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}) = e^{2\pi i x\alpha} e^{2\pi i y\beta} = e^{2\pi i(x\alpha + y\beta)}$$

In conclusion we see there are two families of unitary equivalence classes of representations of H - the 1-dimensional representations $\rho_{\alpha,\beta}$ each corresponding to a pair $(\alpha, \beta) \in \mathbb{R}^2$, and the infinite dimensional representations ρ_λ corresponding to some $\lambda \neq 0$ so:

$$\widehat{H} \cong \mathbb{R}^2 \sqcup (\mathbb{R} \setminus \{0\})$$

for an appropriate choice of $\cong$.

2.2. **Orbits.** We recall the explicit determination of the coadjoint action of H from lecture. Let $\{X^*, Y^*, Z^*\}$ denote the dual basis to $\{X, Y, Z\}$. There is an identification of $\mathfrak{h}^*$ with matrices of the form:

$$\begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}$$

via the map $p : \mathfrak{g}^* \to \mathfrak{gl}(3, \mathbb{R})$:

$$X^* \mapsto \begin{pmatrix} * & * & * \\ 1 & * & * \\ 0 & 0 & * \end{pmatrix}, \quad Y^* \mapsto \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 1 & * \end{pmatrix}, \quad Z^* \mapsto \begin{pmatrix} * & * & * \\ 0 & * & * \\ 1 & 0 & * \end{pmatrix}$$

using:

$$\langle F, X \rangle = \mathrm{Tr}(\rho(F)X) \quad (*)$$

since:

$$\mathrm{Tr}(\begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}) = ax + by + cz$$

We use stars rather than zeroes in the matrix entries as this is really the quotient $\mathfrak{gl}(3,\mathbb{R})/U$ where U are the upper triangular matrices. Checking the coadjoint action using $(*)$ we see:

$$\begin{aligned}\langle Ad_g^*(F), X\rangle = \langle F, Ad_{g^{-1}}X\rangle = \langle F, g^{-1}Xg\rangle &= \mathrm{Tr}(\rho(F)g^{-1}Xg) \\ &= \mathrm{Tr}(Xg\rho(F)g^{-1}) \\ &= \mathrm{Tr}(g\rho(F)g^{-1}X) \\ &= \mathrm{Tr}(\mathrm{Ad}_g(\rho(F))X)\end{aligned}$$

so under the given identification we see that the coadjoint action becomes the adjoint action on matrices. So we compute the action:

$$\begin{aligned}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}^{-1} &= \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}\begin{pmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} * & * & * \\ x+bz & * & * \\ z & y-az & * \end{pmatrix}\end{aligned}$$

so in these coordinates the orbit of $F = (x, y, z) \in \mathfrak{g}^*$ is given by:

$$\Omega_F = \{(x + bz, y - az, z),\ a, b \in \mathbb{R}\}$$

If $z \neq 0$ then Ω_F is the affine plane at height z, and if $z = 0$ then Ω_F is the single point $(x, y, 0)$. In conclusion we see there are two families of orbits - the affine planes corresponding to $z \neq 0$, and the points $(x, y, 0)$ corresponding to a pair $(x, y) \in \mathbb{R}^2$ so:

$$\mathfrak{g}/G \cong \mathbb{R}^2 \sqcup (\mathbb{R} \setminus \{0\})$$

as topological spaces.

2.3. **The correspondence.** Now let $\Omega_F \subset \mathfrak{g}^*$ be a 2-dimensional orbit and first consider the case $F = (0, 0, 1)$. Since $\langle F, Z\rangle = 1$ and $[X, Y] = Z$ we choose $\mathfrak{h}_F = \mathrm{span}\{Y, Z\}$. Then since $M = H/H_F \cong \mathbb{R}$ we let $s : M \to H$ be the section $s(x) = \exp(xX)$ allowing us to identify $H \cong M \times H = H/H_F \times H_F$. Then we compute h_s from $(*)$:

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

so $\beta = b$, $\gamma = c + bx$ and $y = x + a$. Since H is unimodular we then have:

$$\begin{aligned}[\rho'_{F,H}(g)f](x) = \rho_{F,H}(h_s(x, g))f(y) = e^{2\pi i\langle F,(c+bx)Z\rangle}f(x + a) &= e^{2\pi i(c+bx)}f(x + a) \\ &= \rho_1\end{aligned}$$

from 2.1 choosing $\lambda = 1$. The case of arbitrary (x, y, z) with $z \neq 0$ follows from an investigation of the dependence of the induced representation on the choice of coset representative of Ω_F and subalgebra $\mathfrak{h}_F$.

Now let Ω_F be the 0-dimension orbit of $F = (x, y, 0)$. Then $\mathfrak{h}_F = \mathfrak{g}$ so the induced representation of $\rho_{F,H}$ on G is simply $\rho_{F,H}$ itself (there is nothing to "induce"). More specifically, $\rho'_{F,H} = \rho_{F,H}$ is the 1-dimensional representation on G:

$$\begin{aligned}\rho'_{F,H}(\exp(v)) = e^{2\pi i \langle F,v \rangle} &= e^{2\pi i} = e^{2\pi i (x v_1 + y v_2)} \\ &= \exp^{2\pi i x v_1} \exp^{2\pi i y v_2} \\ &= \rho_{x,y}\end{aligned}$$

where $v = (v_1, v_2, v_3) \in \mathfrak{g}$.

References

[1] Alexandre Kirillov, *Lectures on the Orbit Method*. American Mathematical Society, 2004.