The Fibonacci Sequence

A *sequence* is a list of numbers that never ends (e.g. $1, 3, 5, 7, 9, 13, \ldots$) The *Fibonacci Sequence* is an exciting sequence of numbers we will talk about today. The first few numbers are:

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	$\ldots$
0	1	1	2	3	5	8	13	21	34	$\ldots$

The rule that makes the Fibonacci Sequence is "the next number is the sum of the previous two". This kind of rule is sometimes called a *recurrence relation.* Mathematically, this is written as:

$$f_n = f_{n-1} + f_{n-2}$$

One strange fact about Fibonacci numbers is that they can be used to convert kilometers to miles:

$$\begin{aligned} 3\text{ mi} &\approx 5\,\text{km} \\ 5\text{ mi} &\approx 8\,\text{km} \\ 8\,\text{mi} &\approx 13\text{ km} \\ 13\,\text{mi} &\approx 21\text{ km} \\ 21\,\text{mi} &\approx 34\text{ km} \end{aligned}$$

We will explain how this works by using a really powerful idea called "*generating functions*" which let us attack these problems. Generating functions involve using algebra to solve *infinite sums.* Before we jump into Fibonacci, we will start with some warm up problems to get the hang of it.

Exercise 0: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = ?$

Exercise 1: $1 + x + x^2 + x^3 + x^4 + \ldots = ?$

TECHNICAL NOTE: Technically this sum only makes sense when $-1 < x < 1$, but don't worry about that!

Exercise 2: $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots = ?$

Exercise 3: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = ?$

Generating Functions

For any sequence of numbers, there is a *generating function* associated with that sequence. (By a function, I mean an expression that depends on x.) The rule for the generating function is to multiply each term of the sequence by x^n, and finally do the infinite sum of all these terms. This is written mathematically as:

Definition: The generating function for the sequence $a_0, a_1, a_2, a_3, \ldots$ is $S = a_0 \cdot x^0 + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + \ldots$

Here are some examples. We already did these as exercises before!

Example 1: The generating function for the sequence $1, 1, 1, 1, 1 \ldots$ is $S = \frac{1}{1-x}$.

We must evaluate the infinite sum $S = 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + \ldots$. We use the "shifting trick":

$$\begin{array}{rrrrr} S = & 1+ & x+ & x^2+ & x^3 + \ldots \\ - \quad xS = & & x+ & x^2+ & x^3 + \ldots \\ \hline \therefore (1-x)\, S = & 1+ & 0+ & 0+ & 0 + \ldots \end{array}$$

$$\therefore S = \tfrac{1}{1-x}$$

Example 2: The generating function for the sequence $1, 2, 4, 8, 16 \ldots$ is $S = \frac{1}{1-2x}$.

We must evaluate the infinite sum $S = 1 \cdot x^0 + 2 \cdot x^1 + 4 \cdot x^2 + 8 \cdot x^3 + \ldots$.

Notice that we can rewrite each term as a power of $2x$, namely $S = 1 \cdot x^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots$. Now we can use the same shifting trick as before (Alternatively, you can "plug in $2x$ into Example 1):

$$\begin{array}{rrrr} S = & 1+ & (2x)+ & (2x)^2 + \ldots \\ - \quad 2xS = & & 2x+ & (2x)^2 + \ldots \\ \hline \therefore (1-2x)\, S = & 1+ & 0+ & 0 + \ldots \end{array}$$

$$\therefore S = \tfrac{1}{1-2x}$$

Example 3: The generating function for the sequence $1, 2, 3, 4, 5, 6 \ldots$ is $S = \frac{1}{(1-x)^2}$.

We must evaluate the infinite sum $S = 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + \ldots$. We can use the same shifting trick to reduce this to the sum we already computed in example 1:

$$\begin{array}{rrrrrl} S = & 1+ & 2x+ & 3x^2+ & 4x^3 + \ldots & \\ - \quad xS = & & x+ & 2x^2+ & 3x^3 + \ldots & \\ \hline \therefore (1-x)\, S = & 1+ & x+ & x^2+ & x^3 + \ldots & \\ = & \dfrac{1}{1-x} & & & & \text{from Example 1} \end{array}$$

$$\therefore S = \tfrac{1}{(1-x)^2}$$

Fact 1: The generating function for the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8 \ldots$ is $S = \frac{x}{1-x-x^2}$.

We must evaluate the infinite sum $S = 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 2 \cdot x^3 + 3 \cdot x^4 \ldots$. Since the recurrence relation for the Fibonacci numbers involves the last *two* numbers, we must use a "double" shifting trick:

$$\begin{array}{rrrrrrr} S = & 0+ & x+ & x^2+ & 2x^3+ & 3x^4+ & 5x^5 + \ldots \\ - \quad xS = & & 0+ & x^2+ & x^3+ & 2x^4+ & 3x^5 + \ldots \\ - \quad x^2 S = & & & 0+ & x^3+ & x^4+ & 2x^5 + \ldots \\ \hline \therefore \left(1-x-x^2\right) S = & 0+ & x+ & 0+ & 0+ & 0+ & 0 + \ldots \end{array}$$

$$\therefore S = \frac{x}{(1-x-x^2)}$$

The reason all the 0's appear is from the recurrence relation for the Fibonacci numbers; in other words the fact that the next Fibonacci number is the sum of the previous two.

