RICCI FLOW

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1. Introduction.

1.1. Riemannian Manifolds. Let (M, g) be a Riemannian manifold of dimension n. Since g is a symmetric $(0, 2)$ tensor (ie. g is a smooth section of $\mathrm{Sym}^2_+ T^*M$ over M), we can write it in a coordinate chart (U, x) as:

$$g = g_{ij}(x_1, ..., x_n)dx^i dx^j$$

where i) here (and everywhere) we are using Einstein's summation convention and ii) for brevity we drop the $\otimes$ when it is not necessary. Adding a dependence on a new parameter t (often referred to as 'time') to our metric g, we get a family of Riemannian metrics g_t on M that we can write in local coordinates as:

$$g_t = g_{ij}(x_1, ..., x_n, t)dx^i dx^j$$

ie. g is a smooth section of $\mathrm{Sym}^2_+ T^*M$ seen as a vector bundle over $M \times [0, \infty)$ now as opposed to M. Each metric g_t on M gives rise to a preferred torsion free connection ∇ that is compatible with the metric (the *Levi-Civita* connection) which is then used to define the $(1, 3)$ curvature tensor:

$$R_t(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

From this we can define the *Ricci curvature tensor.* For $p \in M$ let $\{e_1, ..., e_n\}$ be an orthonormal (with respect to g_t) basis of T_pM, and define the (symmetric $(0, 2)$) Ricci tensor by:

$$\mathrm{Ric}^t_p(v, w) = \sum_{i=1}^{n} g_t(R_t(e_i, w)v, e_i)$$

and we say that (M, g) is *Einstein* with Einstein constant λ when $\mathrm{Ric}_p(v, w) = \lambda g(v, w)$ for all $p \in M$, $v, w \in T_pM$.

1.2. Vector Calculus on Riemannian Manifolds. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient of f, denoted ∇f, is often defined to be the vector field whose components are the first partial derivatives of f:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Because general smooth manifolds do not possess global coordinate frames (such as $\frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$) this particular definition does not lend itself well to a coordinate free generalization on M. However, with a simple change of perspective we note that:

$$Df = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) = (\nabla f)^t$$

thus $\langle \nabla f, v\rangle = Df(v)$. We see then that we could define ∇f to be the vector field on $\mathbb{R}^n$ such that $\langle \nabla f, v\rangle = Df(v)$ for all v. This definition is coordinate independent although it involves a choice of inner product. The setting of Riemannian geometry is then sufficient to define the gradient of $f \in C^\infty(M)$.

1.2.1. *Definition.* If (M, g) is a Riemannian manifold and $f \in C^\infty(M)$ we define the *gradient* of f to be the vector field $\nabla f \in \Gamma(TM)$ such that $g(\nabla f, v) = df(v)$.

The next step after defining the gradient of a smooth function is to then look at second derivatives - the Hessian. As was the case with the gradient, the classical $\mathbb{R}^n$ definition of the Hessian as "the matrix of second order partial derivatives" is not well suited to generalization to smooth manifolds. But notice that 1) for any smooth (in fact C^2 is sufficient) function $f \in C^\infty(\mathbb{R}^n)$, the second order partials commute so classical $\mathbb{R}^n$ Hessian matrix is symmetri and 2) each entry (c_{ij}) of the Hessian matrix corresponds to the choice of a pair of partial derivatives - $c_{ij} = \frac{\partial f^2}{\partial x_i \partial x_j}$. But a symmetric matrix whose (i, j) entry corresponds to the choice of some pair of basis vectors is exactly a symmetric bilinear form. Then to define the Hessian of $f \in C^\infty(M)$ we ought to find the relevant symmetric bilinear form.

