RIEMANNIAN GEOMETRY

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1. Riemannian Metrics

Definition 1. A **Riemannian metric** on a smooth manifold M is the assignment of an inner product g_p to T_pM for every $p \in M$ such that for every $X, Y \in \mathcal{X}(M)$ the function $M \to \mathbb{R}$ defined by $p \mapsto g_p(X(p), Y(p))$ is smooth. We sometimes use the notation $\langle , \rangle_p$ for g_p and sometimes omit the p.

Remark 2. Let V be a real finite dimensional vector space and $B(V)$ the space of bilinear maps $V \times V \to \mathbb{R}$. The bilinear map $V^* \times V^* \to B(V)$ given by $(f, g) \mapsto ((u, v) \mapsto f(u)g(v))$ descends to an isomorphism $V^* \otimes V^* \to B(V)$. On the other hand, every bilinear map $V \times V \to \mathbb{R}$ descends to a linear map $V \otimes V \to \mathbb{R}$, i.e. an element of $(V \otimes V)^*$. Thus we see that $V^* \otimes V^*$ is canonically isomorphic to $(V \otimes V)^*$ even though V is not canonically isomorphic to V^*.

By restricting, this provides a canonical identification between symmetric bilinear maps $V \times V \to \mathbb{R}$ and $\mathrm{Sym}^2(V^*)$. For $u, v \in V$ let uv denote $\mathrm{Sym}(u \otimes v) \in \mathrm{Sym}^2(V)$ (and similar for V^*). If $\{e_i\}$ is a basis for V with dual basis $\{f_i\}$ for V^* then the symmetric bilinear B is identified with:

$$\begin{aligned}
\sum_{i,j} B(e_i, e_j) f_i \otimes f_j &= \sum_k B(e_k, e_k) f_k \otimes f_k + \sum_{i \neq j} B(e_i, e_j) f_i \otimes f_j \\
&= \sum_k B(e_k, e_k) f_k \otimes f_k + \sum_{i<j} B(e_i, e_j)(f_i \otimes f_j + f_j \otimes f_i) \\
&= \sum_{i \leq j} B(e_i, e_j) f_i f_j.
\end{aligned}$$

A Riemannian metric can therefore be identified with a section of $\mathrm{Sym}^2(T^*M)$.

Example 3.

(1) We can make $\mathbb{R}^n$ into a Riemannian manifold by letting $g_p(u, v) = \langle u, v \rangle$ be the standard Euclidean inner product at each $T_p\mathbb{R}^n \cong \mathbb{R}^n$.
(2) Given a Riemannian manifold (M, g_M) and a smooth submanifold $N \subset M$, we can make N into a Riemannian manifold by defining $g_N(u, v) = g_M(u, v)$.
(3) Given a Riemannian manifold (M, g_M) and an immersion $f : N \to M$ we can make N into a Riemannian manifold by defining $g_N(u, v)_p = g_M(df_p(u), df_p(v))_{f(p)}$. That f is an immersion guarantees us that g_N is an inner product.
(4) Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$. For $x \in G$ denote by L_x the diffeomorphism $L_x(y) = xy$. We can turn an inner product $g_\mathfrak{g}$ on $\mathfrak{g}$ into a *left invariant* Riemannian metric g (i.e. $g(u, v)_y = g(d_yL_x(u), d_yL_x(v))_{xy}$ for all $x, y \in G$, $u, v \in T_yG$) on G by defining $g(u, v)_x = g_\mathfrak{g}(dL_{x^{-1}}(u), dL_{x^{-1}}(v))$. A *right invariant* metric is defined and constructed in a similar way. A metric that is both right and left invariant is called *bi-invariant.* If G has a bi-invariant metric g we have the important relation:

$$(*) \qquad g([X, Y], Z) = g(X, [Y, Z])$$

It is not hard to show conversely that the left invariant metric associated to any inner product $g_{\mathfrak{g}}$ on $\mathfrak{g}$ that satisfies (*) is bi-invariant. Then the bi-invariant metrics on G are classified by the inner products on $\mathfrak{g}$ satisfying (*).

If G is compact let g_L be a left invariant Riemannian metric and ω a volume form (Lie groups are always orientable so such an ω exists). Moreover we can take $\omega = \omega_1 \wedge \cdots \omega_n$ where ω_i are left-invariant 1-forms. Define a new metric g by:

$$g_p(u,v) = \frac{\int g_L(d_pR_x(u), d_pR_x(v)) \cdot \omega}{\int_G \omega}$$

One can check that g is both left and right invariant, showing that every compact Lie group has a bi-invariant Riemannian metric. After showing that $Ad_g : G \to G$ is an isometry we can compute the differential at e and conclude that $ad_X : \mathfrak{g} \to \mathfrak{g}$ is an isometry. One then can use this fact to show (*) above.

(5) Let G be given by:

$$G = \{f : \mathbb{R} \to \mathbb{R} \ : \ f(t) = yt + x, \quad x, y \in \mathbb{R}\}$$

One can show that G is a Lie group with the group operation given by composition, and is diffeomorphic to:

$$\mathbb{H}^+ = \{(x,y) \in \mathbb{R}^2 \ : \ y > 0\}$$

as a smooth manifold. The identity element is given by $f(t) = t$ corresponding to $(0,1) \in \mathbb{R}^2$. Let g_L denote the left invariant metric on G induced by the Euclidean metric on $\mathfrak{g} \cong T_{(0,1)}\mathbb{R}^2$. This turns $\mathbb{H}^+$ into a Riemannian manifold not isometric to $\mathbb{H}^+$ with the Euclidean metric sometimes called the *Poincare half-plane model.*

(6) If (M,g) is a Riemannian manifold and G is a group acting freely and properly on M by isometries (i.e. for all $g \in G$ the map $g : M \to M$ given by $g(x) = g \cdot x$ is an isometry) then the collection of cosets, denoted $\overline{M} = M/G$, is a smooth manifold such that the quotient map $\pi : M \to \overline{M}$ is a submersion. In this case there exists a unique metric $\overline{g}$ on $\overline{M}$ making $(\overline{M}, \overline{g})$ into a Riemannian manifold and π a Riemannian submersion.

1.1. **The metric tensor.** Given a frame $\{X_i\}$ and coframe $\{\sigma^i\}$ the identification of the metric with a section of $\mathrm{Sym}^2(T^*M)$ allows us to represent the metric as:

$$\sum_{i,j} g(X_i, X_j)\sigma^i \otimes \sigma^j = \sum_{i \leq j} g(X_i, X_j)\sigma^i\sigma^j$$

We simplify sums using the **Einstein summation convention**:

$$g(X_i, X_j)\sigma^i\sigma^j := \sum_{i \leq j} g(X_i, X_j)\sigma^i\sigma^j$$

And further simplify $g_{ij} = g(X_i, X_j)$ so we get metric tensor the metric tensor notation:

$$g_{ij}\sigma^i\sigma^j$$

Example 4.

(1) The standard metric tensor on $\mathbb{R}^n$ in the identity chart is $\sum_{i=1}^n dx_i dx_i$ and the matrix $g_{ij}(p)$ is given by the identity.

(2) In polar coordinate on $\mathbb{R}^2 \setminus \{(0,x) \ : \ x > 0\}$ given by $x = r\cos\theta$, $y = r\sin\theta$ we have $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy = \sin\theta dr + r\cos\theta d\theta$ so:

$$\begin{aligned}dxdx &= (\cos\theta dr - r\sin\theta d\theta)(\cos\theta dr - r\sin\theta d\theta)\\ &= \cos^2\theta drdr + r^2\sin^2\theta d\theta d\theta - r\cos\theta\sin\theta drd\theta - r\cos\theta\sin\theta d\theta dr\end{aligned}$$

and similarly:

$$dydy = \sin^2\theta drdr + r^2\cos^2\theta d\theta d\theta + r\cos\theta\sin\theta drd\theta + r\cos\theta\sin\theta d\theta dr$$

so the standard metric tensor in polar coordinates is given by:

$$g = dxdx + dydy = drdr + r^2 d\theta d\theta$$

with matrix g_{ij} given by:

$$\begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}$$

(3) Given $I \subset \mathbb{R}$ an interval and a smooth curve $\gamma : I \to \mathbb{R}^2$, $\gamma(t) = (r(t), z(t))$ such that $r(t) > 0$ for all t, let $f : I \times [0, 2\pi] \to \mathbb{R}^3$ be the function:

$$f(t,\theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

Then $M = f(I \times [0, 2\pi])$ is a smooth 2 dimension submanifold of $\mathbb{R}^3$ that we will call the *surface of revolution for* γ. Restricting the standard metric on $\mathbb{R}^3$ to M (as in example 3.2.ii) we compute the metric tensor on M:

$$\begin{aligned}dx &= \dot{r}\cos\theta dt - r\sin\theta d\theta\\ dy &= \dot{r}\sin\theta dt + r\cos\theta d\theta\\ dz &= \dot{z}dt\end{aligned}$$

so in these coordinates:

$$g = dx^2 + dy^2 + dz^2 = (\dot{r}^2 + \dot{z}^2)dtdt + r^2 d\theta d\theta$$

If γ is *parametrized by arc length* (meaning $\dot{\gamma} = 1$) we have $\dot{r}^2 + \dot{z}^2 = 1$ so in this case:

$$g = dt^2 + r^2 d\theta^2.$$

As a particular case of this example, if $\gamma(\phi) = (\sin(\phi), \cos(\phi))$ then the surface of revolution is S^2 with the metric tensor:

$$g_{\phi,\theta} = d\phi^2 + \sin^2(\phi)d\theta^2$$

Let $A : S^2 \to S^2$ be defined by $A(p) = -p$. T_pS^2 is given by $p^\perp = (-p)^\perp$ so $T_pS^2 = T_{-p}S^2$. Then we can see $d_pA : T_pS^2 \to T_{-p}S^2 = T_pS^2$ is $d_pA(v) = -v$. From this we conclude that for $u, v \in T_pS^2$:

$$g_{-p}(d_pA(u), d_pA(v)) = g_{-p}(-u, -v) = g_{-p}(u, v) = g_p(u, v)$$

so A is an isometry of S^2. In fact replacing 2 with n changes nothing, so the antipodal map is always an isometry of S^n. Another way to see that A is an isometry on S^2 is to see what happens to the metric tensor. We note first that the inverse of f is given by:

$$(\theta,\phi) = f^{-1}(x,y,z) = (\arctan(\frac{\sqrt{x^2+y^2}}{z}), \arctan(\frac{y}{x}))$$

then since $A(x,y,z) = (-x,-y,-z)$ after applying f we see:

$$(\theta,\phi) \mapsto f \circ A(x,y,z) = (\arctan(\frac{\sqrt{(-x)^2+(-y)^2}}{-z}), \arctan(\frac{-y}{-x}),) = (-\theta,\phi).$$

Then since $d(-\theta) = -d\theta$ we have that the metric tensor becomes:

$$g = d\phi^2 + \sin^2(\phi)d(-\theta)^2 = d\phi^2 + \sin^2(\phi)d\theta^2$$

(4) We can construct $\mathbb{R}P(2)$ as the quotient of S^2/A, i.e. $x \sim x$ and $x \sim A(x) = -x$. Then we write $\pm x = \{x, -x\}$ for an element in $\mathbb{R}P(2)$ where $x \in S^2$ and write let $\pi : S^2 \to \mathbb{R}P(2)$ denote the projection map $\pi(x) = \pm x$.

We make $\mathbb{R}P(2)$ into a Riemannian manifold with the metric $g(u,v)_{\pm x} = g_{S^2}((d_x\pi)^{-1}(u), (d_x\pi)^{-1}(v))_x$. This is well defined by what was done in the previous example. Moreover it is clear from the definition of this metric that π is locally an isometry. Similar to the previous example the 2 can be replaced with n to see that $\mathbb{R}P(n)$ can be made into a Riemannian manifold such that $\pi : S^n \to \mathbb{R}P(n)$ is a local isometry.

(5) More generally than the previous example note that $U(1) \subset \mathbb{C}$ acts by isometries on S^{2n+1} and $\mathbb{C}^{n+1}$ via scalar multiplication. Then since $\mathbb{C}P(n) = S^{2n+1}/U(1)$ we get a Riemannian metric on $\mathbb{C}P(n)$ such that the projection map $\pi : S^{2n+1} \to \mathbb{C}P(n)$ is a local isometry. We call this the *Fubini-Study* metric except when $n = 1$, in which case we call it the *Hopf fibration*.

(6) If G is a Lie group with $\{X_1, .., X_n\}$ an orthonormal basis for $\mathfrak{g}$ with respect to the inner product $g_{\mathfrak{g}}$ and $\{Y_1, ..., Y_n\}$ the left-invariant vector fields on G associated to X_i, then the left invariant metric g is given by:

$$g = \omega_{Y_1}^2 + \cdots + \omega_{Y_n}^2$$

where $\omega_{Y_i} \in \Omega^1(G)$ is the 1-form dual to Y_i, i.e. $\omega_i(Y_j) = \delta_{ij}$.

(7) If $M = M_1 \times M_2$ (as in example 3.2.iv) where M_1 has metric tensor g_1 in U_1 and M_2 has metric tensor g_2 in U_2, then the metric tensor on M in $U_1 \times U_2$ is simply $g_1 + g_2$.

(8) If $M = I \times S^n$, where $I \subset \mathbb{R}$ is an interval with the standard metric denoted dt^2 and S^n has the standard metric which we will denote by ds^2, then the metric tensor on M given by $dt^2 + \varphi^2(t)ds_n^2$ where $\varphi \in C^\infty(I)$ is called a *warped product metric*. If φ satisfies the second order ODE:

$$\varphi''(t) + k\varphi(t) = 0$$
$$\varphi'(0) = 1$$
$$\varphi(0) = 0$$

for some $k \in \mathbb{R}$ then we denote it by $\varphi = sn_k^2$. If we parametrize $S^n \subset \mathbb{R}^n \times \mathbb{R}$ by $F : [0,\pi) \times S^{n-1} \to \mathbb{R}^n \to \mathbb{R}$ with $F(\theta, v) = (\sin(\theta)v, \cos(\theta))$ a brief calculation shows that the standard metric on $\mathbb{R}^{n+1}$ restricted to S^n is given by:

$$ds_n^2 = dx_1^2 + \cdots + dx_n^2 = d\theta^2 + \sin^2(\theta) ds_{n-1}^2$$

thus the standard metric on S^n is given by a warped product metric. Moreover the solution to the above ODE with $k = 1$ is $\varphi(t) = \sin(t)$, so S^n is the warped product metric associated to sn_1^2. One can check that $\mathbb{R}^n$ with the usual metric is the warped product metric associated to sn_0^2.

(9) If $M = I \times S^n \times S^m$ then metrics of the form $g = dt^2 + \varphi^2(t) ds_p^2 + \psi^2(t) ds_q^2$ are called *doubly warped product metrics*. Using the diffeomorphism $\mathbb{R}^{n+1} \setminus \{0\} \cong (0, b) \times S^n$ we can see that under suitable conditions we may extend a metric on $I \times S^n$ to a metric on $\mathbb{R}^n$ (as we only need to define it on a single point). This corresponds to the extension of $I = (0, b)$ to $I = [0, b)$. or $I = (0, b]$. Using the diffeomorphism $S^{n+1} \setminus \{p, q\} \cong I \times S^n$ we can see that under suitable conditions we may extend metric on $I \times S^n$ to a metric on S^{n+1} (as we only need to define it on two points). Finally using an embedding $I \times S^n \times S^m \to S^{n+m+1}$ (one example described below) we can use g to get a metric on S^{n+m+1}.

We show how to write the standard metric on S^n as a doubly warped product. Write $n = p + q + 1$. Since $S^p \subset \mathbb{R}^{p+1}$ and $S^q \subset \mathbb{R}^{q+1}$ the function:

$$\begin{aligned} (0, \frac{\pi}{2}) \times S^p \times S^q &\to \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \\ (t, x, y) &\mapsto (x \sin(t), y \cos(t)) \end{aligned}$$

is an embedding whose image is the unit sphere in $\mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{p+q+2}$, i.e. S^{p+q+1}. In this case we see that:

$$g = dt^2 + \sin^2(t) ds_p^2 + \cos^2(t) ds_q^2$$

(10) Using the product construction can define a metric on $T^2 = S^1 \times S^1$ from the standard metric on S^1. Alternatively consider the subset of $\mathbb{R}^3$ given by:

$$T^2 = \{(x, y, z) \ : \ (\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1\}$$

and give it the metric induced as a submanifold of $\mathbb{R}^3$. These two manifolds are diffeomorphic but *not* isometric as Riemannian manifolds. We refer to the first one as the *flat torus*.

(11) Consider the Lobachevksy metric from before:

$$G = \{g(t) = yt + x \ : \ (x, y) \in \mathbb{R}^2\}$$

from before. Using the identity coordinates we have:

$$g_{ij}(x, y) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

with metric tensor:

$$g = \frac{1}{y^2} dx^2 + \frac{1}{y^2} dy^2.$$

One can show that this is equivalent to the warped product metric on $I \times S^1$ given by $dt^2 + sn_{-1}^2(t) ds_1^2$. Another useful way of writing g is as follows. Write $z = x + iy$ so $\overline{z} = x - iy$. Then $dz = dx + idy$ and $d\overline{z} = dx - idy$.

$$\frac{dzd\overline{z}}{(z-\overline{z})^2} = \frac{(dx+idy)(dx-idy)}{(2iy)^2} = -\frac{dx^2 - idxdy + idydx + dy^2}{4y^2} = -\frac{dx^2+dy^2}{4y^2} = -\frac{1}{4}g$$

thus $g = -4\frac{dzd\overline{z}}{(z-\overline{z})^2}$.

1.2. **The gradient.** Now that we have a metric we can associate to each $f \in C^\infty(M)$ a vector field $\nabla f \in \mathcal{X}(M)$, which we will call the *gradient* of f.