The Golden Ratio

Define two numbers φ and β to be the roots of the quadratic equation $x^2 - x - 1$. (This quadratic equation appeared "in reverse" in the denominator for the generating function of the Fibonacci numbers).

By the quadratic equation, these are:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887\ldots$$
$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.6180339887\ldots$$

The number φ is called the *Golden Ratio* and has a number of exciting properties (go see the Wikipedia page for more info!). Both φ and β are intimately related to the Fibonacci sequence because they appear in the generating function!

Fact 2: $1 - x - x^2 = (1-\varphi x)(1-\beta x)$

You can check this fact by expanding it out using "FOIL" and the fact that $\varphi\beta = -1$ and $\varphi + \beta = 1$. Try it here!

Fact 3: Let f_n be the n-th Fibonacci number. Then we have the explicit formula: $f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \beta^n\right)$

The trick we use to get this result is to rewrite the generating function for the Fibonacci numbers $S = \frac{x}{(1-x-x^2)}$ we found in Fact 1 using the factorization $1 - x - x^2 = (1-\varphi x)(1-\beta x)$ we found in Fact 2. By some algebraic manipulations, (This is called the *partial fractions* trick!) we find that:

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\frac{1}{1-\varphi x} + \frac{1}{\sqrt{5}}\frac{1}{1-\beta x}$$

Now we recognize that $\frac{1}{1-\varphi x}$ and $\frac{1}{1-\beta x}$ look a lot like the answer from Example 2. These are infinite sums!

$$\frac{1}{1-\beta x} = 1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \ldots$$
$$\frac{1}{1-\varphi x} = 1 + \varphi x + \varphi^2 x^2 + \varphi^3 x^3 + \ldots$$

Putting this all together gives:

$$\begin{aligned}
& f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots \\
= & \frac{x}{1-x-x^2} \\
= & \frac{1}{\sqrt{5}}\frac{1}{1-\beta x} - \frac{1}{\sqrt{5}}\frac{1}{1-\varphi x} \\
= & \frac{1}{\sqrt{5}}\left(1 + \varphi x + \varphi^2 x^2 + \varphi^3 x^3 + \ldots\right) - \frac{1}{\sqrt{5}}\left(1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \ldots\right) \\
= & 0 + \frac{1}{\sqrt{5}}(\varphi - \beta)\, x + \frac{1}{\sqrt{5}}\left(\varphi^2 - \beta^2\right) x + \frac{1}{\sqrt{5}}\left(\varphi^3 - \beta^3\right) x^3 + \ldots
\end{aligned}$$

Since these two "polynomials" are equal for every value of x, each term must individual by equal. Reading off the coefficients of the power x^n gives the final result,

$$f_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \beta^n\right)$$

Converting Miles to Kilometers

Factoid 1: Fibonacci numbers can be used to convert miles to kilometers by: $f_n\,\text{km} \approx f_{n-1}\,\text{mi}$

The secret of this factoid is an amazing coincidence between the numerical value of φ and the number of kilometers in a mile, and the fact that $|\beta| < 1$. Firstly, notice that:

$$\begin{aligned} \varphi &= 1.6180\ldots \\ \frac{1\text{mi}}{1\text{km}} &= 1.6093\ldots \end{aligned}$$

Because these two values are close, the approximation 1 mi $\approx \varphi\,$km is pretty good (to about 1%). Now notice that since $\beta < 1$, that β^n is really small as n gets larger; $\beta^n \approx 0$. So we have some more approximations:

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left(\varphi^n - \beta^n\right) \approx \frac{1}{\sqrt{5}}\varphi^n \\ f_{n-1} &= \frac{1}{\sqrt{5}}\left(\varphi^{n-1} - \beta^{n-1}\right) \approx \frac{1}{\sqrt{5}}\varphi^{n-1} \\ \therefore f_n &\approx \varphi f_{n-1} \end{aligned}$$

Along with 1mi $\approx \varphi$km, this means that f_{n-1}km $\approx f_n$mi. This works best if n is not-too-small, because when n is large, our approximation that $\beta^n \approx 0$ becomes more accurate. $n = 5$ is already quite a good approximation ($\beta^4 \approx 0.0002$). The first couple listed for you, starting at $n = 5$, so that you can travel to Canada without fear!

$$\begin{aligned} 3\ \text{mi} &\approx 5\,\text{km} \\ 5\ \text{mi} &\approx 8\,\text{km} \\ 8\,\text{mi} &\approx 13\ \text{km} \end{aligned}$$

If you want to covert numbers not on this list, you can bootstrap from the above approximations. For example, starting from 5mi $\approx$ 8km, you can do:

$$100\text{mi} = 20 \cdot 5\text{mi} \approx 20 \cdot 8\text{km} = 160\text{km}$$

If you found this interesting...
Here are some great websites that you can check out where you can get more!

Wikipedia links:

- Fibonacci number
- Golden ratio
- Recurrence relation (advanced)
- Generating function (advanced)

Other links:

- "Doodling in Math: Spirals, Fibonacci, and Being a Plant [1 of 3]" by Vi Hart.
  http://www.khanacademy.org/math/vi-hart/v/doodling-in-math–spirals–fibonacci–and-being-a-plant–1-of-3
- "Exercise - Write a Fibonacci Function" by Salman Khan (has a computer science flavor)
  https://www.youtube.com/watch?v=Bdbc1ZC-vhw