WRITTEN EXAM WORKSHOP - LIMITS AND CONVERGENCE TESTS

1. Theory

Definition 1. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say $\lim_{n\to\infty} x_n = L$ or write $x_n \to L$ if:

Exercise 2. What's the definition in terms of Cauchy sequences?

Example 3. If x_n is increasing and bounded above, $x_n \leq B$, then $\lim x_n$ exists.

Definition 4. Let $(a_n)_{n=1}^{\infty}$be a sequence of real numbers. We say $\sum_{n=1}^{\infty} a_n = L$ if:

Exercise 5. What's the definition in terms of Cauchy sequences? *This is the best way to think about converging series!!!!*

Exercise 6. For what values of $x \in \mathbb{R}$, is $\sum_{n=0}^{\infty} x^n < \infty$?

Theorem 7. *[COMPARISON TEST] Suppose that* $0 \leq a_n \leq b_n$. *If* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\sum_{n=1}^{\infty} a_n < \infty$ *too. On the other hand, if* $\sum_{n=1}^{\infty} a_n = \infty$, *then* $\sum_{n=1}^{\infty} b_n = \infty$ *too.*

Proof. □

Corollary 8. *[RATIO TEST] If* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *exists. If* $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *exists.*

Proof. □

Corollary 9. *[ROOT TEST] If* $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *exists.*

Proof. □

Theorem 10. *[INTEGRAL TEST] Suppose that f is a monotone decreasing function. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.*

Proof. □

Example 11. $\sum \frac{1}{n^p}$ converges for $p > 1$.

Theorem 12. *[CAUCHY CONDENSATION] Suppose a_n is a decreasing sequence and we are interested in the convergence of $\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} 2^n a_{2^n} \leq 2\sum_{n=1}^{\infty} a_n$.*

Proof. □

Example 13. For what α, β does $\sum_{n=1}^{\infty} \left(n^{\alpha} \log n^{\beta}\right)^{-1}$ converge?

Theorem 14. *[DIRICHLET TEST] Suppose (a_n) is a positive decreasing sequence with $a_n \to 0$. Suppose b_n is a sequence which has the property that $\exists M$ s.t.$|\sum_{n=1}^{m} b_n| < M$ $\forall m \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.*

Proof. [from Wikipedia] A bit harder! Uses "summation by parts": $\sum_{k=m}^{n} f_k(g_{k+1} - g_k) = [f_{n+1}g_{n+1} - f_m g_m] - \sum_{k=m}^{n} g_{k+1}(f_{k+1} - f_k)$, which is used to show that: $\sum_{k=1}^{N} a_k b_k = a_N B_N - \sum_{n=0}^{N-1} B_n(a_{n+1} - a_n)$ where $B_N = \sum_{k=1}^{n} b_k$. From here: □

Example 15. Show that $\sum_{n=1}^{M} \sin(n)$ is bounded. (Hint: use some trig identities to get a telescoping sum!) Now show that $\sum_{n=1}^{\infty} \frac{1}{n}\sin(n)$ converges.

2. Problems

Problem 16. (Sept '03 #3) Determine which of these converge: a) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ c) $\sum_{n=2}^{\infty} \frac{\cos(\log(n))}{n\log(n)}$

Problem 17. (Jan '05 #1) Let (x_n) be a sequence converging to a. Show that $\left(x_1 x_2^2 \cdots x_n^n\right)^{1/n^2} \to \sqrt{a}$.

Problem 18. (Sept '09 #1) Suppose $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Show that, for any increasing sequence (a_n) of positive numbers with $a_n \to \infty$, we have:

$$\lim_{N\to\infty} \frac{1}{a_N} \sum_{n=1}^{N} a_n x_n = 0$$