1.2.2. *Definition.* If (M, g) is a Riemanniain manifold and $f \in C^\infty(M)$ we define the *Hessian* of f to be the symmetric $(0, 2)$ tensor:

$$\text{Hess} f(X, Y) = \frac{1}{2} L_{\nabla f} g(X, Y)$$

A quick computation shows that this agrees with our usual Hessian on $\mathbb{R}^n$:

$$\begin{aligned}
L_{\nabla f} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= \nabla f g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}) - g([\nabla f, \frac{\partial}{\partial x_i}], \frac{\partial}{\partial x_j}) - g(\frac{\partial}{\partial x_i}, [\nabla f, \frac{\partial}{\partial x_j}]) \\
&= -g(\nabla f \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} \nabla f, \frac{\partial}{\partial x_j}) - g(\frac{\partial}{\partial x_i}, \nabla f \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \nabla f) \\
&= g(\frac{\partial}{\partial x_i} \nabla f, \frac{\partial}{\partial x_j}) + g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \nabla f) \\
&= \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial^2 f}{\partial x_j \partial x_i} \\
&= 2 \frac{\partial^2 f}{\partial x_i \partial x_j}
\end{aligned}$$

The usual Laplacian of a smooth function on $\mathbb{R}^n$ is given by the trace of the Hessian matrix, so taking the trace of the symmetric Hessian tensor on M gives us a way to define the Laplacian on $C^\infty(M)$.

Our next step is to extend these concepts to the metric itself. What should the gradient of the Riemannian metric be? We defined the gradient of a smooth function by

$$g(\nabla f, X) = df(X) = X(f)$$

In principle this equation can also help us generalize the gradient to g. Plugging g in for f and letting $X, Y, Z \in \mathcal{X}(M)$:

$$\langle \nabla g(Y, Z), X\rangle = dg(Y, Z)(X) = X(g(Y, Z))$$

But recall that the Levi-Civita connection for g satisfies:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

then we see that the Levi-Civita connection ∇ for g already itself acts as a gradient of the metric. What about the Hessian of g? We defined the Hessian of a function f (ie, a $(0,0)$ tensor) as the $(0,2)$ tensor:

$$\mathrm{Hess} f(X,Y) = g(\nabla_X \nabla f, Y) = (\nabla_X \nabla f)(Y) = [\nabla_X, \nabla f](Y)$$

.

So it seems to make sense to define Hess$g$ to be a $(0,4)$ tensor by combining these ideas:

$$\begin{aligned}
\mathrm{Hess} g(X,Y,Z,W) &= (\nabla_X \nabla g)(Y,Z,W) \\
&= [\nabla_X, \nabla g](Y,Z,W) \\
&= \nabla_X(\nabla g(Y,Z,W)) - \nabla g(\nabla_X Y, Z, W) - \nabla g(Y, \nabla_X Y, W) - \nabla g(Y,Z,\nabla_X W) \\
&= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)
\end{aligned}$$

This is simply the $(0,4)$ version of the curvature tensor for (M,g).

Finally, we come to the Laplacian of the metric. Since the Laplacian of $f : \mathbb{R}^n \to \mathbb{R}$ is the trace of Hess f, the Laplacian of the metric ought to be the trace of the curvature tensor. But we have already defined such a notion - the Ricci curvature.

1.3. **The Ricci Flow.** Now that we have suitable generalizations of the gradient, the Hessian, and the Laplacian to Riemannian manifolds, we can now make sense of classical PDE's. For $u \in C^\infty(\mathbb{R}^n \times [0,\infty))$, written $u(x,t)$ for $x \in \mathbb{R}^n$ and $t \in [0,\infty)$ the heat equation is the PDE:

$$\frac{\partial}{\partial t} u + \Delta u = 0$$

where Δu denotes the Laplacian of u. If g_t is a one parameter family of Riemannian metrics parametrized by t, what should a heat equation for g_t look like? We discussed above that the Ricci curvature is the right candidate to act as the Laplacian of a the metric, so the heat equation for g_t ought to have the form:

$$\frac{\partial}{\partial t} g_t + \mathrm{Ric}^t = 0$$

ie.

$$\frac{\partial}{\partial t} g_t = -\mathrm{Ric}^t$$

which leads to our main definition.

1.3.1. *Definition.* Given a Riemannian manifold (M,g), the *Ricci flow* is the PDE:

$$\frac{d}{dt} g_t = -2\mathrm{Ric}^t$$

with initial condition $g_0 = g$. The -2 coefficient is mainly there for conventional and historical purposes, any negative number would suffice (positive numbers too, but the time interval would become negative).