Definition 5. Given $f \in C^\infty(M)$ we define a vector field called the **gradient** of f, denoted $\nabla f \in \mathcal{X}(M)$, implicitly by the equation:

$$g_p(v, \nabla f_p) = d_p f(v), \quad v \in T_pM.$$

If $(x_1, ..., x_n)$ is a chart near p we have that:

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial i}\frac{\partial}{\partial j}$$

where g^{ij} is the inverse matrix to g_{ij}. If $U \subset M$ is open and $f : U \to \mathbb{R}$ satisfies $|\nabla f| = 1$ on U then we will call f a **distance function**, equivalently $|\nabla f|^2 = \langle \nabla f, \nabla f\rangle = 1$. If we consider the Hamiltonian $\mathcal{H}(q,p) = \frac{1}{2}\langle p, p\rangle_q$ on M then we see that distance functions are solutions to the Hamilton-Jacobi equation associated to $\mathcal{H}$. Satisfying such a PDE is a mildly restrictive condition (for example, which functions satisfy this on $M = \mathbb{R}^2$?).

Lemma 6. *For $U \subset M$ open, $r : U \to \mathbb{R}$ is a distance function if and only if r is a Riemannian submersion.*

Proof. Suppose $r : U \to \mathbb{R}$ is smooth and fix $p \in U$. Then $d_p(v) = g_p(\nabla r, v)\partial_t = 0$ if and only if $v \in \nabla r$ hence $v \in \ker d_p r$ if and only if $v \in \operatorname{span}(\nabla r)$. But $v = c\nabla r$ if and only if:

$$d_p r(v) = d_p r(c\nabla r) = cd_p r(\nabla r) = cg_p(\nabla r, \nabla r)\partial_t = c|\nabla r|^2 \partial_t$$

By definition r is a Riemannian submersion if and only if $g_p(u,v)_U = g_{r(p)}(d_p(u), d_p(v))_{\mathbb{R}}$. Combining this with the above we have that r is a Riemannian submersion if and only if $c|\nabla| = c|\nabla r|^2$ i.e. $|\nabla r| = 1$ as desired.

□

Example 7.
Let $M = \mathbb{R}^3$, $U = M \setminus \{0\}$, and $f : U \to \mathbb{R}$ given by:

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}$$

Then ∇f is the usual ($\mathbb{R}^n$ version) gradient given by:

$$\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}$$

thus:

$$|\nabla f| = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

so f is a distance function.

More generally let $M \subset \mathbb{R}^n$ be a smooth submanifold and define $f : \mathbb{R}^n \to \mathbb{R}$ by:

$$f(x) = d(x, M) = \inf_{y \in M}\{|x - y|\}$$

Then one can show that there exists an open $M \subset U \subset \mathbb{R}^n$ such that $|\nabla f| = 1$ on U, hence f is a distance function there. From this we conclude that every smooth submanifold of $\mathbb{R}^n$ is a level set of some distance function. Letting $M = (0) \in \mathbb{R}^3$ then $f(x) = d(x, M)$ is the function from the previous example and $f^{-1}(r)$ gives the sphere of radius r.
As a specific case of the previous example assume $M \subset \mathbb{R}^n$ is an orientable $n-1$ dimension smooth submanifold and let $f : U \to \mathbb{R}$ be the distance function such that $f^{-1}(0) = M$. Since M is orientable it has a unit normal vector field $N : M \to TM^{\perp}$ (where $TM^{\perp}$ denotes the normal bundle). Since $M = f^{-1}(0)$ we know from multivariable calculus that T_pM has a basis given by $\ker D_pf$, i.e. $v \in T_pM$ if and only if $D_pf(v) = 0$ if and only if $\langle v, \nabla f\rangle = 0$. But since M is an $n-1$ dimension submanifold, $T_pM^{\perp}$ is dimension 1 and as was previously noted it is spanned by the unit normal N. Then we conclude that $\nabla f = \pm N$ thus it is possible to pick a sign for N such that $\nabla f = N$ on M.

We conclude by summarizing the above as:

Every smooth submanifold of $\mathbb{R}^n$ is the level set of a distance function, and the unit normal vector fields of orientable hypersurfaces are (up to a sign) the gradient vector fields of these distance functions.

2. Connections and curvature

2.1. Connections.

Definition 8. A map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ satisfying:

$$i)\nabla_{fX+gZ}Y = f\nabla_X Y + g\nabla_Z Y \quad \text{(tensorial in X)}$$
$$ii)\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$$
$$iii)\nabla_X(fY) = f\nabla_X Y + X(f)Y$$

is called an **affine connection on M**. If $\gamma : I \to M$ is a smooth curve and $\nabla_{\dot{\gamma}(t)}X(\gamma(t)) = 0$ for all $t \in I$ then we say that X is **parallel along** γ. It doesn't matter that $\dot{\gamma}$ isn't a vector field on X as ∇ is tensorial in that argument.

Definition 9. Let $\gamma : I \to M$ be a smooth curve. A lift of γ to a map $I \to TM$ is called a **vector field along** γ.

Example 10. The tangent vector $\dot{\gamma}$ defines a vector field along γ.

Proposition 11. *Let M be a smooth manifold with affine connection ∇. If $\gamma : I \to M$ is a smooth curve and $V : I \to TM$ is a vector field along γ then there exists a unique vector field $D_\gamma V$ along γ such that:*

$$D_\gamma(V+W) = D_\gamma V + D_\gamma W$$
$$D_\gamma(fV) = \dot{f}V + fD_\gamma V$$
$$D_\gamma V = \nabla_{\dot{\gamma}} Y$$

where $Y \in \mathcal{X}(M)$ is any extension of V to a vector field on M, i.e. $V(t) = Y(\gamma(t))$. Recall that ∇ is tensorial in the lower argument so we need not extend $\dot{\gamma}$ to a vector field on M.

Proof. Let $p \in M$ and let $(x_1, ..., x_n)$ be a chart for U near p. In these coordinates write $V = \sum_{i=1}^n v_j \frac{\partial}{\partial x_i}$ and $\gamma = (c_1, ..., c_n)$. Define:

$$D_\gamma V = \sum_{k=1}^n \frac{dv_k}{dt}\frac{\partial}{\partial x_k} + \sum_{i,j} \frac{dc_i}{dt} v_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$

□

Definition 12. If T is a $(0,m)$ tensor we define the **covariant derivative** of T to be the $(0,m+1)$ tensor:

$$\nabla T(Y_1, ..., Y_m, Z) = Z(T(Y_1, ..., Y_m)) - \sum_{i=1}^m T(Y_1, ..., \nabla_Z Y_i, ..., Y_m)$$

If $\nabla T \equiv 0$ we say that T is **parallel**.

Example 13.

(1) If f is a $(0,0)$ tensor (i.e. $f \in C^\infty(M)$) then the Covariant derivative of f is simply the gradient:

$$\nabla f(Z) = Z(f)$$

hence the choice of notation makes sense.

(2) For any Riemannian metric g it is always the case that $\nabla g \equiv 0$ hence g is always parallel.

(3) For any coordinate vector field $\frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$, the covariant derivative $\nabla\frac{\partial}{\partial x_i}$ is given by:

$$\nabla\frac{\partial}{\partial x_i}(Z) = Z(\frac{\partial}{\partial x_i}) = 0$$

so coordinate vector fields are parallel. More generally one can show any vector field of the form $V = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ where a_i are constants is parallel (the converse is also true).

Proposition 14. *Given a smooth manifold M with an affine connection ∇, let $\gamma : I \to M$ be a smooth curve and $v_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field V along γ such that $V(t_0) = v_0$ called the* ***parallel transport of*** *v_0 along γ.*

Proof. Let $\gamma : I \to M$ be a smooth curve and $v_0 \in T_{\gamma(t_0)}M$. Suppose there exists a single chart x for an open U such that $\gamma(I) \subset U$. In these coordinates write $v_0 = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ and $\gamma(t) = (x_1(t), ..., x_n(t))$. Consider the differential equation:

$$\sum_{i=1}^n \frac{dv_i}{dt}\frac{\partial}{\partial x_i} + \sum_{i,j=1}^n \frac{dx_i}{dt} v_j \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = 0$$

Since $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} \in \mathcal{X}(U)$ we can write $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}$ and so the system becomes:

$$\sum_{i=1}^n \frac{dv_i}{dt}\frac{\partial}{\partial x_i} + \sum_{i,j=1}^n \frac{dx_i}{dt} v_j \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k} = 0$$

which we re arrange to:

$$\sum_{k=1}^n \left(\frac{dv_k}{dt} + \sum_{i,j=1}^n v_j \frac{dx_i}{dt}\Gamma_{ij}^k \right) \frac{\partial}{\partial x_k} = 0$$

Since the $\frac{\partial}{\partial x_k}$ are linearly independent the above equation gives us n linear first order ODE's:

$$\frac{dv_k}{dt} + \sum_{i,j=1}^n v_j \frac{dx_i}{dt}\Gamma_{ij}^k = 0$$

.

The Picard existence and uniqueness theorem for linear first order ODE's gives us a unique solution $V = \sum_{i=1}^n v_i$ defined for all time satisfying the initial condition $V(t_0) = v_0$.

If $\gamma(I)$ is not contained in a single chart, for any $t \in I$ the interval $[t_0, t]$ is compact so it can be covered with finitely many open sets U_i, in each of which we may define V. By uniqueness, how V is defined in each U_i must agree when the U_i's have nonempty intersection, thus giving us a definition of V on all of I.

□

Definition 15. For an affine connection ∇ on M, we have the $(1, 2)$ tensor $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

called the **torsion tensor** measuring the torsion of the connection ∇. If $T(X, Y) \equiv 0$ we call ∇ **torsion free**.

Remark 16. Recall that $\nabla_X Y$ is tensorial in X. One could antisymmetrize ∇ to make a new operator $\overline{\nabla}_X Y = \nabla_X Y - \nabla_Y X$ so that $\overline{\nabla}_X Y = -\overline{\nabla}_Y X$, removing the tensorial nature of the X argument and giving us a of skew symmetric directional derivative. But observe that:

$$\overline{\nabla}_X Y = \nabla_X Y - \nabla_Y X = T(X,Y) + [X,Y] = T(X,Y) + \mathcal{L}_X Y$$

showing that antisymmetrizing a torsion free connection reduces to the Lie derivative.

Theorem 17. *There exists a unique, torsion free, affine connection ∇ on M such that:*

$$\frac{d}{dt}\langle X, Y\rangle = \langle \frac{DX}{dt}, Y\rangle + \langle X, \frac{DY}{dt}\rangle$$

*called the **Levi-Civita connection**.*

Proof. The directional derivative of vector fields on $\mathbb{R}^n$ satisfies:

$$2g(\nabla_X Y, Z) = \mathcal{L}_Y g(X,Z) + (d\omega_Y)(X,Z)$$

so defining our connection on M implicitly from this equation seems fruitful. We first note the useful *Koszul formula*:

$$\begin{aligned}
2g(\nabla_X Y, Z) &= \mathcal{L}_Y g(X,Z) + (d\omega_Y)(X,Z)\\
&= Yg(X,Z) - g([Y,X],Z) - g(X,[Y,Z]) + X\omega_Y(Z) - Z\omega_Y(X) - \omega_Y([X,Z])\\
&= Yg(X,Z) - g([Y,X],Z) - g(X,[Y,Z]) + Xg(Y,Z) - Zg(Y,X) - g(Y,[X,Z])\\
&= Yg(X,Z) + Xg(Y,Z) - Zg(Y,X) - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X)
\end{aligned}$$

Now to check that ∇ is torsion free we use the Koszul formula:

$$\begin{aligned}
2g(\nabla_X Y - \nabla_Y X, Z) =& Yg(X,Z) + Xg(Z,Y) - Zg(Y,X) - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X)\\
&- Xg(Y,Z) - Yg(Z,X) + Zg(X,Y) + g([X,Y],Z) + g([Y,Z],X) - g([Z,X],Y)\\
&= 2g([X,Y],Z)
\end{aligned}$$

and since X, Y, Z are arbitrary we conclude $T(X,Y) = \nabla_X Y - \nabla_Y X = 0$. Checking that ∇ is metric we again use the Kozsul formula and one sees:

$$2g(\nabla_X Y, Z) + 2g(Y, \nabla_X Z) = 2Xg(Y,Z).$$

To see uniqueness suppose $\overline{\nabla}$ is a torsion free, metric connection. Then by the Kozsul formula again:

$$\begin{aligned}
2g(\nabla_X Y, Z) &= Yg(X,Z) + Xg(Y,Z) - Zg(Y,X) - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X)\\
&= g(\overline{\nabla}_Y X, Z) + g(X, \overline{\nabla}_Y Z) + g(\overline{\nabla}_X Z, Y) + g(Z, \overline{\nabla}_X Y)\\
&- g(\overline{\nabla}_Z Y, X) - g(Y, \overline{\nabla}_Z X) + g(\overline{\nabla}_Z Y, X) - g(\overline{\nabla}_Y Z, X)\\
&- g(\overline{\nabla}_Y X, Z) + g(\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_X Z, Y) + g(\overline{\nabla}_Z X, Y)\\
&= 2g(\overline{\nabla}_X Y, Z)
\end{aligned}$$

thus $\nabla_X Y = \overline{\nabla}_X Y$.

□

The Kozsul formula used in the previous proof is so important we extract it as a separate result.

Corollary 18 (Koszul formula)**.** *The Levi-Civita connection ∇ on M satisfies:*

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)$$

If ∇ is an affine connection on M then $\nabla_X Y = Z$ for some $Z \in \mathcal{X}(M)$. If $(x_1, ..., x_n)$ is a chart for U near p then:

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

We call $\Gamma_{ij}^k : U \to \mathbb{R}$ the *Christoffel symbols* of ∇ in U. If ∇ is torsion free then $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k. Because of this, a torsion free is sometimes referred to as a *symmetric* connection. In the case that ∇ is the Levi-Civita connection we have the important formula for computing the Christoffel symbols:

$$* \quad \Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^{n} g^{mk} \left(\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right) \quad *$$

Example 19.

(1) A quick glance at the above formula shows that the Levi-Civita connection on $\mathbb{R}^n$ with the Euclidean metric has $\Gamma_{ij}^k = 0$ for all i, j, k - recall that g_{ij} was the identity matrix so $\frac{\partial}{\partial x_i} g_{ab} = 0$ for all i, a, b.

(2) Let $G = \mathbb{H}^+$ as in example 3.3.xi. We compute the Christoffel symbols using the above formula. First recall that:

$$g_{ij} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

so:

$$g^{ij} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

hence $g^{11} = g^{22} = y^2$ and $g^{12} = g^{21} = 0$. Moreover it is clear that $\frac{\partial}{\partial x} g_{ab} = 0$ for all a, b. Then we immediately conclude that $\Gamma_{11}^1 = \Gamma_{22}^1 = \Gamma_{12}^2 = 0$. However:

$$\begin{aligned}\Gamma_{11}^2 = \frac{1}{2} \sum_{m=1}^{n} g^{m2} (\frac{\partial}{\partial x} g_{1m} + \frac{\partial}{\partial y} g_{m1} - \frac{\partial}{\partial x_m} g_{11}) &= \frac{1}{2} g^{12} (\frac{\partial}{\partial y} g_{11} - \frac{\partial}{\partial x} g_1 1) + \frac{1}{2} g^{22} (\frac{\partial}{\partial y} g_{21} - \frac{\partial}{\partial y} g_{11}) \\ &= +\frac{1}{2} y^2 (-\frac{\partial}{\partial y} \frac{1}{y^2}) \\ &= \frac{1}{y}\end{aligned}$$

A similar calculation shows that $\Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}$.

(3) Let G be the *Heisenberg* group, ie the 3×3 matrices of the form:

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with the operation of matrix multiplication. There is a natural choice of coordinates $G \to \mathbb{R}^3$ given by $M \mapsto (x, y, z)$. Take the inner product on $\mathfrak{g}$ given by $g_{\mathfrak{g}} = \delta_{ij}$ and let g be the left-invariant Riemannian metric on G induced by $g_{\mathfrak{g}}$. One can compute that in these coordinates:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2+1 & -x \\ 0 & -x & 1 \end{pmatrix}$$

Applying the formula from the previous section we find that:

$$\begin{aligned}
&\Gamma^k_{1,1} = \Gamma^k_{2,2} = \Gamma^k_{3,3} = 0 \\
&\Gamma^1_{1,2} = \Gamma^1_{1,3} = 0 \\
&\Gamma^1_{2,3} = \Gamma^2_{1,3} = \frac{1}{2} \\
&\Gamma^2_{1,2} = \frac{x}{2} \\
&\Gamma^2_{2,3} = 0 \\
&\Gamma^3_{1,2} = \frac{x^2-1}{2} \\
&\Gamma^3_{1,3} = -\frac{x}{2} \\
&\Gamma^3_{2,3} = 0
\end{aligned}$$

(4) Let $\gamma : I \to \mathbb{R}^2$ be a curve $\gamma(t) = (r(t), z(t))$ and let M be the surface of revolution for γ. Then:

$$\begin{aligned}
&\Gamma^1_{1,1} = 0 \\
&\Gamma^1_{1,2} = 0 \\
&\Gamma^1_{2,2} = \frac{1}{2}(-\frac{\partial}{\partial t} r^2) = -rr' \\
&\Gamma^2_{1,1} = 0 \\
&\Gamma^2_{1,2} = \frac{1}{2r^2}(\frac{\partial}{\partial t} r^2(t)) = \frac{rr'}{r^2} = \frac{r'}{r} \\
&\Gamma^2_{2,2} = 0
\end{aligned}$$

2.2. **Various tensor quantities.** Armed with the means to differentiate vector fields along eachother, we now generalize a few concepts from $\mathbb{R}^n$ to M.

Definition 20. Given $f : M \to \mathbb{R}$ we define the **Hessian** of f, denoted Hess$f$, to be the symmetric $(0,2)$ tensor $\frac{1}{2}\mathcal{L}_{\nabla f} g$. The $(1,1)$ version of the Hessian is given by $S(X) = \nabla_X \nabla f$. We note the identity:

$$\text{Hess} f(X,Y) = \frac{1}{2} L_{\nabla f} g(X,Y) = g(S(X), Y) = g(\nabla_X \nabla f, Y) = (\nabla_X \nabla f)(Y)$$

The usual Hessian of f on $\mathbb{R}^n$ is given by the symmetric $n \times n$ matrix $H_{ij} = (\frac{\partial^2 f}{\partial x_i \partial x_j})$. Then the trace of H_{ij} gives us the usual **Laplacian** $\Delta f = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f$. We generalize the Laplacian to M in this way.

