The Fibonacci Sequence

A sequence is a list of numbers that never ends (e.g. $\{1, 2, 3, 4, 5, 6, \ldots\}$) The Fibonacci Sequence is a neat sequence of numbers which are generated by starting with:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \end{aligned}$$

And then getting the next number by adding up the previous two:

$$f_n = f_{n-1} + f_{n-2}$$

This gives the sequence whose first few terms are $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$. This sequence has a number of cool properties and seems to pop up just about everywhere. One really strange fact about Fibonacci numbers is that they can be used to convert kilometers to miles by just shifting over by one in the sequence:

$$\begin{aligned} 3 \text{ mi} &\approx 5\,\text{km} \\ 5 \text{ mi} &\approx 8\,\text{km} \\ 8\,\text{mi} &\approx 13 \text{ km} \\ 13\,\text{mi} &\approx 21 \text{ km} \\ 21\,\text{mi} &\approx 34 \text{ km} \end{aligned}$$

Why does this work?!??!?! How can we get more information about the Fibonacci numbers?!?! Is there a way to get to the n-th Fibonacci sequence without having to compute all of the ones that came before?!?!? These are all questions we will try and answer in this talk.

The tool we will use is a really powerful idea called "generating functions" which let us attack these problems. Generating functions involve taking some "infinite sums" so we will start with some warm up problems to get the hang of it.

Exercise 0: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = ?$

Exercise 1: Show that $1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$

TECHNICAL NOTE: Technically this formula is true only when $-1 < x < 1$, but don't worry about that, its not important right now

Exercise 2: $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots =?$

Exercise 3: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots =?$