1.3.2. *Examples.*

(1) Let $(M,g) = (S^n, s_n^2)$ with the induced metric as a submanifold of $\mathbb{R}^{n+1}$. In tensor notation we can write the Euclidean metric on $\mathbb{R}^{n+1}$ as a warped product $g = dr^2 + r^2 ds_n^2$. If we add a time dependence $f^2(t)$ to the metric on S^n (which is constant so this is essentially a separation of variables) this becomes $g = dr^2 + r^2 f^2(t) ds_n^2$, and after a brief computation we conclude that for a fixed t, $(S^n, g_t) = (S^n, f^2(t)s_n^2)$ has constant sectional curvature equal to $1/f^2(t)$ and thus $\mathrm{Ric}^t = \frac{n-1}{f^2(t)} g_t = (n-1)s_n^2$. The Ricci flow equation

$$\frac{d}{dt} f^2(t) s_n^2 = -2\mathrm{Ric}^t = -2(n-1)s_n^2.$$

ie. $\frac{d}{dt} f^2(t) = -2(n-1)$. But this is a separable ODE and is easily solvable:

$$\begin{aligned} 2\frac{df}{dt}(t)f(t) &= -2(n-1) \\ 2df(t)f(t) &= -2(n-1)dt \\ f^2(t) &= -2(n-1)t + C \end{aligned}$$

and since we would like $g_0 = s_n^2$ we see that $f^2(0) = 1$ so $f(t) = \sqrt{1-2(n-1)t}$. In conclusion we see that the solution to the Ricci flow PDE on S^n with initial condition $g_0 = s_n^2$ is given by:

$$(S^n, (1-2(n-1)t)s_n^2).$$

But the induced metric on rS^n (spheres of radius r) from R^{n+1} is given by $r^2 s_n^2$ so we see that the Ricci flow on S^n has given us spheres of radius $\sqrt{1-2(n-1)t}$, and that they degenerate to a point when $t = 1/2(n-1)$.

Inspection of the previous example reveals that the solution depended entirely on the fact that S^n is Einstein. Then we can generalize this to say that if (M,g) is Einstein with Einstein constant λ then a solution to the Ricci flow is given by $g_t = (1-2\lambda t)g$. By tensorality of the Ricci curvature we can further generalize this to the situation that $\mathrm{Ric} = f \cdot g$ for $f \in C^\infty(M)$ as $g_t = (1-2ft)g$ satisfies $\frac{d}{dt} g_t = -2fg = -2\mathrm{Ric}$.

(2) Let (M,g) be a Riemannian manifold, $\varphi_t : M \to M$ a one parameter family of diffeomorphisms and $c : \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$ a smooth map with $c(0) = 1$. A Ricci flow of the form $g_t = c(t)\varphi_t^* g_{t_0}$ is called a *Ricci soliton*. If $c'(0) > 0$ it is called *expanding*, if $c'(0) < 1$ it is called *shrinking* and if $c'(0) = 0$ it is called *static* or *steady*.

(3) Let $M = \mathbb{R}^2$ and $g = \frac{dx^2+dy^2}{1+x^2+y^2} = k(x,y)^{-1}(dx^2+dy^2)$. Since $\mathbb{R}^2$ has global coordinate fields ∂_x, ∂_y we can compute the Ricci tensor globally in these coordinates using the formula (written with Einstein convention):

$$\mathrm{Ric}_{ij} = \frac{\partial}{\partial x_k}\Gamma_{ij}^k + \Gamma_{km}^k \Gamma_{ij}^m - \frac{\partial}{\partial x_j}\Gamma_{ik}^k - \Gamma_{jm}^k \Gamma_{ik}^m \qquad (*)$$

