PDE Oral Exam study notes

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Abstract. These are some study notes that I made while studying for my oral exams on the topic of PDEs. I took these notes from parts of the textbooks by Fritz John [**2**] and Lawrence C. Evans [**1**] (both books are unsurprisingly called "Partial Differential Equations"). Please be extremely caution with these notes: they are rough notes and were originally only for me to help me study and are not complete or guaranteed to be free of errors. I have made them available to help other students on their oral exams.

Contents

Four Important PDEs

These are notes from Chapter 2 of [**1**].

1.1. Transport Equation

The transport equation is:

$$u_t + b \cdot \nabla u = 0$$

where b is a fixed vector.
Solve this by writing it in "integral surface form" namely:

$$(1, b_1, \ldots, b_n, 0) \cdot (u_t, u_{x_1}, \ldots u_{x_n}, -1) = 0$$

$(u_t, u_{x_1}, \ldots u_{x_n}, -1)$ is the normal to the integral surface $z = u(t, \vec{x})$ so the above says that tangent along integral curves satisfy $(1, b_1, \ldots, b_n, 0)$. This gives us the system:

$$\begin{aligned} 1 &= \frac{\mathrm{d}t}{\mathrm{d}s} \implies t = s + t_i(0) \\ b_i &= \frac{\mathrm{d}x_i}{\mathrm{d}s} \implies x_i = sb_i + x_i(0) \\ 0 &= \frac{\mathrm{d}z}{\mathrm{d}s} \implies z = z_0 \end{aligned}$$

So if we are given intial data we can solve. This is called the **method of characteristics**.

1.2. Laplace's Equation

Laplace's equation is:

$$\Delta u = 0$$

A function satisfying $\Delta u = 0$ is called a **Harmonic function.** Poisson's equation is:

$$-\Delta u = f$$

1.2.1. Fundamental Solution. Find the radial solution that is spherical symmetric. If $\psi(r)$ depends only on r, then $\Delta\psi = 0$ becomes $r^{1-n}\partial_r \left(r^{n-1}\partial_r\psi\right) = 0$. Solving this gives a spherically symetric solution, (its like $\log r$ if $n = 2$ and otherwise its like $r^{-(n-2)}$)

$$\psi(r) = \begin{cases} \frac{1}{\omega_n}\frac{r^{2-n}}{2-n} & \text{for } n > 2 \\ \frac{1}{2\pi}\log r & \text{for } n = 2 \end{cases}$$

With this solution in hand, the solution to Poisson's equation is given by:

$$u(x) = \int \psi(x-y)f(y)\mathrm{d}y = f * \psi = \int f(x-y)\psi(y)\mathrm{d}y$$

(Think of electric charge) You might be tempted to just take Δ of both sides, but this is not justified since ψ has a singularity at $x = y$.

PROPOSITION 1.1. *$u = f * \psi$ is C^2 when f is C^2 and has $\Delta u = f$*

PROOF. Check by taking limits that $\frac{\partial^2 u}{\partial x \partial y} = \psi * \frac{\partial^2 f}{\partial x \partial y}$ so we see u is C^2. To check $\Delta u = f$, fix an $\epsilon > 0$ and split the integral into interior and exterior ball of radius ϵ.

On the exterior, ψ is legit so by doing a differentiation by parts we get some boundary terms on the ball. (The main integral dies sicne $\Delta \psi = 0$ here). These are bounded by the surface area of the ball, which is $O(\epsilon)$

On the interior of the ball, bound by the volume of the ball, get something like $\int_{\partial B(x,\epsilon)} f(y)\mathrm{d}S(y) \to f(x)$ □

1.2.2. Mean-value formulas.

THEOREM 1.1. *(Mean-value theorem) If $u \in C^2(U)$ is harmonic, then:*

$$u(x) = \frac{1}{\omega_n r^n} \int\limits_{\partial B(x,r)} u dS = \int\limits_{B(x,r)} u dy$$

PROOF. Set $\phi(r) = \int_{\partial B(x,r)} u\mathrm{d}S$ and check using greens formula that $\phi' = \frac{r}{n}\int_{B(x,r)} \Delta u(y)\mathrm{d}y$ so ϕ is consant. But also $\lim_{r\to 0}\phi(r) = 0$ is clear.

In Fritz-John he does this by getting making a Green-like function:

$$G(x,\xi) = K(x,\xi) - \psi(r_0) = \psi(|x-\xi|) - \psi(r_0)$$

And then will have (again using Green's identity):

$$u(x_0) = \int\limits_{\Omega} G(x,x_0)\Delta u(x)\mathrm{d}x + \frac{1}{\omega_n r_0^{n-1}} \int\limits_{\partial\Omega} u(x)\mathrm{d}S_x$$

□

THEOREM 1.2. *(Converse to the mean value property) If $u \in C^2$ has:*

$$u(x) = \frac{1}{\omega_n r^n} \int\limits_{\partial B(x,r)} u dS$$

For every ball, then u is harmonic.

PROOF. Otherwise find a ball so that $\Delta u > 0$ in the ball and then $\phi'(r) > 0$ is a contraditction. □

1.2.3. Properties of Harmonic Functions.

THEOREM 1.3. *(Strong Maximum Principle)*
If $u \in C^2(U) \cap C(\overline{U})$ is harmonic then:

$$\max_{\overline{U}} u = \max_{\partial U} u$$

Furthermore if the maximum is achieved at an interior point, then u must be constant in U.