The **Laplacian** of f, denoted Δf is given by the trace of the Hessian and the **divergence** of a vector field X is given by the trace of the map $Y \to \nabla_Y X$.

2.3. **Curvature.** Having sufficiently generalized the gradient, Hessian, and Laplacian to functions and vector fields on M we now attempt to extend these definitions to tensors. Recall that:

$$(*)\quad \nabla g(X,Y,Z) = \langle \nabla g(Y,Z), X\rangle = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

so we see that the Levi-Civita connection ∇ for g already itself acts a gradient of the metric. What about the Hessian and Laplacian of g? We defined the Hessian of a function f (ie, a $(0,0)$ tensor) as the $(0,2)$ tensor:

$$(\triangle)\quad \mathrm{Hess} f(X,Y) = g(\nabla_X \nabla f, Y) = (\nabla_X \nabla f)(Y) = [\nabla_X, \nabla f](Y)$$

.

so it makes sense to define Hess$g$ to be a $(0,4)$ tensor by combining $(*)$ and $\triangle$ letting $f = g$:

$$\begin{aligned}\mathrm{Hess} g(X,Y,Z,W) &= (\nabla_X \nabla g)(Y,Z,W)\\ &= [\nabla_X, \nabla g](Y,Z,W)\\ &= \nabla_X(\nabla g(Y,Z,W)) - \nabla g(\nabla_X Y, Z, W) - \nabla g(Y, \nabla_X Y, W) - \nabla g(Y,Z,\nabla_X W)\\ &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)\end{aligned}$$

where the last equality follows from a very long and tedious expansion of ∇g and a simplification using properties of the Levi-Civita connection. With this heuristic in mind (guided by a search for an appropriate Hessian of the metric) we are led to the following definition.

Definition 21. The **curvature tensor** R for (M,g) is the $(1,3)$ tensor:

$$R(X,Y)Z = R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

that can be turned into a $(0,4)$ tensor (also denoted R) in the usual way by:

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

2.3.1. *Proposition.* The curvature tensor R satisfies the following four identities:

(1) $R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z)$
(2) $R(X,Y,Z,W) = R(Z,W,X,Y)$
(3) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$
(4) $(\nabla_Z R)(X,Y)(W) + (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W = 0$

Given the first two properties in the previous proposition we see that we don't need to evaluate R on all n^4 possible combinations of 4-tuples of basis vectors in each T_pM - there exists a smaller collection of them we can check to get a full description of the curvature. The following is one such useful reformulation of the curvature tensor - as an operator on bivectors.

Definition 22. Given a Riemannian manifold (M,g) we extend g to an inner product on the space of bivectors $\Lambda^2 M$ by $g(x\wedge y, v\wedge w) = g(x,v)g(y,w) - g(x,w)g(y,v)$. If $\{e_1, ..., e_n\}$ is an orthonormal basis for T_pM then $\{e_i \wedge e_j \mid i < n\}$ forms an orthonormal basis for $\Lambda^2_p M$. We also extend R to a symmetric bilinear form $R : \Lambda^2_p M \times \Lambda^2_p M \to \mathbb{R}$ given by $R(x\wedge y, z\wedge w) = R(x,y,w,z)$ and further extend this to a bilinear map on $\Lambda^2 M \times \Lambda^2 M \to \mathbb{R}$ by $R(X\wedge Y, Z\wedge W) = R(X,Y,W,Z)$. This induces a self adjoint linear operator $\mathfrak{R} : \Lambda^2 M \to \Lambda^2 M$ defined implicitly by:

$$g(\mathfrak{R}(X\wedge Y), Z\wedge W) = R(X\wedge Y, Z\wedge W) = R(X,Y,W,Z)$$

that we call the **curvature operator**. As $\mathfrak{R}$ is self adjoint there exists an orthonormal basis of eigenvectors for $\mathfrak{R}$. If all of the eigenvalues are positive (resp. nonnegative) we say that $\mathfrak{R}$ is **positive** (resp. **nonnegative**).Since we have a symmetric bilinear form $R : \Lambda^2_p M \times \Lambda^2_p M \to \mathbb{R}$, recall from section 2.2 that there is an associated quadratic form $S : \Lambda_p^2 M \to \mathbb{R}$ given by $S(\sigma) = g(\mathfrak{R}(\sigma), \sigma)$.

2.3.2. *Remark.* Given a basis $\beta = \{e_1, ..., e_n\}$ for a vector space V we get a basis $\beta' = \{e_i \wedge e_j \mid i < j\}$ for $\Lambda^2 V$ and if β is orthonormal then β' is too. If $T : \Lambda^2 V \to \Lambda^2 V$ is a self adjoint linear operator we know from linear algebra that there exists an orthonormal basis of eigenvectors for T. It is not true however that these eigenvectors are necessarily of the form $e_i \wedge e_j$ with $\{e_1, ..., e_n\}$ an orthonormal basis for V - in general we will need to take linear combinations.

Definition 23. Given $p \in M$ and $u, v \in T_pM$ we define the **sectional curvature at p with respect to the two plane σ generated by** u, v to be:

$$\sec_p(\sigma) = \frac{R(v, u, u, v)}{g(u, u)g(v, v) - g(u, v)^2}$$

If we pick an orthonormal basis for σ we see that $\sec_p(\sigma) = S(\sigma) = g(\mathfrak{R}(\sigma), \sigma)$.

Theorem 24. *The following are equivalent:*

(1) $\sec_p(\sigma) = k$ *for all* $\sigma \subset T_pM$
(2) $R(u, v)w = k(g(u, w)v - g(v, w)u)$ *for all* $u, v, w \in T_pM$
(3) $\mathfrak{R}(\omega) = k\omega$ *for all* $\omega \in \Lambda_p^2 M$

If (M, g) satisfies any of the above for all $p \in M$ we say M has **constant curvature** k. If $k = 0$ we call M **flat**. One can easily see $\mathbb{R}^n$ with the Euclidean metric is flat.

It is easy to see that if $\mathfrak{R}$ is positive then $\sec > 0$. The converse, however, is not true. To see this, note that $S\big|_{Gr_2(T_pM)} = \sec_p$. So it can happen that the restriction of S to $Gr_2(T_pM)$ is positive, but since $Gr_2(T_pM) \subsetneq \Lambda^2(T_pM)$ there may be $\sigma \in \Lambda^2(T_pM)$ with $S(\sigma) = 0$ but $\sigma \notin Gr_2(T_pM)$.

Definition 25. The $(0, 2)$ tensor Ric given by:

$$\mathrm{Ric}(Y, Z) = \mathrm{tr}(R(X, Y)Z)$$

(we are taking the trace of the map $T : T_pM \to T_pM$ where $T(x) = R(x, y)z$). Since the curvature was the appropriate generalization of the Hessian to the metric tensor of M, this is the appropriate generalization of the Laplacian. If $\{e_1, ..., e_n\}$ is an orthonormal basis for T_pM then we have the $(1, 1)$ version of Ric:

$$\mathrm{Ric}(X) = \sum_{i=1}^{n} R(X, e_i)e_i$$

Since the $(0, 2)$ and $(1, 1)$ versions of Ric are related by:

$$\mathrm{Ric}(X, Y) = g(\mathrm{Ric}(X), Y)$$

we let $\{X, e_2, ..., e_n\}$ be an orthonormal basis for T_pM and see:

$$\mathrm{Ric}_p(X, X) = g(\sum_{i=1}^{n} R(X, e_i)e_i, X) = \sum_{i=1}^{n} g(R(X, e_i)e_i, X) = \sum_{i=2}^{n} \sec(X, e_i)$$

so the Ricci curvature can be viewed as the sum of the sectional curvatures. If all the eigenvalues λ_j of the $(1, 1)$ version satisfy $\lambda_j \geq k$ then we say that $\mathrm{Ric} \geq k$. Equivalently $\mathrm{Ric} \geq k$ when $\mathrm{Ric}(X, X) \geq kg(X, X)$. If $\mathrm{Ric}(X) = kX$ for all X then we call M an **Einstein manifold** with **Einstein constant** k. In $(0, 2)$ language this is $\mathrm{Ric}(X, Y) = kg(X, Y)$. It is clear from the definitions that manifolds with constant curvature k are Einstein with Einstein constant $(n-1)k$ and manifolds with positive sectional curvature have positive Ricci curvature (although we will see that the converse of both statements need not hold).

Definition 26. Let $\{e_1, ..., e_n\}$ be an orthonormal frame. Then the **scalar curvature** is defined by:

$$\mathrm{scal} = \sum_{i \neq j} \sec(e_i, e_j)$$

Note that this does not depend on the choice of orthonormal frame.

3. Curvature formulas

If M is a Riemannian manifold, $U \subset M$ is open and $r : U \to \mathbb{R}$ is a distance function, denote the level sets of r in U by U_r. Recall that r is a Riemannian sumbersion so U_r is an embedded submanifold for all $r \in \mathbb{R}$. We write ∇^r for the connection on U_r, g_r for the metric on U_r by restricting g, and R^r for the curvature tensor. In this section we will develop some equations involving ∇^r and R^r on one side, but ∇ and R on the other. This will allow us to compute R given R^r, and conversely to compute R^r given R. Because $R \equiv 0$ on $\mathbb{R}^n$ with the Euclidean metric, this will greatly aid us in computing the curvature of embedded hypersurfaces. For readability we will sometimes write the gradient of r (formerly denoted ∇r) as ∂_r.

3.1. **Riemannian Submersions.**

Definition 27. Given a Riemannian manifold M, an open $U \subset M$ and a distance function $r : U \to \mathbb{R}$, we call the $(1, 1)$ tensor S defined by:

$$S(X) = \nabla_X \partial_r$$

the **shape operator** or **second fundamental form**.

Note the similarity with the (1,1) version of the Hessian; in fact $\text{Hess}r(X, Y) = g(S(X), Y)$. Since ∂_r is normal to U_r, $S(X)$ will tell us information about g_r by computing changes in the unit normal to U_r. This is an *extrinsic* quantity - $S(X)$ tells us about U_r *as a Riemannian submanifold of* $\mathbb{R}^n$ (since it depends on r). This is in contrast to the curvature tensor R, defined intrinsically in terms of the metric and Levi-Civita connection.

Given $X, Y \in \mathcal{X}(U_r)$, we decompose the connection $\nabla_X Y = (\nabla_X Y)^T + (\nabla_X Y)^N$. The tangential part is equal to the Levi-Civita connection on U_r, $(\nabla_X Y)^T = \nabla^r_X Y$, and the normal part is a $(1, 2)$ tensor sometimes also referred to as the second fundmantal form for r. There is a third operator sometimes referred to as the second fundamental form $II(X) = g((\nabla_X X)^N, \partial_r)$.

Theorem 28. *Given an open $U \subset M$ and distance function $r : U \to \mathbb{R}$ we have that for all $X \in \mathcal{X}(U)$:*

$$\nabla_{\partial_r} S(X) + S(S(X)) = -R(X, \partial_r)\partial_r$$

Proof.

$$\begin{aligned}
\nabla_{\partial_r} S(X) + S(S(X)) &= \nabla_{\partial_r} \nabla_X \partial_r - \nabla_{\nabla_{\partial_r} X} \partial_r + \nabla_{\nabla_X \partial_r} \partial_r \\
&= \nabla_{\partial_r} \nabla_X \partial_r - \nabla_{[\partial_r, X]} \partial_r \quad (*) \\
&= -R(X, \partial_r)\partial_r - \nabla_X \nabla_{\partial_r} \partial_r \\
&= -R(X, \partial_r)\partial_r
\end{aligned}$$

where the final equality follows from $S(\partial_r) = \nabla_{\partial_r} \partial_r = 0$.

□

Theorem 29 (Gauss Equations)**.** *For $X \in \mathcal{X}(U)$ we write $X = X^T + X^N$ where $X^T \in TU_r$ and $X^N \in TN = (TU_r)^\perp$. Then for $X, Y, Z, W \in \mathcal{X}(U) \cap \mathcal{X}(U_r)$:*

$$(R(X, Y)Z)^T = R^r(X, Y)Z + g(S(X), Z)S(Y) - g(S(Y), Z)S(X)$$
$$R(X, Y, Z, W) = R^r(X, Y, Z, W) + g(S(X), Z)g(S(Y), W) - g(S(Y), Z)g(S(X), W)$$

Theorem 30 (Codazzi-Mainardi Equations)**.** *Given X, Y, Z as in the previous theorem:*

$$R(X,Y,Z,\partial_r) = g(\nabla_Y S(X) - \nabla_X S(Y), Z) = g(\nabla_Y S(X), Z) - g(\nabla_X S(Y), Z)$$

Theorem 31 (Gauss's Theorema Egregium)**.** *If $M \subset \mathbb{R}^3$ is a hypersurface then for all $p \in M$:*

$$\sec_p(\sigma) = \det S$$

where S is the shape operator for M.

Proof. Let $\{X, Y\}$ be an orthonormal basis for σ. Since $R \equiv 0$ on $\mathbb{R}^3$ we have by theorem 5.1.3:

$$\sec_p(\sigma) = R^r(X,Y,Y,X) = g(S(X),X)g(S(Y),Y) - g(S(X),Y)g(S(Y),X) = \det S$$

□

This says that the product of the eigenvalues for S, (sometimes referred to as the **Gaussian curvature** of M) an extrinsic quantity defined in terms of the distance function r, is actually intrinsic to M and invariant under the choice of embedding.

Proposition 32. *The curvature tensor can be recovered from the sectional curvature by:*

$$\begin{aligned}6R(X,Y,V,W) = {} & R(X+W, Y+V) - R(X, Y+V) - R(W, Y+V) - R(X+W, V) \\ & - R(X+W, Y) + R(X,V) + R(W,Y) - R(X+V, Y+W) + R(X, Y+W) \\ & + R(V, Y+W) + R(X+V, Y) + R(X+V, W) - R(V,Y) - R(X,W)\end{aligned}$$

4. Computing the curvature

In this section we compute some curvature quantities for a selection of Riemannian metrics.

Example 33.

(1) We first compute the curvature tensor R of $S^n(r)$. Let $r : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{R}$ be given by $r(v) = |v|$. As was described in section 3, r is a distance function with $U_r = S^n(r)$, so we compute R^r. Recall (3.3.1.viii) that the metric tensor for $\mathbb{R}^{n+1}$ was given by the warped product metric associated to sn_0^2. The $k = 0$ solution to that ODE is $\varphi(r) = r$ hence we may write:

$$g = dr^2 + r^2 ds_n^2 = dr^2 + g_r$$

Since $R \equiv 0$ on $\mathbb{R}^n$ the first Gauss equation (theorem 5.1.3) says:

$$0 = R^r(X,Y)Z + g(S(X),Z)S(Y) - g(S(Y),Z)S(X)$$

i.e.

$$R^r(X,Y)Z = g(S(Y),Z)S(X) - g(S(X),Z)S(Y)$$

As was noted in definition 5.1.1 $g(S(X),Y) = \text{Hess}r(X,Y)$, so we compute the Hessian:

$$\begin{aligned}\text{Hess}r = \frac{1}{2}\mathcal{L}_{\partial_r}g = \frac{1}{2}(\mathcal{L}_{\partial_r}(dr^2) + \mathcal{L}_{\partial_r}(r^2ds_n^2)) &= \frac{1}{2}(\mathcal{L}_{\partial_r}(dr)dr + dr\mathcal{L}_{\partial_r}(dr) + \partial_r(r^2)ds_n^2 + r^2\mathcal{L}_{\partial_r}(ds_n^2)) \\ &= \frac{1}{2}(d(\mathcal{L}_{\partial_r}r)dr + drd(\mathcal{L}_{\partial_r}r) + 2rds_n^2) \\ &= \frac{1}{2}(d(1)dr + drd(1) + 2rds_n^2) \\ &= rds_n^2 \\ &= \frac{g_r}{r}.\end{aligned}$$

Now since:

$$\partial_r = \frac{1}{r}\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$$

then:

$$S(X) = \nabla_X \partial_r = \nabla_X(\frac{1}{r}\sum_{i=1}^{n+1} x_i\frac{\partial}{\partial x_i}) = \frac{X}{r}$$

so S is the identity. Putting all of this into our equation for R^r:

$$\begin{aligned}R^r(X,Y)Z = g(S(Y),Z)S(X) - g(S(X),Z)S(Y) &= \text{Hess}r(Y,Z)S(X) - \text{Hess}r(X,Z)S(Y) \\ &= \left(\frac{g_r(Y,Z)}{r}\right)\left(\frac{X}{r}\right) - \left(\frac{g_r(X,Z)}{r}\right)\left(\frac{Y}{r}\right) \\ &= \frac{1}{r^2}\left(g_r(Y,Z)X - g_r(X,Z)Y\right)\end{aligned}$$

hence we apply theorem 4.3.3 and conclude $S^n(r)$ has constant curvature $1/r^2$ and is therefore also Einstein with $\text{Ric}(X) = \frac{n-1}{r^2}X$.

(2) Let M_1, M_2 be Riemannian manifolds with metrics g_1, g_2 and $M = M_1 \times M_2$ with the product metric. Since $TM = TM_1 \times TM_2$ any vector field $X : M \to TM$ can be decomposed into $X = X_1 + X_2$ where $X_1 : M \to TM_1$ and $X_2 : M \to TM_2$. Let R_1, R_2 denote the curvature tensors in M_1, M_2. Under this decomposition we have:

$$R(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2) = R_1(X_1, Y_1, Z_1) + R_2(X_2, Y_2, Z_2)$$

It is easy to see from this formula that for all $p \in M$ there exists u, v such that $\sec_p(u, v) = 0$.