A brief computation gives $\mathrm{Ric} = \frac{dx^2+dy^2}{(1+x^2+y^2)^2} = k(x,y)^{-2}(dx^2+dy^2)$. Noting that a function f of the form $f(x,y,t) = (k(x,y)+\lambda(t))^{-1}$ would satisfy $\frac{d}{dt} f = -\lambda'(t)(k(x,y) + \lambda(t))^{-2}$ we try to find such a solution to the Ricci flow. Such λ must satisfy $\lambda(0) = 0$ and $\lambda'(0) = 1$. Moreover, using $(*)$ again we see that $g_t = (k(x,y)+\lambda(t))^{-1}(dx^2+dy^2)$ has Ricci curvature $\mathrm{Ric}_t = 2(\lambda(t)+1)(k(x,y)+\lambda(t))^{-2}$. Since we need $-\lambda'(t)(k(x,y)+\lambda(t))^{-2} = \frac{d}{dt} f = -2\mathrm{Ric}_t = -4(\lambda(t)+1)(k(x,y)+\lambda(t))^{-2}$ we have reduced the problem to the ODE $\lambda' = 4(\lambda+1)$ with initial condition $\lambda(0) = 0$, which is easily solvable by $\lambda(t) = e^{4t}$. Thus a solution to the Ricci flow on (M,g) is given by:

$$g_t = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$$

Let $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi_t(x, y) = (e^{-2t}x, e^{-2t}y) = e^{-2t}(x, y)$. Then:

$$D_{(x,y)} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

so φ_t is a diffeomorphism. Moreover we see $\varphi_t^* g(e_i, e_i) = \frac{e^{-4t}}{1+x^2e^{-4t}+y^2e^{-4t}} = \frac{1}{e^{4t}+x^2+y^2} = g_t(e_i, e_i)$ and $\varphi_t^* g(e_1, e_2) = 0$ so $g_t = \varphi_t^* g$. Thus (M, g) is a static Ricci soliton.

The most important application of the Ricci flow thus far has been its use in proving Thurston's *geometrization conjecture*. By the 1960's it had been shown that every smooth, compact, orientable 3-manifold has a unique decomposition into a connected sum of *prime* 3-manifolds, where prime means that it cannot be written as a nontrivial connected sum. This reduced the classification of 3-manifolds to the prime 3-manifolds. Thurston conjectured that every prime, closed 3-manifold could be decomposed into a different type of connected sum (where the gluing is along T^2 rather than S^2) with finitely many pieces, each piece having the geometric structure of one of the eight *Thurston geometries*. He then proved [4] the conjecture in the case that M is a *Haken* manifold, (something we will not define here).

Shortly thereafter, Richard Hamilton introduced [1] the Ricci flow. As we saw in example (1) above, the Ricci flow on S^n degenerates it to a point in finite time. Given that the flow acts locally, any piece of a manifold that looks like S^n (in the sense of Riemannian geometry) will degenerate to a point in finite time. But the first goal was to decompose 3-manifolds into pieces glued together by spheres! Collapsing pieces that look like S^2 to a point is progress - what appears on 'either side' of the collapsed S^2 ought to be the prime pieces we desire.

At this point, we need to further decompose the prime pieces into the Thurston geometries that are glued together along T^2. One could see this process as dual to the previous one. Since the gluing is expected to happen along T^2 and $T^2 \times I$ is flat, running the Ricci flow near regions that resemble $T^2 \times I$ will appear to have no effect. However, the pieces that remain end up expanding so if we divide the flow by a time dependent scaling factor to keep the total volume constant, the parts that look like $T^2 \times I$ collapse and the Thurston geometries are all that remain.

2. Existence.

In order for this program to be successful, we first need to know that solutions to the Ricci flow actually exist (in general). To do this, we first need to be more precise about PDE's on manifolds.

2.1. **PDE's/PDO's on Manifolds.**

2.1.1. *Definition.* Let E, F be smooth vector bundles over M of ranks m, n (respectively). We say that $P : \Gamma(E) \to \Gamma(F)$ is a *differential operator of order* k and write $P \in DO^k(E, F)$ if

(1) For all $u \in \Gamma(E)$, $\text{supp}Pu \subset \text{supp}u$.
(2) For any $U \subset M$ and trivializations $E|_U$, $F|_U$, P is given by:

$$(Pu)(x) = \sum_{|\alpha| \leq k} A_\alpha(x)(D^\alpha u)(x)$$

where the sum is taken over multi indices α and $A_\alpha(x) : E|_U \to F|_U$ is an $n \times m$ matrix valued function of x.

