Generating Functions

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Generating functions are a way to associate every sequence of numbers with a function. The way we associate a sequence of numbers to a function is by putting the n-th term of the sequence as the coefficient in front of x^n and adding it all up. That is, we associate the sequence $A = \{a_0, a_1, a_2, \ldots\}$ with the function $F_A(x)$ defined by:

$$F_A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$F_A(x)$ is called the *generating function* of the sequence A. Here are some common sequences and their associated generating functions.

Example 1: $F_{\{1,1,1,1,\ldots\}}(x) = \frac{1}{1-x}$

TECHNICAL NOTE: Technically this formula is true only when $-1 < x < 1$, but don't worry about that, its not important right now

Let $S = F_{\{1,1,1,1,\ldots\}}(x)$. Then:

$$\begin{aligned} S = &\quad 1 + x + x^2 + \ldots \\ xS = &\quad\quad\; x + x^2 + \ldots \\ \therefore (1-x)\,S = &\quad 1 + 0 + \;0 + \ldots \end{aligned}$$

$$\therefore S = \frac{1}{1-x}$$

Example 2: $F_{\{1,2,4,8,16,\ldots\}}(x) = \frac{1}{1-2x}$

$$\begin{aligned} F_{\{1,2,4,8,16,\ldots\}}(x) &= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots \\ &= 1 + (2x) + (2x)^2 + (2x)^3 + \ldots \\ &= F_{\{1,1,1,1,\ldots\}}\,(2x) \\ &= \frac{1}{1-2x} \end{aligned}$$

Example 3: $F_{\{1,2,3,4,\ldots\}}(x) = \frac{1}{(1-x)^2}$

Let $S = F_{\{1,2,3,4,\ldots\}}(x)$.We do the same "shifting" trick as before:

$$\begin{aligned} S = &\quad 1 + 2x + 3x^2 + \ldots \\ xS = &\quad\quad\;\; x + 2x^2 + \ldots \\ \therefore (1-x)\,S = &\quad 1 + \;x + \;x^2 + \ldots \\ = &\quad \frac{1}{1-x} \qquad \text{(from Example 1)} \end{aligned}$$

$$\therefore S = \frac{1}{(1-x)^2}$$

Fibonacci

We can find the generating function for the Fibonacci numbers using the same trick! This will let us calculate an explicit formula for the n-th term of the sequence. Recall that the Fibonacci numbers are given by $f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$.

Fact 1: $F_{\{f_0,f_1,f_2,f_3,\ldots\}}(x) = \frac{x}{1-x-x^2}$

To make the notation a bit simpler, lets write $F(x) = F_{\{f_0,f_1,f_2,f_3,\ldots\}}(x)$. Now, to get the desired result, we do the same shifting trick and use the properties of the Fibonacci sequence.

$$\begin{aligned}
F(x) &= f_0 + f_1x + f_2x^2 + f_3x^3 + \ldots \\
xF(x) &= f_0x + f_1x^2 + f_2x^3 + \ldots \\
x^2F(x) &= f_0x^2 + f_1x^3 + \ldots \\
\therefore \left(1 - x - x^2\right) F(x) &= f_0 + (f_1 - f_0)x + 0x^2 + 0x^3 + \ldots \\
&= x
\end{aligned}$$

$$\therefore F(x) = \frac{x}{1-x-x^2}$$

Fact 2: $f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \psi^n\right)$ where $\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$.

Note: φ andψ here are special because $-\varphi, -\psi$ are the two roots of the quadratic $1 - x - x^2$ which comes from the quadratic equation. We already know that the quadratic equation $1 - x - x^2$ has something to do with Fibonacci numbers from Fact 1.

This comes right out of the generating function we calculated in Fact 1, and using some factoring:

$$\begin{aligned}
1 - x - x^2 &= -\left(x + \varphi\right)\left(x + \psi\right) \\
&= -\varphi\psi\left(\frac{1}{\varphi}x + 1\right)\left(\frac{1}{\psi}x + 1\right)
\end{aligned}$$

Now we use the fact that $\varphi\psi = -1,\ \frac{1}{\varphi} = -\psi, \frac{1}{\psi} = -\varphi$ (just check using $\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$) to further simplify:

$$\begin{aligned}
1 - x - x^2 &= -(-1)\left(-\psi x + 1\right)\left(-\varphi x + 1\right) \\
&= \left(1 - \psi x\right)\left(1 - \varphi x\right)
\end{aligned}$$