(3) Consider $T^2 = S^1 \times S^1$ with the product metric induced by the standard metric on S^1 (the *flat torus*). Using example 2 we decompose the curvature tensor into $R = R_1 + R_2$. If $X, Y : M \to TS^1$ then $Y = fX$ for some $f \in C^\infty(M)$. Thus:

$$R(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2) = R_1(X_1, Y_1, Z_1) + R_2(X_2, Y_2, Z_2) = 0$$

since for example:

$$R_1(X_1, Y_1, Z_1) = R_1(X_1, fX_1, gX_1) = fgR_1(X_1, X_1, X_1) = 0$$

So $R \equiv 0$ on T^2 justifying the name. A similar calculation can be done for T^n.

(4) $(S^n(\frac{1}{\sqrt{p}}) \times S^m(\frac{1}{\sqrt{q}}))$. Example 1 gave us that the metric on $S^n(r)$ is the metric on $S^n = S^n(1)$ scaled by r^2 (the equality $g_r = r^2 ds_n^2$). Then $M = S^n(\frac{1}{\sqrt{p}}) \times S^m(\frac{1}{\sqrt{q}}) = (S^n \times S^m, \frac{1}{p} ds_n^2 + \frac{1}{q} ds_m^2)$ as Riemannian manifolds. By the previous example we have $R = R_n + R_m$ where R_n and R_m were computed in example 1. If $X_n, Y_n, Z_n : M \to TS^n$ and $X_m, Y_m, Z_m : M \to TS^m$ then:

$$\begin{aligned}
R(X_n, Y_n, Z_n) &= R_n(X_n, Y_n, Z_n) = p(g_n(Y_n, Z_n)X_n - g_n(X_n, Z_n)Y_n) \\
R(X_m, Y_m, Z_m) &= R_m(X_m, Y_m, Z_m) = q(g_m(Y_m, Z_m)X_m - g_m(X_m, Z_m)Y_m) \\
R(X_n, X_m, Z_n + Z_m) &= R_n(X_n, 0, Z_n) + R_m(0, X_m, Z_m) = 0 = R(X_m, X_n, Z_n + Z_m)
\end{aligned}$$

The final equation tells us that the only way for M to have constant sectional curvature would be that it is identically zero everywhere (this could also be realized by what is written at the end of example 2). If we choose X_n, Y_n linearly independent at some $p \in M$ then $g_n(Y_n, Z_n)X_n - g_n(X_n, Z_n)Y_n \neq 0$ unless both of the coefficients are. But if $n > 1$ we can certainly always choose Z_n in such a way that at least one of the coefficients is nonzero (for example $Z_n = Y_n$) so M can never have constant curvature for $n > 1$. By symmetry this applies to m thus a product of spheres cannot have constant curvature if either of $n, m > 1$. If $n = m = 1$ we are reduced to the previous example where we saw that indeed $S^1 \times S^1$ (with the product metric) has constant curvature 0.

The above also tell us that:

$$\begin{aligned}
\mathfrak{R}(X_n \wedge Y_n) &= pX_n \wedge Y_n \\
\mathfrak{R}(X_m \wedge Y_m) &= qX_m \wedge Y_m \\
\mathfrak{R}(X_n \wedge Y_m) &= 0
\end{aligned}$$

Now by example 2 again we have that:

$$\begin{aligned}\mathrm{Ric}(X_n) &= (n-1)pX_n\\ \mathrm{Ric}(X_m) &= (m-1)qX_m\\ \mathrm{Ric}(X_n + X_m) &= (n-1)pX_n + (m-1)qX_m\end{aligned}$$

so M is Einstein if and only if $p(n-1) = q(m-1)$. Thus we have an example of an Einstein manifold that does not have constant curvature by taking $p = q = 1$ and any $n = m > 1$, having Einstein constant $n - 1 = m - 1$. It is an open problem whether there exists a different metric on $S^2 \times S^2$ with strictly positive sectional curvature.

(5) (Surfaces of revolution). Let $\gamma(t) = (\varphi(t), \psi(t))$ be a curve into $\mathbb{R}^2$ and let $M \subset \mathbb{R}^3$ be the surface of revolution for γ with metric tensor $(\dot\varphi^2 + \dot\psi^2)dt^2 + \varphi^2 d\theta^2$ on $I \times S^1$. Since this is not the product metric we cannot proceed as in the previous examples (R does not factor into $R_1 + R_2$).

First parametrize γ by arc length so our metric takes on the form $dt^2 + \varphi^2(t)d\theta^2$. Then letting $r : M \to \mathbb{R}$ be $r(t, \theta) = t$ we see $\partial_r = \nabla r = \frac{\partial}{\partial t}$ so:

$$g(\nabla r, \nabla r) = dt^2(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \equiv 1$$

hence r is a distance function with level sets $r^{-1}(k) = \{k\} \times S^1$. Since M has dimension 2 we need only compute $R(\partial_\theta, \partial_r, \partial_r)$ via theorem 5.1.2:

$$R(\partial_\theta, \partial_r, \partial_r) = -\nabla_{\partial_r} S(\partial_\theta) - S(S(\partial_\theta)) = -\nabla_{\partial_r} S(\partial_\theta) + S(\nabla_{\partial_r}\partial_\theta) - S(S(\partial_\theta))$$

where ∂_θ is counterclockwise angular field on S^1. We computed the Christoffel symbols for M in example 4.1.8.iv so we have:

$$S(\partial_\theta) = \nabla_{\partial_\theta}\partial_r = \frac{\varphi'}{\varphi}\partial_\theta.$$

thus we conclude

$$R(\partial_\theta, \partial_r)\partial_r = -\frac{\varphi''}{\varphi}\partial_\theta.$$

The sectional curvature is then:

$$\sec = \frac{R(\partial_\theta, \partial_r, \partial_r, \partial_\theta)}{\varphi^2} = \frac{g(-\frac{\varphi''}{\varphi}\partial_\theta, \partial_\theta)}{\varphi^2} = \frac{-\frac{\varphi''}{\varphi}\varphi^2}{\varphi^2} = -\frac{\varphi''}{\varphi}$$

and the Ricci curvatures:

$$\begin{aligned}\mathrm{Ric}(\partial_r) &= \frac{\varphi''}{\varphi}\partial_r\\ \mathrm{Ric}(\partial_\theta) &= -\frac{\varphi''}{\varphi}\partial_\theta\end{aligned}$$

(6) Consider $M = \mathbb{H}^+$ with curvature tensor:

$$\frac{dx^2}{y^2} + \frac{dy^2}{y^2}.$$

This is equivalent to the warped product:

$$dt^2 + sn_{-1}^2 ds_1^2$$

on $I \times S^1$. An inspection of the previous example reveals that this is merely a specific case with $\varphi = sn_{-1}^2$. The ODE defining $\varphi = sn_{-1}^2$ is:

$$\varphi'' - \varphi = 0, \quad \text{equivalently } \frac{\varphi''}{\varphi} = 1$$
$$\varphi'(0) = 1$$
$$\varphi(0) = 0$$

so we immediately conclude the curature tensor is:

$$R(\partial_\theta, \partial_r)\partial_r = -\partial_\theta$$

so the sectional curvature is $\sec = -1$ and we see M has constant curvature -1.

(7) More generally suppose G is a Lie group with a bi-invariant metric. Then:

$$\nabla_X Y = \frac{1}{2}[X, Y]$$
$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$$
$$R(X, Y, Z, W) = \frac{1}{4}g([X, Y], [W, Z]) \geq 0$$

The first follows from the Koszul formula and the second two follow from the first.

(8) Let H be the Heisenberg group from example 4.1.8.iii. We first note that (using the calculation of the Christofel symbols done there) that $[X, Y] = Z$ but:

$$\nabla_X Y(x, y, z) = \frac{x}{2}Y + \frac{x^2 - 1}{2}Z \neq Z$$

hence (by the previous example) our metric is not bi-inariant.

(9) (Berger spheres) The *special unitary group* $SU(2)$ is defined by:

$$\begin{aligned} SU(2) &= \{M \in M_2(\mathbb{C}) \mid MM^* = I,\ \det(M) = 1\} \\ &= \{\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : |z|^2 + |w|^2 = 1\} \\ &= S^3 \end{aligned}$$

and is a 3 dimension (real) Lie group. The Lie algebra $\mathfrak{su}(2)$ is given by:

$$\begin{aligned} \mathfrak{su}(2) &= \{M \in M_2 \mid M^* = -M,\ \operatorname{tr}(M) = 0\} \\ &= \{\begin{pmatrix} ix & iz + y \\ iz - y & -ix \end{pmatrix} \mid x, y, z \in \mathbb{R}\} \end{aligned}$$

with basis:

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

If we let g_1 be the inner product on $\mathfrak{su}(2)$ satisfying $g_1(X, Y) = g_1(X, Z) = g_1(Y, Z) = 0$ and $g_1(X, X) = g_1(Y, Y) = g_1(Z, Z) = 1$ (i.e. making $\{X, Y, Z\}$ into an orthonormal basis) then the induced left-invariant metric on $SU(2)$ makes $SU(2)$ into a Riemannian manifold isometric to S^3 with its usual round metric.

If we instead let g be the inner product on $\mathfrak{su}(2)$ such that $g(X,Y) = g(X,Z) = g(Y,Z) = 0$ (i.e. X, Y, Z are orthogonal) and $g(Y,Y) = g(Z,Z) = 1$ but $g(X,X) = \epsilon^2 > 0$ we call $SU(2)$ with the induced left-invariant metric g_ϵ the ϵ-Berger sphere. The metric tensor for $(SU(2), g_\epsilon)$ is:

$$\epsilon^2 dX^2 + dY^2 + dZ^2$$

By the Koszul formula $\nabla_V V = 0$. Computing the rest:

$$\begin{aligned}
\nabla_X Y &= (2-\epsilon^2)Z \\
\nabla_Y X &= -\epsilon^2 Z \\
\nabla_X Z &= (\epsilon^2 - 2)Y \\
\nabla_Z X &= \epsilon^2 Y \\
\nabla_Y Z &= X \\
\nabla_Z Y &= -X
\end{aligned}$$

therefore:

$$\begin{aligned}
R(X,Y)Y &= \epsilon^2 X \\
R(Z,X)X &= \epsilon^4 Z \\
R(Y,Z)Z &= (4-3\epsilon^2)Y
\end{aligned}$$

and:

$$R(X,Y)Z = R(Y,Z)X = R(Z,X)Y = 0$$

.

Finally we compute $\mathfrak{R}$. Since $g(X,X) = \epsilon^2$ and $g(Y,Y) = g(Z,Z) = 1$ we have that $\{\frac{X}{\epsilon}, Y, Z\}$ forms an orthonormal basis for $\mathfrak{su}(2)$ that we take (by abuse of notation) to be left invariant vector fields on $SU(2)$. This defines an orthonormal basis $\{\frac{X}{\epsilon} \wedge Y, \frac{X}{\epsilon} \wedge Z, Y \wedge Z\}$ for $\Lambda^2 M$. Now:

$$\begin{aligned}
&g(\mathfrak{R}(\frac{X}{\epsilon} \wedge Y), \frac{X}{\epsilon} \wedge Y) = R(\frac{X}{\epsilon}, Y, Y, \frac{X}{\epsilon}) = \frac{1}{\epsilon^2} R(X,Y,Y,X) = \frac{1}{\epsilon^2} g(\epsilon^2 X, X) = \epsilon^2 \\
&g(\mathfrak{R}(\frac{X}{\epsilon} \wedge Y), \frac{X}{\epsilon} \wedge Z) = R(\frac{X}{\epsilon}, Y, Z, \frac{X}{\epsilon}) = 0 \\
&g(\mathfrak{R}(\frac{X}{\epsilon} \wedge Y), Y \wedge Z) = R(\frac{X}{\epsilon}, Y, Z, Y) = 0
\end{aligned}$$

so $\mathfrak{R}(\frac{X}{\epsilon} \wedge Y) = \epsilon^2 \frac{X}{\epsilon} \wedge Y$, equivalently $\mathfrak{R}(X \wedge Y) = \epsilon^2 X \wedge Y$. Similar calculations will show that $\mathfrak{R}(Z \wedge X) = \epsilon^2 Z \wedge X$ and $\mathfrak{R}(Y \wedge Z) = (4 - 3\epsilon^2) Y \wedge Z$. Thus we have an orthonormal basis of eigenvectors to diagonalize $\mathfrak{R}$ so $\epsilon^2 \leq \sec_p \leq 4 - 3\epsilon^2$ for all $p \in SU(2)$.

(10) If $F : M \to N$ is a Riemannian submersion we can compute the curvature tensor for N in terms of the curvature tensor for M. Given $p \in M$, $d_pF : T_pM \to T_{F(p)}N$ is surjective so $T_pM = \ker(d_pF) \oplus \ker(d_pF)^\perp = V_p \oplus H_p \cong V_p \oplus T_{F(p)}M$. Then we can write $v \in T_pM$ as $v = v^V + v^H$ and call v^V the **vertical part** and v^H the **horizontal part**. Thus if

$X \in \mathcal{X}(N)$ there is a unique $\overline{X} \in \mathcal{X}(M)$ with $\overline{X}^V \equiv 0$ such that $dF(\overline{X}) = X$ called the (basic) **horizontal lift** of X. With this notation we have the following result:

Theorem 34 (O'Neill's formula)**.** *If R is the curvature tensor for N and $\overline{R}$ is the curvature tensor for M then:*

$$R(X,Y,Y,X) = \overline{R}(\overline{X},\overline{Y},\overline{Y},\overline{X}) + \frac{3}{4}\left|[\overline{X},\overline{Y}]^V\right|^2$$

We saw earlier that the quotients of Riemannian manifolds by certain group actions give us Riemannian submersions, and we can now use the O'Neill formula to compute the curvature tensor for them. To this end we will do the calculation for complex projective space.

By the isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we can see $S^{2k-1} \subset \mathbb{C}^k$. We have an action of $S^1 \subset \mathbb{C}$ on S^{2k-1} by scalar multiplication that satisfies the conditions given previously. Under this action we have $\mathbb{C}P(n) = S^{2n+1}/S^1$ with $\pi : S^{2n+1} \to \mathbb{C}P(n)$ a Riemannian submersion. We can write the metric on S^{2n+1} (letting $p = 2n-1$, $q = 1$) as:

$$g = dr^2 + \sin^2(r)ds^2_{2n-1} + \cos^2(r)d\theta^2$$

and see S^1 acting independently on S^{2n-1} and S^1. We then write the metric on $\mathbb{C}P(n)$ as:

$$\overline{g} = dr^2 + \sin^2(r)(g + \cos^2(r)h)$$

which we call the **Fubini-Study metric**. Using the O'Neill formula we see that the sectional curvatures in $\mathbb{C}P(n)$ must be larger than 1 as S^m has constant curvature equal to 1. Conversely fix $p \in \mathbb{C}P(n)$, let $X, Y \in T_p\mathbb{C}P(n)$ be orthonormal and denote by N the unit vector field on S^{2n+1} that is tangent to the action of S^1. Then one can show that $g(\frac{1}{2}[\overline{X},\overline{Y}], V) = g(\overline{Y}, i\overline{X}) \leq 1$ so by the O'Neill formula $\sec_p(X,Y) = 1 + \frac{3}{4}|[\overline{X},\overline{Y}]^V|^2 = 1 + 3\overline{g}(i\overline{X},\overline{Y}) \leq 4$. Thus we conclude that for all $p \in \mathbb{C}P(n)$ we have $1 \leq \sec_p \leq 4$.

To compute the Ricci curvature fix $p \in M$, let $X \in T_p\mathbb{C}P(2)$ satisfy $|X| = 1$ and extend X to an orthonormal basis $X, E_2, ..., E_{2n}$ such that $i\overline{X} = \overline{E_2}$. Then by the O'Neill formula:

$$\begin{aligned}
\mathrm{Ric}_p(X,X) &= \sum_{i=2}^{2n} \sec_p(X,E_i) \\
&= \sec_p(X,E_2) + \sum_{i=3}^{2n} \sec_p(X,E_i) \\
&= 1 + 3|g(\overline{E_2}, i\overline{X})|^2 + \sum_{i=3}^{2n}(1 + 3|g(\overline{E_i}, i\overline{X})|^2) \\
&= 1 + 3 + \sum_{i=3}^{2n} 1 \\
&= 4 + (2n-2) \\
&= 2n + 2
\end{aligned}$$

thus $\mathbb{C}P(2)$ is Einstein with Einstein constant $2n+2$. Finally consider an orthonormal basis for $T_p\mathbb{C}P(2)$ of the form X, iX, Y, iY. Then the following basis β:

$$X \wedge iX + Y \wedge iY, \quad X \wedge iX - Y \wedge iY$$
$$X \wedge Y + iX \wedge iY, \quad X \wedge Y - iX \wedge iY$$
$$X \wedge iY + Y \wedge iX, \quad X \wedge iY - Y \wedge iX$$

diagonalizes $\mathfrak{R}$, whose matrix in these coordinates is:

$$[\mathfrak{R}]_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

To see this, we first evaluate $R(Y, iY, iX, X)$ as it will come up frequently in the computation. Using proposition 5.1.6 we have:

$$\begin{aligned} 6R(Y, iY, iX, X) = & 4\sec(X+Y, iX+iY) - 2\sec(Y, iX+iY) - 2\sec(X, iX+iY) \\ & - 2\sec(X+Y, iX) - 2\sec(X+Y, iY) + \sec(Y, iX) + \sec(X, iY) \\ & - 4\sec(Y+iX, iY+X) + 2\sec(Y, iY+X) + 2\sec(iX, iY+X) \\ & + 2\sec(Y+iX, iY) + 2\sec(Y+iX, X) - \sec(iX, iY) - \sec(Y, X) \end{aligned}$$

and applying the formula for sec above the right hand side is 12 so $R(Y, iY, iX, X) = 2$. Armed with this we will do an example calculation for $\mathfrak{R}$. First assume that the β given are all eigenvectors for $\mathfrak{R}$. Then:

$$\begin{aligned} & g(\mathfrak{R}(X \wedge iX \pm Y \wedge iY), X \wedge iX \pm Y \wedge iY) \\ & = R(X, iX, iX, X) + R(Y, iY, iY, Y) \pm 2R(X, iX, iY, Y) \\ & = \sec(X, iX) + \sec(Y, iY) \pm 2R(X, iX, iY, Y) \\ & = 8 \pm 4 \end{aligned}$$

and:

$$g(\mathfrak{R}(X \wedge iX \pm Y \wedge iY), X \wedge iX \pm Y \wedge iY) = cg(X \wedge iX \pm Y \wedge iY, X \wedge iX \pm Y \wedge iY) = 2c$$

$2c = 8 \pm 4$ i.e. $c = 4 \pm 2$ giving us the 6 and one of the 2 entries in the matrix above.