Given $P \in DO^k(E, F)$ and $f_1, ..., f_k \in C^\infty(M)$, it is a consequence of polarization that $\text{ad}(f_1) \cdots \text{ad}(f_k)P \in \Gamma(E^* \otimes F)$ is symmetric in the f_i so $\text{ad}(f_1) \cdots \text{ad}(f_k)P = \text{ad}(f)^k P$. Furthermore, by Taylor's theorem the value of $\text{ad}(f)^k P$ at x depends only on $d_x f$. Thus, the following is well defined.

2.1.2. *Definition.* Let $\pi : T^*M \to M$ denote the bundle projection. If $P \in DO^k(E,F)$, the *principal symbol* $\sigma_P \in \Gamma(\pi^*(E^* \otimes F))$ is given by:

$$\sigma_P(\xi) = \frac{1}{k!}\mathrm{ad}(f)^k P$$

where f is any function satisfying $d_x f = \xi$.

2.1.3. *Remark.* The above definition involves two standard identifications - the first being $E^* \otimes F \cong \mathrm{Hom}(E,F)$ and the the second being that sections of a pullback bundle $f^*s \in \Gamma(f^*E)$ are given by composing f with section s of E: $f^*s = s \circ f$.

2.1.4. *Example.* In local trivializations, $Pu = \sum_{|\alpha| \leq k} A_\alpha(x)(D^\alpha u)$. Then $\mathrm{ad}(f)^k P = k! \sum_{|\alpha|=k} A_\alpha(x)(D^\alpha f)$ so σ_P is given at $x \in U$ by

$$\sigma_P(x,\xi) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha$$

2.1.5. *Definition.* Let (M,g) a Riemannian manifold and E a vector bundle over M with the induced bundle metric g_E on E and $g_\wedge$ on T^*M. Then $P \in DO^{(k)}(E,E)$ is said to be *strongly parabolic* if there exists $\delta > 0$ such that for all coordinate charts $U \subset M$, $x \in U$, $\xi \in T^*_x U \setminus 0$, and $v \in E_x$:

$$g_E(\sigma_P(x,\xi)v,v) > \delta|\xi|^2_\wedge |v|^2_E$$

2.1.6. *Example.* Let (M,g) be $\mathbb{R}^2$ with the canonical metric and E be the trivial rank 1 bundle so $\Gamma(E) \cong C^\infty(\mathbb{R}^2)$. Let $P : \Gamma(E) \to \Gamma(E)$ be the differential operator $P(u) = \partial_x^2 u + \partial_y^2 u$. In this case P is of the form from example 2.1.4, with $A_\alpha = (0)$ except when $\alpha = (2,0)$ and $\alpha = (0,2)$ in which case $A_\alpha = (1)$. Then $\sigma_P : T^*M \to \mathrm{Hom}(\mathbb{R},\mathbb{R})$ is given by:

$$\sigma_P((x,y),(\xi_1,\xi_2)) = (\xi_1^2 + \xi_2^2)$$

Choosing $\delta = \frac{1}{2}$ we see that for $(x,y) \in M$, $\xi \in T^*_{(x,y)}M$, $v \in E_{(x,y)} \cong \mathbb{R}$:

$$g_E(\sigma_P(x,\xi)v,v) = (\xi_1^2 + \xi_2^2) g_E(v,v) = |\xi|^2_\wedge |v|^2_E > \delta|\xi|^2_\wedge |v|^2_E$$

so P is strongly parabolic.

2.1.7. *Definition.* Let $L : \Gamma(E) \to \Gamma(E)$ be a smooth map. If there exists a linear transformation $D[L] : \Gamma(E) \to \Gamma(E)$ satisfying:

$$D[L](v) = \frac{d}{dt} L(\gamma(t))|_{t=0}$$

for all $v \in \Gamma(E)$ and $\gamma : (-1,1) \to \Gamma(E)$ such that $\gamma'(0) = v$ we call $D[L]$ the *linearization of* L. It is a fact that if $D[L]$ exists then $D[L] \in DO^{(k)}(E,E)$ for some k. We say that L is *strongly parabolic* when $D[L] \in DO^{(k)}$ is strongly parabolic.