Now we do a trick known as "partial fractions" to simplify $F(x)$:

$$\begin{aligned}
F(x) &= \frac{x}{1 - x - x^2} \\
&= \frac{x}{\left(1 - \psi x\right)\left(1 - \varphi x\right)} \\
&= \frac{\frac{1}{\sqrt{5}}}{1 - \psi x} - \frac{\frac{1}{\sqrt{5}}}{1 - \varphi x}
\end{aligned}$$

(To find these "partial fractions" one must solve a linear system of equations) These look very much like what we did in Example 1 and 2! By the same argument, we can see that:

$$\begin{aligned}
\frac{1}{1 - \psi x} &= 1 + \psi x + \psi^2x^2 + \psi^3x^3 + \ldots \\
\frac{1}{1 - \varphi x} &= 1 + \varphi x + \varphi^2x^2 + \varphi^3x^3 + \ldots
\end{aligned}$$

So we have then:

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(1+\varphi x+\varphi^2x^2+\varphi^3x^3+\ldots\right)-\frac{1}{\sqrt{5}}\left(1+\psi x+\psi^2x^2+\psi^3x^3+\ldots\right) \\ &= 0+\frac{1}{\sqrt{5}}(\varphi-\psi)\,x+\frac{1}{\sqrt{5}}\left(\varphi^2-\psi^2\right)x+\frac{1}{\sqrt{5}}\left(\varphi^3-\psi^3\right)x^3+\ldots \end{aligned}$$

Since $F(x) = f_0 + f_1x + f_2x^2 + f_3x^3 + \ldots$, comparing coefficients gives us the result we want!

Factoid 1: Fibonacci numbers can be used to convert miles to kilometers by: $f_n\,\mathrm{km} \approx f_{n-1}\,\mathrm{mi}$

The secret of this factoid is an amazing coincidence between the numerical value of $\varphi$and the number of kilometers in a mile, and the fact that $\psi < 1$. Firstly, notice that:

$$\begin{aligned} \varphi &= 1.6180\ldots \\ \frac{1\mathrm{mi}}{1\mathrm{km}} &= 1.6093\ldots \end{aligned}$$

Because these two values are close, the approximation 1 mi $\approx \varphi\,$km is pretty good (to about 1%). Now notice that since $\psi < 1$, that ψ^n is really small as n gets larger; $\psi^n \approx 0$. So we have some more approximations:

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left(\varphi^n-\psi^n\right)\approx\frac{1}{\sqrt{5}}\varphi^n \\ f_{n-1} &= \frac{1}{\sqrt{5}}\left(\varphi^{n-1}-\psi^{n-1}\right)\approx\frac{1}{\sqrt{5}}\varphi^{n-1} \\ \therefore f_n &\approx \varphi f_{n-1} \end{aligned}$$

Along with 1mi $\approx \varphi$km, this means that f_{n-1}km $\approx f_n$mi. This works best if n is not-too-small, because when n is large, our approximation that $\psi^n \approx 0$ becomes more accurate. $n = 5$ is already quite a good approximation ($\psi^4 \approx 0.0002$). Here are the first couple listed for you, starting at $n = 5$, so that you can travel to Canada without fear!

$$\begin{aligned} 3\text{ mi} &\approx 5\,\text{km} \\ 5\text{ mi} &\approx 8\,\text{km} \\ 8\,\text{mi} &\approx 13\text{ km} \\ 13\,\text{mi} &\approx 21\text{ km} \\ 21\,\text{mi} &\approx 34\text{ km} \end{aligned}$$

If you found this interesting...

Here are some great websites that you can check out where you can get more!

Wikipedia links:

- Fibonacci number
- Golden ratio
- Recurrence relation (advanced)
- Generating function (advanced)

Other links:

- "Doodling in Math: Spirals, Fibonacci, and Being a Plant [1 of 3]" by Vi Hart.
  http://www.khanacademy.org/math/vi-hart/v/doodling-in-math–spirals–fibonacci–and-being-a-plant–1-of-3
- "Exercise - Write a Fibonacci Function" by Salman Khan (has a computer science flavor)
  http://www.khanacademy.org/science/computer-science/v/exercise—write-a-fibonacci-function