5. Hypersurfaces and isometric immersions.

If M is an Riemannian manifold of dimension $n-1$ we would like to know when we can find a Riemannian embedding or at least an isometric immersion $F: M \to \mathbb{R}^n$. Below we shall assume we have found one and derive several necessary conditions to act as obstructions.

Definition 35. Given an isometric immersion $F: M \to \mathbb{R}^n$ if M is orientable we can choose a unit vector field $N \in \mathcal{X}(\mathbb{R}^n)$ normal to M. If M is not orientable we can instead pass to the oriented double cover and fix N this way. Since $|N(x)| = 1$ we can view N as a map $G: M \to S^{n-1}$ called the **Gauss map**. Then we see that $dG(v) = S(v)$.

Theorem 36. *Suppose $F: M \to \mathbb{R}^n$ is an isometric immersion with $n > 2$ and M is compact without boundary. If the eigenvalues of S are always positive then $G: M \to S^{n-1}$ is a diffeomorphism and F is an embedding.*

Proof. If the eigenvalues of S are always positive, in particular they are never zero. Since $dG = S$ we have that dG is invertible on M hence it is a local diffeomorphism. As M is compact and S^n is connected this gives us that G is surjective. For $x \in S^n$ compactness of M gives us that $G^{-1}(x)$ is finite (of order m). For each $p_i \in G^{-1}(x)$ we can find pairwise disjoint neighbourhoods P_i (as M is Hausdorff) and by choosing smaller neighbourohods U_i if needed the inverse function theorem gives us open neighbourhoods V_i of x such that $G: U_i \to V_i$ is a diffeomorphism. Then it is easy to see that:

$$\bigcap_{i=1}^{m} V_i \setminus G(M \setminus \bigcup_{i=1}^{m})$$

is an evenly covered neighbourhood of x hence G is a covering map. But S^n is simply connected so G must be a diffeomorphism.

It remains to show that F is an embedding. M is compact and F is an immersion so we need only show that F is injective. Fix $p_0 \in M$ and let $g: M \to \mathbb{R}$ be given by:

$$f(p) = d(F(p), T_{F(p_0)}F(M)) = \langle F(p) - F(p_0), N(p_0)\rangle$$

where N is the unit normal to $F(M)$. Hence:

$$df(v) = \langle dF(v), N(p_0)\rangle.$$

But F is an immersion so $df(v) \neq 0$ so the critical points for f are the two points p_1, p_2 such that when $N(p_i) = \pm N(p_0)$, one of which is $p_2 = p_0$. Moreover $f(p_1) \neq f(p_0)$ else $f \equiv 0$. Then with no loss in generality we may assume $f \geq 0$. Now if $F(p) = F(p_0)$ then $f(p) = f(p_0) = 0$ so $p = p_0$, thus F is injective hence an embedding. □

Proposition 37. *Suppose $F: M \to \mathbb{R}^n$ is an isometric immersion and let $p \in M$. Let $\{e_1, ..., e_n\}$ an orthonormal basis of eigenvectors for $S: T_pM \to T_pM$ with eigenvalues λ_i. Then $\mathfrak{R}(e_i \wedge e_j) = \lambda_i\lambda_j e_i \wedge e_j$.*

Proof. Since $S(e_i) = \lambda_i e_i$ by theorem 5.1.3 we have:

$$\begin{aligned}
g(\mathfrak{R}(e_i \wedge e_j), e_k \wedge e_\ell) &= g(S(e_j), e_\ell)g(S(e_i), e_k) - g(S(e_i), e_\ell)g(S(e_j), e_k) \\
&= \lambda_j\lambda_i g(e_j, e_\ell)g(e_i, e_k) - \lambda_i\lambda_j g(e_i, e_\ell)g(e_j, e_k) \\
&= \lambda_j\lambda_i g(e_i \wedge e_j, e_k \wedge e_l) \\
&= \delta_{i,k}\delta_{j,\ell}\lambda_i\lambda_j
\end{aligned}$$

□

Since $\lambda_i\lambda_j = g(\mathfrak{R}(e_i \wedge e_j), e_i \wedge e_j) = \sec(e_i, e_j)$ the previous result allows us to conclude that any Riemannian hypersurface with positive sectional curvature must have positive curvature operator. This shows that $\mathbb{C}P(2)$ does not have a Riemannian embedding into $\mathbb{R}^5$ as we saw that $\mathbb{C}P(2)$ has 0 as an eigenvalue but $1 \leq \sec \leq 4$.

Now we give a result that can be used to rule out the existence of isometric immersions for manifolds of dimension $n \geq 3$.

Proposition 38. *Suppose $F : M \to \mathbb{R}^{n+1}$ is an isometric immersion and $n \geq 3$. If $\mathfrak{R} : \Lambda^2(T_pM) \to \lambda^2(T_pM)$ is positive then we can compute $S : T_pM \to T_pM$ independently of F.*

Proof. Let $\beta = \{e_1, e_2, e_3\}$ be an orthonormal basis for T_pM and let $(s_{ij}) = [S]_\beta$. Since $(s_{ij}) = g(S(e_i), e_j)$ we can apply the previous proposition with no loss in generality to conclude that all the eigenvalues are positive. By theorem 5.1.3 we have:

$$g(\mathfrak{R}(e_i \wedge e_j), e_k \wedge e_\ell) = g(S(e_j), e_\ell)g(S(e_i), e_k) - g(S(e_i), e_\ell)g(S(e_j), e_k)$$

This allows us to compute S^{-1} by first computing the cofactor matrix (where the subscripts are taken mod 3):

$$(c_{ij}) = (-1)^{i+j}(s_{i+1,j+1}s_{i+2,j+2} - s_{i+2,j+1}s_{i+1,j+2})$$

along with $\det S = \det(c_{ij})^{n-1}$.

□

To see an application of the previous result we will show that the Berger spheres cannot be isometrically immersed into $\mathbb{R}^4$. We saw that $\mathfrak{R}$ is positive if $0 < \epsilon < 1$. Following the computation in the proof in the previous proposition we see that:

$$S(\frac{X}{\epsilon}) = \frac{\epsilon^2}{\sqrt{4-3\epsilon^2}}\frac{X}{\epsilon}$$
$$S(Y) = \sqrt{4-3\epsilon^2}Y$$
$$S(Z) = \sqrt{4-3\epsilon^2}Z$$

We know $(\nabla_Y S)(Z) = (\nabla_Z S)(Y)$, but:

$$(\nabla_Y S)(Z) = (\sqrt{4-3\epsilon^2} - \frac{\epsilon^2}{\sqrt{4-3\epsilon^2}})\epsilon X \neq -(\sqrt{4-3\epsilon^2} - \frac{\epsilon^2}{\sqrt{4-3\epsilon^2}})\epsilon X = (\nabla_Z S)(Y)$$

Now we show that $n \geq 3$ in the previous proposition was necessary. Let M be the warped product with metric tensor $dt^2 + (a\sin(t))^2 d\theta^2$. Then as was computed earlier we have that M has constant curvature equal to 1. But letting $x(t) = \int_0^t \sqrt{1 - a\cos(x)^2}dx$ and $y(t) = a\sin(t)$ we have $\dot{x}^2 + \dot{y}^2 = 1$ and $y = a\sin(t)$ so we can write M as a surface of revolution $F(t,\theta) = (\cos\theta y(t), \sin\theta y(t), x(t))$. Computing a basis for T_pM we have:

$$T_{F(t,\theta)}M = \text{span}\{(\dot{x}(t), \cos\theta\dot{y}(t), \sin\theta\dot{y}(t)), (0, -\sin\theta, \cos\theta)\}$$

so M has unit normal $N_{F(t,\theta)} = (\dot{y}(t), -\cos\theta\dot{x}(t), -\sin\theta\dot{x}(t))$. Recalling that we have the Gauss map $G = N$ and $dG = S$ we see that $S(\partial_t) \neq \partial_t$ so $S \neq I$. But $M = S^2$ is another surface with constant curvature equal to 1 and on S^2 we have $S = I$, so we conclude that S does depend on F.

Theorem 39 (Gauss-Bonnet)**.** *If M is a compact orientable 2 dimension Riemannian manifold:*

$$\int_M scal\ dV = 4\pi\chi(M)$$

6. Geodesics

Let (M, g) be a Riemannian manifold and $p, q \in M$. By a **path from** p **to** q we mean a continuous map $\gamma : [0, a] \to M$ with $\gamma(0) = p$, $\gamma(a) = q$ such that the restriction of γ to $(0, a)$ is smooth. We define the **length** of γ to be:

$$\ell(\gamma) = \int_0^a |\dot{\gamma}(t)| dt$$

and the distance from p to q by:

$$d(p, q) = \inf_{\gamma} \ell(\gamma)$$

where the infimum is taken over paths from p to q. Observe that d is a metric on M making (M, d) into a metric space. Note also that, although $d(p, q)$ is well defined, it may not be attained by any path from p to q (take for example $M = \mathbb{R}^2 \setminus 0$ with the Euclidean metric, and $p = (-1, 0)$, $q = (1, 0)$). Before continuing we make an observation about smooth curves into $\mathbb{R}^n$. Let $p, q \in \mathbb{R}^n$ and $\alpha(t) = (1 - t)p + tq$ defined on $[0, 1]$. Then:

$$\ell(\alpha) = \int_0^1 |\dot{\alpha}| dt = \int_0^1 |q - p| dt = |q - p| = d(q, p)$$

Let $r : \mathbb{R}^n \to \mathbb{R}$ be the function $r(x) = d(p, x)$. We know from earlier discussions that r is a distance function and α is an integral curve for ∇r. Now suppose $\gamma : [0, 1] \to \mathbb{R}^n$ is some other smooth curve from p to q and $|\dot{\gamma}| > 0$. Then:

$$\ell(\gamma) = \int_0^1 |\dot{\gamma}| dt = \int_0^1 |\dot{\gamma}| \cdot |(\nabla r) \circ \gamma| dt \geq \int_0^1 \langle (\nabla r) \circ \gamma, \dot{\gamma} \rangle dt = \int_0^1 dr(\dot{\gamma}) = r(\gamma(1)) - r(\gamma(0)) = d(p, q)$$

where the inequality comes from Cauchy-Schwarz. Since that inequality is an equality if and only if $k\dot{\gamma} = (\nabla r) \circ \gamma$ then we must have that γ is a reparametrization of α .

6.1. Covariant differentiation and geodesics.

Definition 40. If $\gamma : I \to M$ is a smooth curve and $D_{\dot{\gamma}} \dot{\gamma}(t_0) = 0$ then γ is called **geodesic at** t_0. If γ is geodesic at t for all $t \in I$ then we simply call γ a **geodesic**. In coordinates $(x_1, ..., x_n)$ this is equivalent to the **geodesic equation**:

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt} \frac{dx_j}{dt} = 0$$

for all $k = 1, ..., n$. The image of a geodesic is also called a geodesic.

The geodesic equation has a solution for any pair of initial conditions p, v defined on $(-\delta, \delta)$ for some $\delta > 0$. Since $\Gamma_{ij}^k = 0$ on $\mathbb{R}^n$ we see that geodesics in Euclidean space are exactly the straight lines. An alternative characterization of geodesics is sometimes convenient. Let $X \in \mathcal{X}(TM)$ be given in coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ by:

$$\frac{dx_k}{dt} = y_k$$
$$\frac{dy_k}{dt} = -\sum_{ij} \Gamma_{ij}^k y_i y_j$$

Then $\gamma : I \to M$ is a geodesic if and only if $(\gamma, \dot{\gamma})$ is an integral curve for X. What we conclude is that for any $p \in M$ there exists $\epsilon > 0$ such that for any $v \in T_pM$ with $|v| < \epsilon$ there exist $\delta > 0$

and a unique geodesic $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$. For a choice of $p \in M$, $v \in T_pM$, we denote the unique geodesic through p with velocity v by $\gamma(p, v, t)$. We note here that if γ is a geodesic:

$$\frac{d}{dt} g(\dot{\gamma}, \dot{\gamma}) = 2g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) = 0$$

so $|\dot{\gamma}| = c$ is constant. It follows that:

$$s(t) = \int_0^t |\dot{\gamma}| dt = \int_0^t c\ dt = ct$$

i.e. the arc length of γ is proportional to the velocity.

Proposition 41. *If $\gamma(p, v, t)$ is defined on $(-\delta, \delta)$ then for any $a > 0$ the geodesic $\gamma(p, av, t)$ is defined on $(-\delta/a, \delta/a)$ and moreover:*

$$\gamma(p, v, at) = \gamma(p, av, t)$$

Proof. Define $c(-\delta/a, \delta/a) \to M$ by $c(t) = \gamma(p, v, at)$. Then $c(0) = \gamma(0) = p$, $\dot{c}(0) = a\dot{\gamma}(0) = av$, and:

$$D_c(\dot{c}) = a^2 \nabla_{\dot{\gamma}(at)} \dot{\gamma}(at) = 0$$

so c is a geodesic through p with velocity av. By uniqueness $c(t) = \gamma(p, av, t)$.

□

This proposition tells us that we have a tradeoff we can make between the domain of definition $(-\delta, \delta)$ for a geodesic γ and the $\epsilon > 0$ of choice for $v \in T_pM$. In particular we can demand that our geodesics γ be defined on (at least) $I = (-2, 2)$. This allows us to make the following definition:

Definition 42. Given $p \in M$ we have the **exponential map at** p, $\exp_p : B(0, \epsilon) \to M$ given by:

$$\exp_p(v) = \gamma(p, v, 1)$$

defined on the appropriate ball of radius ϵ that allows our geodesics to be defined on $(-2, 2)$.

It is clear that $\exp_p$ is smooth and $\exp_p(0) = p$. We also have the following important fact.

Proposition 43. *For all $p \in M$ the exponential map is a local diffeomorphism near $0 \in T_pM$.*

Proof. A quick computation shows:

$$d_0 \exp_p(v) = \frac{d}{dt}\Big|_{t=0}(\exp_p(tv)) = \frac{d}{dt}\Big|_{t=0}\gamma(p, tv, 1) = \frac{d}{dt}\Big|_{t=0}\gamma(p, v, t) = v$$

as claimed. □

Definition 44. Fix $p \in M$ and let $\epsilon > 0$ is such that $\exp_p$ is a diffeomorphism from $B_\epsilon(0)$ onto its image. Then we call $\exp_p(B_\epsilon(0)) = B_\epsilon(p)$ the **normal** (or **geodesic**) *ϵ-neighbourhood* (or *ϵ-ball* or simply just ball) of p. We call the boundary of this ball the **normal** (or **geodesic**) *ϵ-sphere* at p.

Let $\beta = \{e_1, ..., e_n\}$ be an orthonormal basis for T_pM. We have an isomorphism $\varphi : \mathbb{R}^n \to T_pM$ given by $\varphi(x_1, ..., x_n) = x_1e_1 + \cdots x_ne_n$. We can combine this with the exponential map to get a diffeomorphism $\varphi^{-1} \circ \exp^{-1} : B_p(\epsilon) \to \mathbb{R}^n$. We call the coordinate chart $(B_p(\epsilon), \varphi^{-1} \circ \exp^{-1})$ **normal coordinates at p**.

Let U be a normal neighbourhood of p. If $x : U \to \mathbb{R}^n$ is a normal coordinate chart for p then the coordinate basis $\{\frac{\partial}{\partial x_i}\}$ is orthonormal. Conversely if $\{e_1, ..., e_n\}$ is an orthonormal basis for T_pM there exists a unique normal coordinate chart x for U such that $\frac{\partial}{\partial x_i} = e_i$. Finally if x and $\overline{x}$ are two normal coordinate charts on U then $x_j = A_{i,j}\overline{x}_i$ for some orthogonal matrix (A_{ij}). In normal coordinates the metric tensor becomes:

$$g = dr^2 + g_r$$

where g_r is the restriction of g to S^{n-1} (recall earlier discussions of distance functions).

Proposition 45. *Fix $p \in M$ and let (U, x) be normal coordinates for p. Then $g_{ij}(p) = I$ and $\Gamma^k_{ij}(p) = 0$. Moreover if $v = v_1\frac{\partial}{\partial x_1} + \cdots + v_n\frac{\partial}{\partial x_n} \in T_pM$ then $\gamma(p, v, t) = (tv_1, ..., tv_n)$ in these coordinates. Finally all first partial derivatives of g_{ij} vanish at p.*

6.2. **Geodesics as length minimizing curves.**

Theorem 46 (The Gauss Lemma)**.** *Fix $p \in M$ and $\epsilon > 0$ such that $\exp_p : B_0(\epsilon) \to B_p(\epsilon)$ is a diffeomorphism. Let $v \in B_0(\epsilon)$ and write $q = \exp_p(v)$. Define $r : B_p(\epsilon) \to \mathbb{R}$ by $r(q) = |\exp^{-1}(q)|$. Let $\{e_1, ..., e_n\}$ be an orthonormal basis for T_pM and let $x : B_p(\epsilon) \to T_pM$ be the associated normal coordinates. For $v \in B_0(\epsilon)$ identify $T_v(T_pM) \cong T_pM$, write $q = \exp_p(v)$ and define the vector field ∂_r on $B_p(\epsilon)$ by $\partial_r(q) = d_v \exp_p(v)$. Then the following hold:*

(1) r, ∂_r are well defined on $B_p(\epsilon)$ (independent of the choice of normal coordinates).
(2) r, ∂_r are smooth on $B_p(\epsilon) \setminus p$.
(3) r^2 is smooth on $B_p(\epsilon)$.
(4) $\nabla r(q) = \partial_r(q)$.