2.1.8. *Theorem.* Let E be a vector bundle bundle over $M \times [0,\infty)$. If $L : \Gamma(E|_{M\times\{0\}}) \to \Gamma(E|_{M\times\{0\}})$ is strongly parabolic then for all $u_0 \in \Gamma(E|_{M\times\{0\}})$ there exists $T > 0$ and unique $u \in \Gamma(E_{M\times[0,T)})$ such that:

$$Pu = \frac{\partial u}{\partial t}$$

and $u(x,0) = u_0(x)$.

2.1.9. *Example.* Since $P = \partial_x^2 + \partial_y^2$ is strongly parabolic, we see that for any $g : \mathbb{R}^2 \to \mathbb{R}$ we have $T > 0$ and a unique $u : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$ such that:

$$u_t = P(u) = \partial_x^2 u + \partial_y^2 u$$

and $u(x, y, 0) = g(x, y)$. But $P(u) = \nabla u$ hence $u_t - \nabla u = 0$. This is a familiar result - existence of solutions to the heat equation given a boundary condition.

2.2. **The Ricci Flow as a Parabolic Differential Operator.** Given theorem 2.1.8, we would like for the Ricci flow to be given by a parabolic differential operator on (M, g). As defined, this is unfortunately not the case. To see this (and then remedy it) we need to first write the Ricci flow as the appropriate differential operator.

For a Riemannian metric g, $\mathrm{Ric}(g)$ is a symmetric $(0, 2)$ tensor. But the metric g itself is a nondegenerate symmetric $(0, 2)$ tensor, so we can view the Ricci curvature as a map:

$$\mathrm{Ric} : \Gamma(\mathrm{Sym}_+^2(T^*M)) \to \Gamma(\mathrm{Sym}^2(T^*M))$$

between those spaces. One can check in local coordinates that $\mathrm{Ric} \notin DO^{(k)}$ (it is very highly nonlinear) so we need to take the linearization. Let $\gamma : [0, 1] \to \Gamma(\mathrm{Sym}^2(T^*M))$ be such that $\gamma(0) = g$ and write $\gamma'(0) = h = h_{ij} dx^i \otimes dx^j$. It is not hard to verify that:

$$(D[\mathrm{Ric}](h))_{jk} = \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp})$$

with principal symbol:

$$\sigma_{\mathrm{Ric}}(\xi)(h)_{jk} = \frac{1}{2} g^{qp} (\xi_q \xi_j h_{kp} + \xi_q \xi_k h_{jp} - \xi_q \xi_p h_{jk} - \xi_j \xi_k h_{qp})$$

But choosing any $h \in \Gamma(\mathrm{Sym}^2(T^*M))$ with $h_{ij} = \xi_i \xi_j$ for all $1 \leq i, j \leq n$ we see that $\sigma_{\mathrm{Ric}}(\xi)(h) = 0$ so:

$$g_E(\sigma_{\mathrm{Ric}}(\xi)(h), h) = 0 \leq \delta |\xi|_\wedge^2 |h|_{g_E}^2$$

for any $\delta > 0$ hence Ric is not parabolic. To remedy this, we employ the following trick of DeTurck introduced in [6].

2.3. **The DeTurck Trick.** First, a useful proposition.

2.3.1. *Proposition.* The difference of two connections is a tensor.

Proof. Checking on smooth functions:

$$\nabla_X^1 fY - \nabla_X^2 fY = f\nabla_X^1 Y + X(f)Y - f\nabla_X^2 Y - X(f)Y = f\nabla_X^1 Y - f\nabla_X^2 Y = f(\nabla_X^1 Y - \nabla_X^2 Y)$$

and since connections are already tensorial in the lower argument we conclude $\nabla^1 - \nabla^2$ is a tensor. □

Now let $\widetilde{\nabla}$ be an arbitrary connection on M with Christoffel symbols $\widetilde{\Gamma}_{pq}^j$ in some local coordinate U. Then $X_i = -g^{pq} g_{ij} (\Gamma_{pq}^j - \widetilde{\Gamma}_{pq}^j)$ (*) is the local expression (in U) of a $(1, 0)$ tensor (ie. a vector field) X on M. This leads to our next definition.