Proof. i), ii) and iii) follow from earlier discussions. To prove iv) we need to show that $g_q(\partial r(q), v) = d_q r(v)$ for all $q \in U$, $v \in T_qU$. Writing $q = (x_1, ..., x_n)$ in U we note that in these coordinates:

$$r^2 = x_1^2 + \cdots + x_n^2$$

so $2rdr = 2x_1dx_1 + \cdots + 2x_ndx_n$ therefore:

$$d_qr = \frac{1}{r}(x_1dx_1 + \cdots + x_ndx_n).$$

and also:

$$\partial_r = \sum_{i=1}^{n} \frac{x_i}{r}\frac{\partial}{\partial x_i}$$

.

Consider $v = \partial_r$. Then $g_q(\partial r(q), v) = 1$ by proposition 8.1.8. Conversely we see:

$$\begin{aligned} d_qr(v) = \frac{1}{r}(x_1dx_1 + \cdots + x_ndx_n)(\partial_r) &= \frac{1}{r}(x_1dx_1 + \cdots + x_ndx_n)\sum_{i=1}^{n} \frac{x_i}{r}\frac{\partial}{\partial x_i} \\ &= \frac{x_1^2 + \cdots x_n^2}{r^2} \\ &= 1 \end{aligned}$$

so the result is true for $v = \partial_r$. Now let $J = \sum_{i<j=1}^{n} -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i}$ (in $\mathbb{R}^2$ this is the familiar rotational field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$). Then:

$$dr_q(J) = \frac{1}{r}(x_1dx_1 + \cdots + x_ndx_n)(\sum_{i<j}^{n} x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}) = 0$$

To compute $g_q(\partial r(q), J)$ requires a couple steps. First see that $g(\partial r, J)$ is constant along geodesics of the form $\gamma(p, v, t)$ since:

$$\partial_r g(\partial r, J) = g(\nabla_{\partial_r}\partial r, J) + g(\partial r, \nabla_{\partial_r} J) = g(\partial r, \nabla_J \partial_r) = \frac{1}{2}Jg(\partial r, \partial r) = 0$$

Now we compute:

$$|g(\partial_r, J)| \leq |\partial_r| \cdot |J| = |J| \leq \sum_{i<j}^{n} |x_i| \cdot \left|\frac{\partial}{\partial x_j}\right| + |x_j| \cdot \left|\frac{\partial}{\partial x_i}\right| \leq r\left(\left|\frac{\partial}{\partial x_i}\right| + \left|\frac{\partial}{\partial x_j}\right|\right)$$

so by taking $\lim_{x\to p}$ of both sides we see that $g(\partial r(p), J) = 0$ and since it is constant along geodesics based at p we must have that $g(\partial r(q), J) = 0$ for all q.

What we have shown thus far is that for any $q \in U$, $dr_q(\partial_r(q)) = g_q(\partial r(q), \partial r(q))$ and $dr_q(J) = g_q(\partial r(q), J)$. But for all $v \in T_qU$ we can write v as a linear combination of ∂_r and J, concluding the proof. □

Corollary 47. *Fix $p \in M$ and let $\epsilon > 0$ sufficiently small that* $\exp_: B_0(\epsilon) \to B_p(\epsilon)$ *is a diffeomorphism. Then for all $v \in B_0(\epsilon)$ we have* $\exp_p(tv) = \gamma(p, v, t)$ *is the unique shortest path connecting p to* $\exp_p(v)$.

Proof. By the Gauss lemma we see that r is a distance function. Now fix $q \in B_p(\epsilon)$ and suppose $c : [a, b] \to M$ is a smooth curve with $c(a) = p$ and $c(b) = q$. A short computation shows:

$$\ell(c) = \int_a^b |\dot{c}|dt = \int_a^b |\dot{c}| \cdot |\nabla r| \geq \int_a^b dr(\dot{c}) = r(c(b)) = d(p, q)$$

and equality holds if and only if $\dot{c}$ is proportional to ∇r, i.e. c is a reparametrization of exp (hence a geodesic).

□

Remark 48. In short, we conclude that a smooth path γ from p to q is a geodesic if and only if $\ell(\gamma) = d(p, q)$.

6.3. **Computing geodesics.** In this section we will compute the geodesics for a few Riemannian manifolds.

Example 49.

(1) ($\mathbb{R}^n$) As mentioned before for $p, v \in \mathbb{R}^n$ let $\gamma(t) = vt + p$. Then $\ddot{\gamma} \equiv 0$ so γ is the geodesic through p in the direction of v.
(2) (S^2) Define $\gamma : [0, 2\pi] \to [0, \pi] \times [0, 2\pi]$ by $\gamma(t) = (\frac{\pi}{2}, t)$. Then one checks that $D_\gamma(\dot{\gamma}) \equiv 0$ with the Riemannian metric tensor given by $dt^2 + \sin^2(t)d\theta$ so γ is a geodesic. But in S^2, $\gamma(t) = (\cos(t), \sin(t), 0)$ thus the equator is a geodesic on S^2. Since $SO(3)$ acts on S^2 by isometries and brings great circles to great circles we get (by uniqueness) that the geodeics on S^2 are given by great circles.

This example shows that not all geodesics connecting p to q are minimizing geodesics. Consider $p = (1, 0, 0)$ and $q = (0, 1, 0)$. Then $\gamma : [0, \frac{\pi}{2}] \to M$ where $\gamma(t) = (\cos(t), \sin(t), 0)$

is a geodesic with $\gamma(0) = p$, $\gamma(\frac{\pi}{2}) = q$ with $\ell(\gamma) = \frac{\pi}{2}$ and $\alpha(t) = (\cos(t), -\sin(t), 0)$ is another geodesic connecting p to q with $\ell(\alpha) = \frac{3\pi}{2}$.

(3) ($\mathbb{H}^+$) Let $\mathbb{H}^+$ as previously defined. Let $\gamma : [a,b] \to \mathbb{H}^+$ be given by $\gamma(t) = (0,t)$ and suppose $\alpha : [a,b] \to \mathbb{H}^+$ satisfies $\alpha(a) = (0,a)$ and $\alpha(b) = (0,b)$ and write $\alpha(t) = (x(t), y(t))$. Then:

$$\ell(\alpha) = \int_a^b |\dot{\alpha}|dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\frac{dt}{y} \geq \int_a^b \left|\frac{dy}{dt}\right|\frac{dt}{y} \geq \int_a^b \frac{dy}{y} = \ell(\gamma)$$

thus γ is a geodesic. Recall that (by the calculation in example 3.3.1.xi) we can write:

$$g = -4\frac{dzd\overline{z}}{(z-\overline{z})^2}$$

.

Then any map $\varphi : \mathbb{H}^+ \to \mathbb{H}^+$ of the form:

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1$$

is an isometry that sends the y-axis to a semicircle whose diameter lies on the x-axis, and in fact for every $p \in \mathbb{H}^+$ and $v \in T_p\mathbb{H}^+$ we can construct such a circle, giving us a full description of geodesics in $\mathbb{H}^+$.

(4) (Lie groups) Suppose G is a Lie group with bi-invariant metric g. Let X be a left-invariant vector field on G and suppose γ is an integral curve for X through p, i.e. $\dot{\gamma}(t) = X(t)$. Then γ is a geodesic as $\nabla_{\dot{\gamma}}\dot{\gamma} = \nabla_X X = \frac{1}{2}[X,X] = 0$.

When the metric is not bi-invariant, integral curves for left-invariant vector fields are not necessarily geodesics (the computation for $\mathbb{H}^+$ above gives a counterexample).

(5) (Riemannian submersions) Suppose $F : M \to N$ is a Riemannin submersion. Then we may compute the geodesics of N by lifting to M using the following:

(a) Fix $p \in M$ and let $\alpha : (a,b) \to N$ be a geodesic with $\alpha(0) = F(p)$. Then there exists a unique horizontal lift $\gamma : (a,b) \to M$ such that $\gamma(0) = p$ such that γ is a geodesic on M.

(b) If $\gamma : (a,b) \to M$ is a geodesic and $\dot{\gamma}(0)$ is horizontal then $\dot{\gamma}(t)$ is horizontal for all $t \in (a,b)$ and $F \circ \gamma : (a,b) \to N$ is a geodesic and $\ell(F \circ \gamma) = \ell(\gamma)$.

Proof. (a) Given $t_0 \in (a,b)$ there exists $\epsilon > 0$ such that $L = \alpha((t_0 - \epsilon, t_0 + \epsilon))$ is a smooth 1 dimension submanifold of N. Then $V = F^{-1}(L)$ is a smooth 1-dimension submanifold of M. Let X be the horizontal vector field on V defined by:

$$X(x) = (d_xF)^{-1}(\dot{\alpha}(F(x)))$$

Define γ to be the integral curve for X through p. Then one can check that γ is a geodesic.

(b) Let $v = d_{\gamma(0)}F(\dot{\gamma}(0))$ and define $\alpha(t) = \alpha(F(\gamma(0)), v, t)$ to be the geodesic through $F(\gamma(0))$ in the direction of v. Let β be the unique horizontal lift of α from part i. Then $\beta(0) = \gamma(0)$ and $\dot{\beta}(0) = \dot{\gamma}(0)$ so by uniqueness we have $\beta = \gamma$. Since β was a lift of α we have $\alpha = F \circ \beta = F \circ \gamma$ as claimed.

□

We use this to compute the geodesics on $\mathbb{C}P(2)$. First we note that as a real inner product space, under the standard inner product on $\mathbb{C}^3$ we have for all $v \in \mathbb{C}^3$ that $\langle v, iv\rangle = 0$. Then we can extend any $v \neq 0$ to an orthonormal basis on $\mathbb{C}^3$ of the form $\{v, iv, w_3, w_4, w_5, w_6\}$. If $v \in S^5 \hookrightarrow \mathbb{C}^3$ then $T_vS^5 \cong \text{span}\{iv, w_3, w_4, w_5, w_6\}$. If $p : S^5 \to \mathbb{C}P(2)$ is the natural projection $z \mapsto [z]$ then $d_vp(iv) = 0$ so $T_{[v]}\mathbb{C}P(2) \cong \text{span}\{w_3, w_4, w_5, w_6\} = \{v, iv\}^{\perp}$.

Now suppose $[v] \in \mathbb{C}P(2)$ and let $w \in T_{[v]}\mathbb{C}P(2)$ i.e. $w \in \{v, iv\}^{\perp}$. Then by the previous result we have that $\alpha(t) = \gamma([v], w, t) = p(\cos(t)v + \sin(t)w)$ and so all geodesics are periodic with period π as:

$$\alpha(t+\pi) = p(\cos(t+\pi)v + \sin(t+\pi)w) = p(-\cos(t)v - \sin(t)w) = p(\cos(t)v + \sin(t)w) = \alpha(t)$$

Let $\alpha, \gamma : (0, \pi) \to \mathbb{C}P(2)$ be two geodesics with $\alpha(0) = \gamma(0) = [v]$, let $u = \dot{\alpha}(0)$ and $w = \dot{\gamma}(0)$. Then we have two cases. Either $u = \lambda w$ for $\lambda \in \mathbb{C}$ or not. If $u = \lambda w$ then $[u] = [w]$ so $p(u) = p(w)$.

i) If $u = \lambda w$ then:

$$\alpha(\frac{\pi}{2}) = p(u) = p(w) = \gamma(\frac{\pi}{2}).$$

ii) If $u \neq \lambda w$ then $\alpha(\pi) = \gamma(\pi)$ and $\alpha(t) \neq \gamma(t)$ for $0 < t < \pi$.

Notice that a geodesic is only minimizing on $[t_0, t_0 + \frac{\pi}{2}]$ since $\beta(t) = \gamma(t_0 - t)$ will be a shorter path connecting $\gamma(t_0)$ to $\gamma(t_0 + t_1)$ as it has length $t_1 - \frac{\pi}{2}$ for $t_1 > \frac{\pi}{2}$.

6.4. **Riemannian manifolds as a metric space.**

Proposition 50. *The topology induced by the metric $d(p, q) = \inf_\gamma \ell(\gamma)$ is homeomorphic to the manifold topology on M.*

Corollary 51. *M with the metric topology is a complete metric space if and only if for all $p, q \in M$ there exists a geodesic $\gamma : [a, b] \to M$ with $\gamma(a) = p$, $\gamma(b) = q$.*

Theorem 52. *Suppose $U \subset M$ is open and $r : U \to \mathbb{R}$ is a distance function. Then the integral curves for $\partial_r = \nabla r$ are geodesics.*

Proof. Since $|\nabla r| \equiv 1$ we see that $\nabla_{\partial_r}\partial_r \equiv 0$. Then if γ is an integral curve for ∂_r, since $\dot{\gamma} = \partial_r$ we conclude $\nabla_{\dot{\gamma}}\dot{\gamma} \equiv 0$. □

Theorem 53 (Hopf-Rinow)**.** *The following are equivalent:*

1. *For all $p \in M$, $\exp_p$ is defined for all $v \in T_pM$.*
2. *There exists $p \in M$ such that $\exp_p$ is defined for all $v \in T_pM$.*
3. *Every closed and bounded subspace of M is compact.*
4. *(M, d) is a complete metric space.*

Moreover any of the above imply that for all $p, q \in M$ there exists a minimizing geodesic connecting p to q.

7. Jacobi fields and variations.

Let $N \in S^2$ be the north pole. The tangent space $T_N S^2$ has an orthonormal basis $v_1 = \frac{\partial}{\partial x}$, $v_2 = \frac{\partial}{\partial y}$ with respect to the usual round metric on S^2. Notice that:

$$||tv_1 - tv_2|| \geq d(\exp_N(tv_1), \exp_N(tv_2))$$

with strict inequality for $t > 0$. That is to say, two orthogonal geodesics starting at N get "closer together".

Similarly, consider $i \in \mathbb{H}^+$. The tangent vectors $v_1 = \frac{\partial}{\partial x}$, $v_2 = \frac{\partial}{\partial y}$ form an orthonormal basis for $T_i\mathbb{H}^+$ with its hyperbolic metric but this time:

$$||tv_1 - tv_2|| \leq d(\exp_i(tv_1), \exp_i(tv_2))$$

i.e. the geodesics spread further apart.

Finally, $v_1 = \frac{\partial}{\partial x}$, $v_2 = \frac{\partial}{\partial y}$ form an orthonormal basis for $\mathbb{R}^2$ with its Euclidean metric and:

$$||tv_1 - tv_2|| = d(\exp_0(tv_1), \exp_0(tv_2)).$$

Recall that S^2 has constant curvature equal to 1, $\mathbb{H}^+$ has constant curvature equal to -1, and $\mathbb{R}^2$ has constant curvature equal to 0.

7.1. Jacobi fields.

Given $p \in M$ let $\epsilon > 0$ such that $B_p(\epsilon)$ is a normal neighbourhood, let $\gamma_1(t) = \gamma_1(p, v, t)$ and $\gamma_2(t) = \gamma_2(p, w, t)$ for some $v, w \in T_pM$ such that $\exp_p(v)$ and $\exp_p(w)$ are defined. Suppose $\alpha : I \to T_pM$ is a smooth curve with $\alpha(0) = v$, $\dot{\alpha}(0) = w - v$. Then:

$$J(t) = d_{tv} \exp_p(t(w - v))$$

is a vector field along γ_1 and (by the Gauss lemma):

$$\exp_p(J(t)) = \gamma_2(t)$$

So we see $|J(t)|$ measures the rate that γ_1 and γ_2 diverge in M. A quick computation shows that:

$$D^2_{\gamma_1} J + R(\dot{\gamma_1}, J)\dot{\gamma}_1 = 0.$$

Definition 54. Given an open $U \subset M$ and a distance function $r : U \to \mathbb{R}$ a **Jacobi field for** r is a vector field $J \in \mathcal{X}(U)$ such that:

$$\nabla_{\partial_r} \nabla_{\partial_r} J = -R(J, \partial_r)\partial_r$$

Equivalently we can define Jacobi fields as follows. Let $\gamma : [0, a] \to M$ be a geodesic. Then a vector field J along γ is called a **Jacobi field** if it satisfies the **Jacobi equation**:

$$D^2_\gamma J + R(\dot{\gamma}, J)\dot{\gamma} = D_\gamma D_\gamma J + R(\dot{\gamma}, J)\dot{\gamma} = 0.$$

Example 55.

(1) For any geodesic γ, $\dot{\gamma}$ and $t\dot{\gamma}$ are both clearly Jacobi fields.
(2) Suppose M has constant curvature equal to k and let J be a Jacobi field along some γ. Then:

$$g(R(\dot{\gamma}, J)\dot{\gamma}, V) = K(g(\dot{\gamma}, \dot{\gamma})g(J, V) - g(\dot{\gamma}, V)g(J, \dot{\gamma})) = kg(J, V)$$

thus we have $R(\dot{\gamma}, J)\dot{\gamma} = kJ$ therefore we have reduced the Jacobi equation to:

$$D^2_\gamma J + kJ = 0,$$

an easily solvable system.

Proposition 56. *Fix $p \in M$ and let $\gamma : [0,a] \to M$ be a geodesic with $\dot{\gamma}(0) = v$. Let J be a Jacobi field along γ such that $J(0) = 0$. Let $w = D_\gamma J(0)$ and $\alpha : I \to T_pM$ be a smooth curve with $\alpha(0) = av$, $\dot{\alpha}(0) = w$. Define $f(t,s) = \exp_p(\frac{t}{a}\alpha(s))$ and define $\overline{J}(t) = \frac{\partial f}{\partial s}(t,0)$. Then $\overline{J} = J$.*

Thus we see the only Jacobi fields along a geodesic γ satisfying $J(0) = 0$ are the kind described at the beginning of this section.

We now give the first fundamental relationship between curvature and geodesics.

Proposition 57. *Fix $p \in M$ and let $\gamma = \gamma(p,v,t)$. Let $w \in T_v(T_pM)$ satisfy $|w| = 1$ and define $J(t) = d_{tv}\exp_p(tw)$. Then:*

$$|J(t)|^2 = t^2 - \frac{1}{3}R(v,w,v,w)t^4 + r(t)$$

where $r \in \mathcal{O}(t^4)$.

Corollary 58. *If γ is as in the previous proposition but is also parametrized by arc length (i.e. $|\dot{\gamma}(0)| = |v| = 1$) then:*

$$|J(t)|^2 = t^2 - \frac{1}{3}\sec_p(\sigma)t^4 + r(t).$$

where σ is the 2-plane generated by v, w. Hence:

$$|J(t)| = t - \frac{1}{6}\sec_p(\sigma)t^3 + r(t)$$

where $r(t) \in \mathcal{O}(t^3)$.

The equation $|J(t)| = t - \frac{1}{6}\sec_p(\sigma)t^3 + R(t)$ gives the desired relationship between curvature and geodesics. It says that the rate of divergence of geodesics in M originating at p differs from the respective geodesics in T_pM originating at 0 by $\frac{1}{6}\sec_p(\sigma)$ times a third order term. This explains the behaviour outlined previously - on manifolds with positive constant curvature $|J(t)| \leq t$ i.e. the rate of divergence is *smaller* than it would be in T_pM, and on manifolds with negative constant curvature $|J(t)| \geq t$ i.e. the rate of divergence is *larger* than it would be on T_pM.

Proposition 59. *Let J be a Jacobi field along γ. Then:*

$$g(J(t), \dot{\gamma}(t)) = g(J'(0), \dot{\gamma}(0))t + g(J(0), \dot{\gamma}(0))$$

Definition 60. If $\gamma : [0,a] \to M$ is a geodesic and $t_0 \in [0,a)$ we say that $\gamma(0)$ is **conjugate** to $\gamma(t_0)$ if there exists Jacobi field J along γ that is not identically zero such that $J(0) = 0$ and $J(t_0) = 0$.

Proposition 61. *Let $\gamma : [0,a] \to M$ be a geodesic with $v_1 \in T_{\gamma(0)}M$, $v_2 \in T_{\gamma(a)}M$. If $\gamma(0)$ is not conjugate to $\gamma(a)$ then there exists a unique Jacobi field J such that $J(0) = v_1$, $J(a) = v_2$.*

7.2. **Variations.** Having a relationship between the curvature and geodesics, we seek to develop relationships between curvature and topology.

Definition 62. Let $\gamma : [0,a] \to M$ be a smooth curve. A **variation** of γ is a continuous function $f : (-\epsilon,\epsilon) \times [0,a] \to M$ such that $f(0,t) = \gamma(t)$ and there exists a partition $0 = t_0 < t_1 < \cdots < t_{k+1} = a$ such that $f|_{(t_i,t_i+1)}$ is smooth for all $i = 0,...,k$. We say that f is **proper** if $f(s,0) = \gamma(0)$ and $f(s,a) = \gamma(a)$ for all $s \in (-\epsilon,\epsilon)$.

The function $f_s(t) = f(s,t)$ is called a **curve in the variation**. The function $f_t(s) = f(s,t)$ is called a **transversal curve of the variation**. The **variational field of** f is defined to be the vector field $V(t)$ along γ defined by $V(t) = \frac{\partial f}{\partial s}(0,t)$.

Proposition 63. *If V is a piecewise smooth vector field along a smooth curve γ there exists a variation f of γ such that V is the variational field of f. If moreover $V(0) = V(a)$ then f can be chosen to be a proper variation.*

Definition 64. Given a smooth curve γ and a variation f of γ the **length of** f_s denoted $L(s)$ is:

$$L(s) = \int_0^a \left|\frac{\partial f}{\partial t}(s,t)\right| dt$$

and the **energy of** f_s denoted $E(s)$ is:

$$E(s) = \int_0^a \left|\frac{\partial f}{\partial t}(s,t)\right|^2 dt$$

For notational simplicity we also define the following two quantities:

$$L(\gamma) = \int_0^a |\dot{\gamma}(t)| dt, \qquad E(\gamma) = \int_0^a |\dot{\gamma}(t)|^2 dt$$

By the Cauchy-Schwarz inequality on $L^2([0,a])$ we have:

$$L(\gamma)^2 = \left(\int_0^a \left|\frac{\partial f}{\partial t}(s,t)\right| dt\right)^2 \leq \int_0^a dt \int_0^a \left|\frac{\partial f}{\partial t}(s,t)\right|^2 dt = aE(\gamma)$$

with equality if and only if $|\frac{\partial f}{\partial t}(s,t)|$ is constant.

Lemma 65. *Let $p, q \in M$ and let $\gamma : [0,a] \to M$ be a minimal geodesic connecting p to q. If $\alpha : [0,a] \to M$ is a smooth curve connecting p to q then:*

$$E(\gamma) \leq E(\alpha)$$

with equality if and only if α is a minimal geodesic.

Theorem 66 (First Variation)**.** *If $\gamma : [0,a] \to M$ is a piecewise smooth curve and f is a variation of γ then:*

$$\frac{1}{2}E'(0) = -\int_0^a g(V(t), D_\gamma \dot{\gamma}(t)) dt - g(V(0), \dot{\gamma}(0)) + g(V(a), \dot{\gamma}(a)) - \sum_{i=1}^k g(V(t_i), \dot{\gamma}(t_i^+) - \dot{\gamma}(t_i^-))$$

where $\dot{\gamma}(t_i^+), \dot{\gamma}(t_i^-)$ are the right hand and left hand limits, respectively.

Corollary 67. *A piecewise smooth curve $\gamma : [0,a] \to M$ is a geodesic if and only if for every proper variation of γ we have $E'(0) = 0$.*

The first variation formula characterises geodesics as critical points of the energy functional E. If γ is a geodesic connecting a to b then any nearby curves also conencting a to b will have lengths at most equal to that of γ, so $L(0)$ is a local minimum. Since $L(\gamma)^2 \leq aE(\gamma)$ we can see then that being a critical point of E is being a local minimum of E, so one could say that geodesics are energy minimizers.

Theorem 68 (Second Variation)**.** *With the same conditions as theorem 9.2.5:*

$$\frac{1}{2}E''(0) = -\int_0^a g\left(V(t), D_\gamma^2 V(t) + R(\dot{\gamma}(t), V(t))\dot{\gamma}(t)\right) dt - \sum_{i=1}^k g(V(t_i), D_\gamma V(t_i^+) - D_\gamma V(t_i^-))$$

8. Sectional Curvature and comparisons.

We will now use the results of the previous sections to deduce some general topological properties of Riemannian manifolds with nonpositive sectional curvature, and Riemannian manifolds with nonnegative sectional curvature.

8.1. Nonpositive sectional curvature.

Theorem 69 (Ambrose)**.** *If $f: M \to N$ is a local isometry with M complete and N connected then N is complete and f is a covering map.*

Lemma 70. *If (M, g) is complete and $p \in M$, if $d_v \exp_p$ is invertible for all $v \in T_pM$ then $\exp_p$ is a covering map.*

Proof. Since $\exp_p$ is invertible everywhere it is an immersion so we define a metric g' on T_pM by $g'((u,v))_w = g(d_w \exp_p(u), d_w \exp_p(v))_{\exp_p(w)}$. This metric makes $\exp_p$ into a local Riemannian isometry. But $\exp_0 : T_0T_pM \to T_pM$ is defined for all $v \in T_0T_pM$ so (T_pM, g') is complete hence $\exp_p$ is a covering map. □

Theorem 71 (Cartan-Hadamard)**.** *If M is complete and connected with $\sec_p(\sigma) \leq 0$ for all $p \in M$ and 2-plane $\sigma \subset T_pM$, then $\exp_p : T_pM \to M$ is a covering map (hence $\mathbb{R}^n$ is the universal cover).*

Proof. We need to show that $d_w \exp_p$ is invertible for all $p \in M$, $w \in T_pM$. To that end we show $|d_w \exp_p(v)| > 0$. So fix $p \in M$, $w \in T_pM$ and let $\gamma(t) = \exp_p(tw)$. Let J be a Jacobi field along γ such that $J(0) = 0$ and $J'(0) = v$ so that $|J(1)| = |d_w \exp_p(v)|$. Then:

$$\langle J, J\rangle'' = 2\langle J', J'\rangle + 2\langle J'', J\rangle = 2\langle J', J'\rangle - 2\langle R(\dot\gamma, J)\dot\gamma, J\rangle = 2|J'|^2 - 2\sec_p(\dot\gamma, J) \geq 2|J'|^2$$

Integrating this inequality we get:

$$\langle J, J\rangle'(t) - \langle J, J\rangle'(0) \geq 2\int_0^t |J'(t)|^2$$

But $\langle J, J\rangle'(0) = \langle J'(0), J(0)\rangle = 0$ so this becomes:

$$\langle J, J\rangle' \geq 2\int_0^t |J'|^2 > 0$$

thus $\langle J, J\rangle > 0$ i.e. $d_v \exp_p$ is invertible for all $v \in T_pM$.

□

If M is complete and simply connected with $\sec \leq 0$ the previous theorem tells us that $\exp_p$ is a diffeomorphism for all $p \in M$. Hence we can take $B_p(\epsilon) = M$ so $r(x) = d(x,p)$ is smooth on $M \setminus p$ and $f = \frac{1}{2}r^2$ is smooth on M with $\text{Hess} f = dr^2 + r\text{Hess}r$.

Jacobi fields are a useful tool for computing the Hessian of distance functions. To see this suppose γ is a geodesic such that $\gamma(0) = p$ and let J be a Jacobi field along γ with $J(0) = 0$. If $\alpha(t) = \exp_p(t\dot\gamma(0))$ is a minimizing geodesic for $0 \leq t < 1$ then $\dot c(t) = \nabla r(c(t))$ so:

$$\text{Hess}r(J(t), J(t)) = g(\nabla_{J(t)}\nabla r, J(t)) = g(\frac{\partial^2\overline{\gamma}}{\partial s \partial t}, J)(0,t) = g(\frac{\partial^2\overline{\gamma}}{\partial s \partial t}, J)(0,t) = g(\dot J(t), J(t))$$

Since J can be arbitrary this allows us to compute $\text{Hess}r$ in general. But:

$$g(\dot J(t), J(t)) \geq \int_0^t |\dot J(s)|^2 ds > 0$$

so $\text{Hess}r$ is positive definite. Therefore:

$$\frac{d^2}{dt} f(\gamma(t)) = \frac{d}{dt} g(\nabla f, \dot\gamma) = g(\nabla_{\dot\gamma}\nabla f, \dot\gamma) + g(\nabla f, \ddot\gamma) = \text{Hess} f(\dot\gamma, \dot\gamma) > 0$$

Theorem 72 (Cartan)**.** *If M is a complete simply connected Riemannian manifold with $\sec_p(\sigma) \le 0$ then any isometry of finite order has a fixed point.*

Proof. Let F be an isometry of M of order k. Fix $p \in M$. Given $q \in M$ define $f_q(x) = \frac{1}{2} d(x, q)^2$. Let x so that the following function $g(x)$ is minimized:

$$g(x) = \max\{f_p(x), f_{F(p)}(x), ..., f_{F^{k-1}(p)}(x)\}$$

The existence of this x follows from the calculations proceeding the statement of the theorem and moreover x is unique. However:

$$g(F(x)) = \max\{f_p(F(x)), f_{F(p)}(F(x)), ..., f_{F^{k-1}(p)}(F(x))\} = g(x)$$

so by uniqueness $F(x) = x$ as claimed.

□

Corollary 73. *If M is a complete with $\sec \le 0$ then the fundamental group is torsion free.*

Proof. We induce the usual covering metric on $\widetilde{M}$ making it into a simply connected Riemannian manifold with $\sec \le 0$. By theorem 10.1.3 any isometry of $\widetilde{M}$ with finite order has a fixed point. But isometries of $\widetilde{M}$ are a foritori deck transformations thus cannot have any fixed points hence the fundamental group of M cannot have any elements of finite order. □

Theorem 74 (Preissman)**.** *Let M be a compact Riemannian manifold with $\sec_p(\sigma) < 0$ for all $p \in M$, 2-plane $\sigma \subset T_pM$. If $H \le \pi_1(M)$ is nontrivial and abelian then $H \cong \mathbb{Z}$.*

Corollary 75. *If M, N are compact then there cannot exist a Riemannian metric on $M \times N$ with negative sectional curvature.*

Proof. For contradiction suppose g has negative sectional curvature so the fundamental group is torsion free. Since the fundamental group is torsion free, the fundamental theorem of finitely generated abelian group gives us that $\pi_1(M \times N) \cong \mathbb{Z}^m$. If $m \ge 2$ then we would have an abelian subgroup that isn't cyclic, contradicting Preissman's theorem. Then any abelian subgroup A must be equal to 0 or $\mathbb{Z}$. In either case it must factor into subgroups $A = A_M \times A_N$ for $A_m \le \pi_1(M)$, $A_N \le \pi_1(N)$ so one of A_M, A_N must be $\{0\}$. With no loss in generality assume it is A_M. Since A was arbitrary $\pi_1(M)$ has no abelian subgroups so M is simply connected.

The universal cover of $M \times N$ is the cartesian product of their universal covers. But by the Cartan-Hadamard theorem, the universal cover of $M \times N$ is $\mathbb{R}^n$, and by the previous paragraph the universal cover is M is M, and M is compact, a contradiction.

□

8.2. **Comparisons.** Before dealing with positive sectional curvature we need an important inequality relating the curvature to the metric.

Theorem 76 (Riccati / Rauch Comparison)**.** *Let M is a Riemannian manifold of dimension $n+1$ with $k \le \sec_p \le K$ and in exponential coordinates write $g = dr^2 + g_r$. Then:*

$$sn_K^2(r) ds_n^2 \le g_r \le sn_k^2(r) ds_n^2$$
$$\frac{sn'_K(r)}{sn_K(r)} g_r \le Hess r \le \frac{sn'_k(r)}{sn_k(r)} g_r$$

8.3. **Positive sectional curvature.**

Theorem 77 (Bonnet-Synge). *Suppose M is a Riemannian manifold with $0 < k \leq \sec_p(\sigma)$ for all $p \in M$ and 2-plane $\sigma \subset T_pM$. Then any geodesic γ such that $\ell(\gamma) > \frac{\pi}{\sqrt{k}}$ is not length minimizing.*

Proof. An application of the second variation of energy. Let $\gamma : [0, a] \to M$ be a geodesic of length $\ell > \frac{\pi}{\sqrt{k}}$ and let $V(t) = \sin(\frac{\pi}{\ell}t)E(t)$ where E is a parallel field along γ. Since $V(0) = V(a) = 0$ there exists a proper variation f of γ such that V is the variational field. By theorem 9.2.7:

$$\begin{aligned}
\frac{1}{2}E''(0) &= \int_0^a \langle V'(t), V'(t)\rangle dt - \int_0^a (V(t), \dot\gamma(t), \dot\gamma(t), V(t))dt \\
&= (\frac{\pi}{\ell})^2 \int_0^a \cos^2(\frac{\pi}{\ell}t)dt - \int_0^a \sin^2(\frac{\pi}{\ell}t)\sec(E(t), \dot\gamma)dt \\
&\leq (\frac{\pi}{\ell})^2 \int_0^a \cos^2(\frac{\pi}{\ell}t)dt - k\int_0^a \sin^2(\frac{\pi}{\ell}t)dt \\
&< k\int_0^a \cos^2(\frac{\pi}{\ell}t)dt - k\int_0^a \sin^2(\frac{\pi}{\ell}t)dt \\
&= 0
\end{aligned}$$

Thus any curve α in the variation of f has $\ell(\alpha) < \ell(\gamma)$ so γ is not distance minimizing as claimed. □

Corollary 78. *If M is a complete Riemannian manifold and $0 < k \leq \sec_p(\sigma)$ then M is compact with diam$(M) \leq \frac{\pi}{\sqrt{k}}$ and $|\pi_1(M)| < \infty$.*

Proof. By the Bonnet-Synge theorem M is bounded so by the Hopf-Rinow theorem M is compact. With the metric induced by the covering map, applying the Bonnet-Synge theorem to $\widetilde{M}$ allows us to conclude the same things, thus $|\pi_1(M)| < \infty$. □

Theorem 79 (Myers). *If M is a complete Riemannian manifold and $0 < k(n-1) \leq Ric_p(\sigma)$ then M is compact with diam$(M) \leq \frac{\pi}{\sqrt{k}}$ and $|\pi_1(M)| < \infty$.*

Proof. Let $\gamma : [0, a] \to M$ be a geodesic of length $\ell > \frac{\pi}{\sqrt{k}}$ and define $V_i(t) = \sin(\frac{\pi}{\ell}t)E_i(t)$ for $i > 1$ where E_i are parallel along γ and $\dot\gamma(t), E_2(t), ..., E_n(t)$ are an orthonormal basis for $T_{\gamma(t)}M$. By the second variational formula:

$$\begin{aligned}
\sum_{i=2}^n E''(0) &= \sum_{i=2}^n \int_0^a \langle V_i'(t), V_i'(t)\rangle dt - \int_0^a (V_i(t), \dot\gamma(t), \dot\gamma(t), V_i(t))dt \\
&= (n-1)(\frac{\pi}{\ell})^2 \int_0^a \cos^2(\frac{\pi}{\ell}t)dt - \int_0^a \sin^2(\frac{\pi}{\ell}t)\mathrm{Ric}(\dot\gamma(t), \dot\gamma(t))dt \\
&< (n-1)k\int_0^a \cos^2(\frac{\pi}{\ell}t)dt - (n-1)k\int_0^a \sin^2(\frac{\pi}{\ell}t)dt \\
&< 0
\end{aligned}$$

so as before γ is not distance minimizing. By repeating the argument in 10.3.2 we conclude that M is compact with diam$(M) \leq \frac{\pi}{\sqrt{k}}$ and $|\pi_1(M)| < \infty$. □

Theorem 80 (Synge). *Suppose M is compact and $\sec_p(\sigma) > 0$. Then:*

(1) If $\dim(M) = 2k$ and M is orientable then M is simply connected.
(2) If $\dim(M) = 2k+1$ then M is orientable.