2.3.2. *Definition.* Consider the vector bundle $E = \mathrm{Sym}^2(T^*M)$ over $M \times [0, \infty)$ where M is compact. The *Ricci-DeTurck flow* $P : \Gamma(E) \to \Gamma(E)$ is given by $P(h)_{ij} = -2\mathrm{Ric}(h)_{ij} + \nabla_i X^t_j + \nabla_j X^t_i$ where X^t is a time dependent vector field defined in terms of h by (*) in the preceeding paragraph. Since σ_P only depends on the highest order terms of P, by inspecting the principal symbol for Ric and the definition of P we see that for any $h \in \Gamma(E)$, $\sigma_P(\xi)(h)_{ij} = g^{pq}\xi_p\xi_q h_{ij}$ so:

$$g_E(\sigma_P(\xi)(h), h) = |\xi|^2_\wedge |h|^2_E$$

then we let $\delta = \frac{1}{2}$ and conclude that P is strongly parabolic. So by theorem 2.1.8 get $T > 0$ and $h_t \in \Gamma(E_{M\times[0,T)})$ such that $P(h_t) = \partial_t h_t$ and $h_0 = g$. Now since M is compact, there exists (see eg. [5]) a one parameter family of diffeomorphisms $\varphi_t : M \to M$ for $0 \le t < T$ such that:

$$\frac{\partial \varphi_t}{\partial t}(p) = X^t(\varphi(x))$$

and $\varphi_0(x) = Id_M$. Let $g_t = \varphi_t^* h_t$. Then a computation shows:

$$\begin{aligned}
\frac{\partial}{\partial t} g_t = \frac{\partial}{\partial s}(\varphi^*_{t+s} h_t(t+s))_{s=0} &= \varphi_t^*(\frac{\partial}{\partial t} h_t) + \frac{\partial}{\partial s}(\varphi^*_{t+s} h_t)_{s=0} \\
&= \varphi_t^*(P(h_t)) + \frac{\partial}{\partial s}(\varphi^*_{t+s} h_t)_{s=0} \\
&= \varphi_t^*(-2\mathrm{Ric}(h_t) + \mathcal{L}_{X^t} h_t) + \frac{\partial}{\partial s}((\varphi_t^{-1} \circ \varphi_{t+s})^* \varphi_t^* h_t)_{s=0} \\
&= -2\mathrm{Ric}(g_t) + \varphi_t^*(\mathcal{L}_{X^t} h_t) - \mathcal{L}_{(\varphi_t^{-1})_*(X^t)}(\varphi_t^* h_t) \\
&= -2\mathrm{Ric}(g_t)
\end{aligned}$$

and since $\varphi_0 = Id_M$, $g_0 = Id^*(h_0) = g$ ie. g_t is a solution to the Ricci flow on (M, g) for $t \in [0, T)$.

3. Closed 3-manifolds with Positive Ricci Curvature.

The goal of this section is to outline a proof of the following remarkable result of Hamilton: *Every closed Riemannian* 3*-manifold with positive Ricci curvature admits a metric with constant curvature* $k > 0$.

The Killing-Hopf theorem says that the universal cover of a Riemannian manifold with constant curvature k is diffeomorphic to $\mathbb{H}^n$ (hyperbolic space) if $k < 0$, $\mathbb{R}^n$ if $k = 0$, or S^n if $k > 0$. Combining this with Hamilton's result we see that the only closed, simply connected Riemannian 3-manifold with positive Ricci curvature is S^n.

We follow the exposition of [3] (which was itself inspired by the exposition of [5]), although we are only outlining the proof sketch so there are not many details given in what follows.