Corollary 81. *Suppose M is a compact Riemannian manifold with $\dim(M) = 2k$, $\sec_p > 0$ and M is not orientable. Then $\pi_1(M) = \mathbb{Z}_2$.*

Proof. Let $\widetilde{M}$ denote the oriented double cover of M with the induced Riemannian metric. It is clear that $\widetilde{M}$ is compact, orientable, and has $\sec_p > 0$. By Synge's theorem we conclude that $\widetilde{M}$ is simply connected and is thus the universal cover of M. But $\widetilde{M}$ is double cover so $\pi_1(M) \cong \mathbb{Z}_2$ as claimed. $\square$

This tells us, for example, that $\mathbb{R}P(2) \times \mathbb{R}P(2)$ does not admit a metric with $\sec_p > 0$ as $\pi_1(\mathbb{R}P(2) \times \mathbb{R}P(2)) = K_4 \neq \mathbb{Z}_2$. Meanwhile, Preissman's theorem gave us that there cannot exist a metric on $\mathbb{R}P(2) \times \mathbb{R}P(2)$ with $\sec_p < 0$.

Theorem 82. *If $\sec_p \leq K$ for some $K > 0$ then $\exp_p : B_{\frac{\pi}{K}}(0) \to M$ has no critical points.*

Theorem 83. *If M is an orientable Riemannian manifold with $\dim(m) = 2k$ and $0 < \sec_p \leq 1$ then inj$(M) \geq \pi$. If M is not orientable then inj$(M) \geq \frac{\pi}{2}$.*

Theorem 84. *If $f : M \to \mathbb{R}$ is smooth and proper, b is not a critical value and all critical points in $f^{-1}([a, b])$ have index $\geq m$ then:*

$$f^{-1}((-\infty, a]) \subset f^{-1}((-\infty, b])$$

Theorem 85. *If M is a complete Riemannian manifold and $A \subset M$ is a compact submanifold such that every geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0), \gamma(1) \in A$ has index $\geq k$ then A is k-connected.*

Theorem 86 (Sphere theorem)**.** *If M is a closed n dimensional Riemannian manifold with $\sec_p \geq 1$ and inj$_p > \frac{\pi}{2}$ for some $p \in M$ then M is $(n-1)$ connected and homotopy equivalent to S^n.*

Corollary 87. *If M is a closed simply connected n-dimension Riemannian manifold such that $1 \leq \sec_p < 4$ then M is $(n-1)$ connected and homotopy equivalent to S^n.*

Corollary 88. *If M is a closed n-dimension Riemannian manifold with Ric $\geq (n-1)$ and inj$_p > \frac{\pi}{2}$ for some $p \in M$ then M is simply connected.*

Lemma 89. *Let M be a complete n-dimension Riemannian manifold with $\sec_p > 0$.*

(1) If $N \subset M$ is a totally geodesic $n-k$-dimension submanifold then N is $(n-2k+1)$-connected.
(2) If N_1, N_2 are totally geodesic submanifolds of M of dimensions $n - k_1$, $n - k_2$ such that $k_1 \leq k_2$ and $k_1 + k_2 \leq n$ then $N_1 \cap N_2$ is a nonempty totally geodesic $(n - k_1 - k_2)$-connected. submanifold.

9. Bochner techniques

Given a smooth manifold M, the de Rham theorem provides a link between smooth differential forms and the algebraic topology of M. Fixing a Riemannian metric, we will relate its curvature to differential forms on M, allowing us to derive more topological constraints.

9.1. Killing Fields.

Definition 90. Let $X \in \mathcal{X}(M)$ and let $\Phi_t : M \to M$ be the flow of X with $t \in U \subset \mathbb{R}$. If Φ_t is an isometry for all $t \in U$ then we say X is a **Killing field**.

Proposition 91. *A vector field $X \in \mathcal{X}(M)$ is a Killing field if and only if $L_X g \equiv 0$ if and only if $v \mapsto \nabla_x X$ is a skew symmetric $(1,1)$ tensor.*

Proposition 92. *For any $p \in M$, if X is a Killing field then X is uniquely determined by $X(p)$ and $(\nabla X)(p)$.*

Theorem 93. *The zero set of a Killing field is a disjoint union of totally geodesic submanifolds, each of which has even codimension.*

Theorem 94. *The space of Killing fields $\mathfrak{iso}(M)$ is a Lie algebra of dimension $\leq n(n+1)/2$. If M is compact then $\mathfrak{iso}(M)$ is the Lie algebra of $Iso(M)$. Moreover if M is complete and the dimension of $Iso(M)$ is exactly $n(n+1)/2$ then M has constant curvature.*

9.2. Killing fields and negative Ricci curvature.

Proposition 95. *Given a $(1,1)$ tensor T we define $|T|^2 = tr(T \circ T*) = \sum_{i=1}^{n} g(T(E_i), T(E_i))$ where $\{E_1, ..., E_n\}$ is an orthonormal frame. Let X denote a Killing field and define $f : M \to \mathbb{R}$ by $f(p) = \frac{1}{2} g(X(p), X(p))$. Then:*

(1) $\nabla f = -\nabla_X X$
(2) $Hessf(V,V) = g(\nabla_V X, \nabla_V X) - R(V,X,X,V)$
(3) $\Delta f = |\nabla X|^2 - Ric(X,X)$

Theorem 96. *If M is compact, oriented, and $Ric \leq 0$ then every Killing field is parallel. If moreover $Ric < 0$ then every Killing field vanishes identically.*

Corollary 97. *If M is as in the previous theorem and $p = \dim(Iso(M))$ then $\widetilde{M} = \mathbb{R}^p \times N$.*

9.3. Killing fields and positive curvature.

Theorem 98. *If M is a compact even dimension Riemannian manifold with positive curvature then every Killing field has a zero.*

Theorem 99. *If there exists a nontrivial Killing field X on a compact manifold without boundary M then the fundamental group of M has a cyclic subgroup of index $\leq c(n)$.*

Theorem 100. *If X is a Killing field on a compact Riemannian manifold M and N_i are the components for the zero set of X:*

(1) $\chi(M) = \sum_i \chi(N_i)$
(2) $\sum_p b_{2p}(M) \geq \sum_i \sum_p b_{2p}(N_i)$
(3) $\sum_p b_{2p+1}(M) \geq \sum_i \sum_p b_{2p+1}(N_i)$

Corollary 101. *If M is a compact Riemannian manifold of dimension 6 with positive sectional curvature that has a nontrivial Killing field then $\chi(M) > 0$. If M is compact orientable of dimension 4 with positive sectional curvature that has a nontrivial Killing field then $\chi(M) \leq 3$ and therefore is either S^4 or $\mathbb{C}P(2)$.*

Definition 102. The **symmetry rank** of a compact Riemannian manifold is the rank (as a compact Lie group) of $\text{Iso}(M)$. If $\mathfrak{h}(M) \subset \mathfrak{iso}(M)$ is an Abelian subalgebra with $\dim(\mathfrak{h}(M))$ equal to the symmetry rank of M define $\mathfrak{Z}(\mathfrak{h}(M))$ to be the components for the zero sets of the Killing fields in $\mathfrak{h}(M)$.

Proposition 103.

(1) If $N \in \mathfrak{Z}(\mathfrak{h}(M))$ then all Killing fields in $\mathfrak{h}(M)$ are tangent to N.
(2) $N \in \mathfrak{Z}(\mathfrak{h}(M))$ is maximal with respect to inclusion if and only if the restriction of $\mathfrak{h}(M)$ to N has dimesion equal to $\dim(\mathfrak{h}(M)) - 1$.
(3) If $N \in \mathfrak{Z}(\mathfrak{h}(M))$ then N is contained in finitely many maximal sets $N_1, ..., N_m$ and $N = N_1 \cap \cdots \cap N_m$.
(4) If $N, N' \in \mathfrak{Z}(\mathfrak{h}(M))$ then $N \cap N' \in \mathfrak{Z}(\mathfrak{h}(M))$.

9.4. **Hodge.**

Definition 104. Let M be a compact orientable Riemannian n dimension manifold. Write $\omega \in \Omega^k(M)$ as $\omega = f_0 \cdot df_1 \wedge \cdots \wedge df_k$ and define:

$$h(\omega_1, \omega_2) = h(f_0 \cdot df_1 \wedge \cdots \wedge df_k, g_0 \cdot dg_1 \wedge \cdots \wedge dg_k) = f_0 g_0 \det\left(g(\nabla f_i, \nabla h_j)_{1 \leq i,j \leq k}\right)$$

and then extend g to an inner product on $\Omega^k(M)$ by:

$$g(\omega_1, \omega_2) = \int_M h(\omega_1, \omega_2) dV$$

which we finally use to (implicitly) define the **Hodge star operator**:

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

by:

$$g(*\omega_1, \omega_2) = \int_M h(*\omega_1, \omega_2) dV = \int_M \omega_1 \wedge \omega_2$$

Lemma 105. *The Hodge star operator satisfies $*^2 = (-1)^{k(n-k)}$*

10. Symmetric Spaces and Holonomy

10.1. Symmetric Spaces.

Definition 106. Given a Riemannian manifold M and $p \in M$ we define the **isotropy group of M at p** denoted Iso_p to be the isometries $F : M \to M$ such that $F(p) = p$. If for all $p \in M$ there exists $F \in \text{Iso}_p$ such that $d_pF = -I$ then we call M **symmetric**. If for all $p, q \in M$ there exists $F \in \text{Iso}$ such that $F(p) = q$ we say that M is **homogeneous**.

Lemma 107. *If M is symmetric it is homogeneous and complete.*

The converse to the above is not true, but we do have the following:

Lemma 108. *If M is a homogeneous Lie group with a bi-invariant metric then M is symmetric.*

Definition 109. The **rank** of a geodesic γ is the dimension of the vector space of parallel vector fields E_i along γ such that $R(E_i(t), \dot{\gamma})\dot{\gamma} = 0$ for all i, t. The **rank** of M is the minimum rank over all geodesics in M.

Lemma 110. *If M is symmetric then $\nabla R \equiv 0$.*

Proof. Let $A_p \in \text{Iso}_p$ such that $d_pA_p = -I$. Then:

$$d_pA_p(\nabla_X R)(Y, Z, W) = (\nabla_{d_pA_pX} R)(d_pA_pY, d_pA_pZ, d_pA_pW)$$

hence:

$$-(\nabla_X R)(Y, Z, W) = (\nabla_{-X} R)(-Y, -Z, -W) = (\nabla_X R)(Y, Z, W)$$

i.e. $\nabla R = 0$.

□

Definition 111. If $\nabla R \equiv 0$ we call M **locally symmetric**.

Theorem 112 (Cartan)**.** *If M is locally symmetric then for all $p \in M$ there exists an isometry A_p defined in a neighbourhood U of p such that $d_pA_p = -I$. If moreover M is simply connected and complete A_p is defined on all of M so M is symmetric.*

Theorem 113 (Cartan)**.** *Suppose M is a simply connected symmetric Riemannian manifold and N is a complete locally symmetric Riemannian manifold and $\dim(M) = \dim(N)$. Fix $p \in M$ and $q \in N$ and let $T : T_pM \to T_qN$ be an isometry such that $T(R^M(x, y)z) = R^N(T(x), T(y))T(z)$ for all $x, y, z \in T_pM$. Then there exists a unique isometry $\varphi : M \to N$ such that $d_p\varphi = T$.*

Theorem 114. *M is symmetric if and only if there exists a Lie algebra $\mathfrak{g}$ and a linear involution $L : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ decomposes as a direct sum of the eigenspaces for L, and $M = G/K$ where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ is the Lie algebra of K and moreover $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{k}$, and $[\mathfrak{k}, \mathfrak{t}] \subset \mathfrak{t}$.*

Remark 115. We give two proofs of this fact, giving us two descriptions of symmetric spaces involving the decomposition of a Lie algebra. The first is the *algebraic description of symmetric spaces* and the second is the *curvature descripion of symmetric spaces.*

Proof. 1) Suppose M is symmetric, fix $p \in M$, and denote by $\mathfrak{iso}$ the Killing fields on M, and let $\mathfrak{iso}_p$ denote those Killing fields whose flows fix p (i.e. the Lie algebra of Iso_p). The map $X \mapsto (X(p), (\nabla X)(p))$ gives us an injection $\mathfrak{iso} \to T_pM\mathfrak{so}(T_pM)$ that is surjective onto T_pM. This leads to a Lie algebra isomorphism $\mathfrak{iso} \cong T_pM \times \mathfrak{iso}_p$. First note that $X \in T_pM$ if and only if $(\nabla X)(p) = 0$ and $X \in \mathfrak{iso}_p$ if and only if $X(p) = 0$. Then if we denote those X such that $(\nabla X)(p) = 0$ by $\mathfrak{t}_p$ we can write $\mathfrak{iso} = \mathfrak{t}_p \oplus \mathfrak{iso}_p$.

To see how the Lie bracket structure is, suppose $X, Y \in \mathfrak{t}_p$ or $X, Y \in \mathfrak{iso}_p$. Then $[X,Y](p) = 0$ thus $[X,Y] \in \mathfrak{iso}_p$. However if $X \in \mathfrak{t}_p$ and $Y \in \mathfrak{iso}_p$ then $[X,Y](p) = (\nabla Y)(X(p)) \in \mathfrak{t}_p$.

Finally we define the involution L. First let $\sigma : \mathrm{Iso} \to \mathrm{Iso}$ be defined by $\sigma(g) = A_p \circ g \circ A_p$. Then σ is an isomorphism satisfying $\sigma(g) = g$ if and only if $g \in \mathrm{Iso}_p$ and $\sigma^2 = I$ and so $d\sigma(h) = h$ if $h \in \mathfrak{iso}_p$ and $d\sigma(v) = -v$ for $v \in \mathfrak{t}_p$. Thus we may let $L = d\sigma$.

Conversely suppose $L : \mathfrak{g} \to \mathfrak{g}$ is an involution on the Lie algebra $\mathfrak{g}$ and write $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$ where $\mathfrak{t}$ is the eigenspace for eigenvalue -1 and $\mathfrak{k}$ is the eigenspace for eigenvalue 1. Since $L[X_1, X_2] = [L(X_1), L(X_2)] = [-X_1, -X_2] = [X_1, X_2]$ we have that $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{t}] \subset \mathfrak{t}$.

Letting G, K be Lie groups such that $K \leq G$ is compact, $\mathfrak{k}$ is the Lie algebra for K and $\mathfrak{g}$ is the Lie algebra for G we can make G/K into a simply connected Riemannian manifold as we can fix a bi-invariant metric on K inducing a Euclidean inner product on $\mathfrak{g}$. Recall the long exact sequence $\pi_1(K) \to \pi_1(G) \to \pi_1(G/K) \to \pi_0(K) \to \pi_0(G) \to 1$. Since G/K is simply connected $\pi_1(G/K) = 0$ so $\pi_0(K) \to \pi_0(G)$ is an isomorphism and $\pi_1(K) \to \pi_1(G)$ is surjective. This allows us to find an involution $\sigma : G \to G$ such that $d\sigma = L$.

2) If M is symmetric and $p \in M$, let $\mathfrak{r}_p$ denote the Lie algebra generated by maps of the form $R(x,y) : T_pM \to T_pM$ and then define $\mathfrak{g} = T_pM \oplus \mathfrak{r}_p$. If $x, y \in T_pM$ then we let $[x,y] = R(x,y) \in \mathfrak{r}_p$. If $x, y \in \mathfrak{r}_p$ then define $[x,y] = (y \circ x - x \circ y) \in \mathfrak{r}_p$. Finally if $x \in T_pM$ and $y \in \mathfrak{r}_p$ define $[x,y] = y(x) \in T_pM$. Bianchi's first identity will show that the Jacobi identity holds thus $\mathfrak{g}$ is a Lie algebra. Define $L : \mathfrak{g} \to \mathfrak{g}$ to be $L(x) = x$ if $x \in \mathfrak{r}_p$ and $L(y) = -y$ if $y \in T_pM$. Then L is a linear involution and the decomposition into Eigenspaces is as required.

□

10.2. **Holonomy.**

Definition 116. Suppose M is a Riemannian manifold and $\gamma : [a,b] \to M$ is a smooth loop (i.e. $\gamma(a) = \gamma(b) = p$). Then the parallel transport map $P : T_pM \to T_pM$ is a linear isometry. We define the **holonomy group at** p, denoted Hol_p, to be the Lie subgroup of $O(T_pM)$ generated by all such linear isometries (i.e. indexed by the loops γ). We further define the **restricted holonomy group at p** $\mathrm{Hol}_p^0 \trianglelefteq \mathrm{Hol}_p$ to be the connected normal subgroup where each γ is taken to be contractible.

If $E \subset T_pM$ is invariant under the action of Hol_p^0 then $E^\perp$ is also invariant. Hence we can write $T_pM = E_1 \oplus \cdots E_k$ where each E_i is irreducible and invariant. Since parallel transport from p to q will preserve this decomposition we have:

$$TM = \eta_1 \oplus \cdots \oplus \eta_k$$

where each η_i is a distribution.

Theorem 117 (de Rham decomposition)**.** *For all $p \in M$ there exists an open U containing p such that $U = (U_1 \times \cdots U_k)$ as a Riemannian manifold with metric g on U induced by the product metric on the (U_i, g_i) such that $TU_i = \eta_i|_{U_i}$. If moreover M is simply connected then we can take $U = M$.*

Definition 118. Given the above decomposition $TM = \eta_1 \oplus \cdots \eta_k$, if $k = 1$ we say that M is **irreducible**.

Corollary 119. *If M is irreducible and $\nabla R \equiv 0$ then M is Einstein.*

Theorem 120. *If M is symmetric and irreducible then M is Einstein with Einstein constant k. Then either $k > 0$, $k = 0$, or $k < 0$. If $k > 0$ then M is compact with nonnegative curvature* *operator. If $k = 0$ then M is flat so $M = S^1$ or $\mathbb{R}$. If $k < 0$ then M is noncompact with nonpositive curvature operator.*

Theorem 121. *If M is locally symmetric the Lie algebra of Hol_0^p, denoted $\mathfrak{hol}_p$, is generated by $R(v, w)$, i.e. $\mathfrak{hol}_p = \mathfrak{r}_p$ defined earlier in this section. Moreover $\mathfrak{hol}_p \subset \mathfrak{iso}_p$.*

Corollary 122. *If M is irreducible and symmetric $\mathfrak{hol}_p = \mathfrak{iso}_p$.*