3.1. **Normalized Ricci Curvature.**

3.1.1. *Definition.* If M is a compact Riemannian manifold the *normalized Ricci flow* on M is the following PDE:

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}^t + \frac{2s_t}{n} g_t$$

where s_t is the *normalized average scalar curvature*:

$$s_t = \frac{\int_M \mathrm{scal}_t dV_t}{\int_M dV_t}.$$

Solutions to the normalized Ricci flow are given by time dependent metrics of the form $\tilde{g}_t = \psi(t) g_t$ where g_t is a solution to the Ricci flow and $\psi(0) = 1$ where ψ is some rescaling factor to preserve the volume of M. The utility of the normalized Ricci flow comes from the following two propositions, from which Hamilton's theorem will follow immediately.

3.1.2. *Proposition A..* Suppose (M, g) is a compact Riemannian 3-manifold with positive Ricci curvature. Then the normalized Ricci flow $\tilde{g}_t$ exists for all time.

3.1.3. *Proposition B..* If (M, g) is as above, the limit $g_\infty = \lim_{t\to\infty} \tilde{g}_t$ exists and is a Riemannian metric on M with constant curvature $k > 0$.

3.2. Proposition A.

The proof of proposition A is done by the following two lemmas.

3.2.1. *Lemma.* Suppose (M, g) is a compact Riemannian manifold with positive Ricci curvature and let g_t be a solution to the Ricci flow. Then the scalar curvature of the normalized Ricci flow $\tilde{g}_t$ is uniformly bounded, ie:

$$s = \sup_{x\in M, t>0} \mathrm{scal}_t(x) < \infty$$

3.2.2. *Lemma.* If (M, g) is a compact Riemannian manifold and $\tilde{g}_t$ is the normalized Ricci flow, defined on $[0, T)$. Then:

$$\int_0^T s dt = \infty$$

where $s = \sup_{x\in M,t>0} \mathrm{scal}_t(x)$.

3.2.3. *Proof of proposition A..* Since $s < \infty$, $\int_0^T s dt = \infty$ implies $T = \infty$.

3.3. Proposition B..

The proof of proposition B follows from the following series of results.

3.3.1. *Lemma.* If the *traceless Ricci tensor* $E = \mathrm{Ric} - \frac{1}{n}\mathrm{scal}\, g$ is identically 0 then (M, g) is Einstein.

3.3.2. *Lemma.* Let (M, g) be a compact Riemannian 3-manifold with positive Ricci curvature and $\tilde{g}_t$ the normalized Ricci flow. Then there exists $C, k > 0$ such that the traceless Ricci tensor satisfies $|E^t| \le Ce^{-kt}$.

3.3.3. *Theorem.* Suppose (M, g) is a compact Riemannian manifold with smooth, time dependent metrics g_t defined on $[0, T)$ where $g_0 = g$. If there exists $K > 0$ such that:

$$\int_0^T |\frac{\partial}{\partial t} g_t| dt \le K$$

then $\lim_{t\to T} g_t = g_T$ exists, $g_t \to g_T$ uniformly, and:

$$\frac{g}{e^k} \le g_T \le e^k g.$$

3.3.4. *Theorem.* If (M, g) is a compact Riemannian 3-manifold with positive Ricci curvature and $\tilde{g}_t$ is the normalized Ricci flow, there exists $K > 0$ such that:

$$\int_0^\infty |\frac{\partial}{\partial t}\tilde{g}_t| dt < K$$

3.3.5. *Proposition.* If (M, g) is as in the previous theorem, then the limit metric g_T given by theorem 3.3.3 is smooth.

3.3.6. *Proof of proposition B..* By theorem 3.3.4 and 3.3.3, $\lim_{t\to\infty} \tilde{g}_t = g_\infty$ exists and moreover by proposition 3.3.5 g_∞ is smooth. Then we see by lemma 3.3.2:

$$|E_\infty| = \lim_{t\to\infty} |E^t| \leq \lim_{t\to\infty} Ce^{-kt} = 0$$

so by lemma 3.3.1, (M, g_∞) is Einstein. But in dimension $n \leq 4$, Einstein is equivalent to constant curvature so we conclude (M, g_∞) has constant curvature.

3.4. **Proof of Hamilton's theorem.** If M is a closed Riemannian 3-manifold with positive Ricci curvature, by proposition A the normalized Ricci flow $\tilde{g}_t$ exists for all time and by proposition B, g_∞ is a metric on M with constant curvature $k > 0$.

References

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