PROOF. Suppose u has an interior maximum. By using the mean value property, the set where u achieves its maximum is an open set. However this is also $u^{-1}\{M\}$ is a closed set. Hence u must be constantly equal to the maximum M. □

THEOREM 1.4. *(Uniqueness) The initial value problem has at most one solution:*

$$\begin{aligned} -\Delta u &= f \text{ in } U \\ u &= g \text{ in } \partial U \end{aligned}$$

PROOF. Subtract two candidate solutions and apply the maximum (and minimum by multiplying by -1) principles to see the difference is 0. □

1.2.3.1. *Regularity.* We will show that if $u \in C^2$ is harmonic, then actually $u \in C^\infty$. Thus **harmonic functions are automatically infinitely differentiable**.

THEOREM 1.5. *If $u \in C(U)$ is continuous and satifies a mean-value property for every ball $B(x,r) \subset U$ then actually:*

$$u \in C^\infty(U)$$

Note that u may not be smooth or even continuous up the boundary ∂U.

PROOF. Let $u^\epsilon = u * \eta_\epsilon$ be a standard mollifier so that η_ϵ is supported in a ball of radius $< \epsilon$ and $\|\eta_\epsilon\|_{L^1} = 1$. Then $u^\epsilon \in C^\infty(U_\epsilon)$ where $U_\epsilon = \{x \in U \,|\mathrm{dist}(x, \partial U) > \epsilon\}$. By looking in a ball of radius $r < \epsilon$ and using the fact that u satisfies the mean value property we will get that $u = u^\epsilon$. □

REMARK 1.1. In Fritz John we used the integral representation formula $u(x_0) = \int K(x,x_0)\nabla u(x) + \nabla K(x,x_0)u(x)\mathrm{d}x$ to see that C^2 solutions to $\Delta = 0$ were C^∞ and analytic.

1.2.3.2. *Local Estimates.*

THEOREM 1.6. *If u is harmonic in U then:*

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

PROOF. Use the fact that $u_{x_i}$is harmonic too. This satisfies a mean value property. Integrate by parts to compare back to u and work out the estimate $u_{\xi_i}(0) = \frac{n}{a^{n+1}\omega_n}\int_{|x|=a} x_i u(x)\mathrm{d}S_x$. □

1.2.3.3. *Liouville's Theorem.*

THEOREM 1.7. *If u is harmonic and bounded in $\mathbb{R}$ then u is constant.*

PROOF. Have $|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^\infty} \to 0$ □

THEOREM 1.8. *(Representation formula) If $f \in C_c^2(\mathbb{R}^n)$ for $n \geq 3$ then any bounded solution of:*

$$-\Delta u = f$$

has the form:

$$u(x) = \int\limits_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy + C$$

PROOF. Since f is compactly supported, the integral solution u above is bounded. Now, if v is any other solution, then $v-u$ is bounded and has $\Delta(v-u)=0$. By Liouville's theorem, it must be a constant. □

REMARK 1.2. This doesnt work in dimension $n=2$ since the kernal $\Phi(x)=-\frac{1}{2\pi}\log|x|$ is unbounded as $|x|\to\infty$.

1.2.3.4. *Analyticity.*

THEOREM 1.9. *If u is harmonic in U then u is actually analytic in U .*

PROOF. Use the bounds on the derivatives to get a bound:

$$\|D^\alpha u\|_{L^\infty}\leq M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|}|\alpha|^{|\alpha|}$$

By Stirlings formula this is so tight as to have a postive radius of convergence. □

1.2.3.5. *Harnack Inequality.*

THEOREM 1.10. *Say V is a connected open set so that $\overline{V}$ is compact in U. Let u be a* ***non-negative harmonic function*** *in U. Then there is a positive constant C depending only on V so that:*

$$\sup_V u\leq C\inf_V u$$

i.e. $\frac{1}{C}u(y)\leq u(x)\leq Cu(y)$ for all $x,y\in V$.

REMARK 1.3. This says that the value for **non-negative** harmonic functions are comparable; u cannot be very large/small at one point without being large/small everywhere. The intutive idea is taht since V is a positive distance away from ∂U , the **averaging effects** of the Laplace equation smooth out u in this way.

PROOF. Let $r=\frac{1}{4}\mathrm{dist}\,(V,\partial U)$. Choose $x,y\in V$ so that $|x-y|\leq r$ so that $B(x,2r)\supset B(y,r)$. Then:

$$u(x)=\frac{1}{\alpha(n)2^nr^n}\int\limits_{B(x,2r)}u\mathrm{d}z\geq\frac{1}{\alpha(n)2^nr^n}\int\limits_{B(y,r)}u\mathrm{d}z=\frac{1}{2^n}u(y)$$

So this gives a Harnack inequality if $|x-y|$. Since V is precompact, we can cover it with fintely many of these balls and get a constant that works everywhere on V. □

1.2.4. Green's Function. We seek a solution to Green's function:

$$-\Delta u=f\text{ in }U$$

subject to prescribed boundary conditions:

$$u=g\text{ on }\partial U$$

From the Greens identity from vector calc, and our proof that $\Delta\Phi(y-x)=\delta_x$ we know:

$$\begin{aligned}\int\limits_V u(y)\Delta\Phi(y-x)-\Phi(y-x)\Delta u(y)\mathrm{d}y &= \int\limits_{\partial V}u(y)\frac{\partial\Phi}{\partial\nu}(y-x)-\Phi(y-x)\frac{\partial u}{\partial\nu}(y)\mathrm{d}S\\ u(x)-\int\limits_V\Phi(y-x)\Delta u(y)\mathrm{d}y &=\end{aligned}$$

So if we can find a Φ with $\frac{\partial \Phi}{\partial \nu} \equiv 0$ on the boundary V this will give us a solution. (We still need to maintain $\Delta\Phi(y-x) = \delta_y$ for $y \in V$)

The idea is to add on a corrector function:

$$\begin{cases} \Delta\phi^x = 0 & \text{in } U \\ \phi^x = \Phi(y-x) & \text{on } \partial U \end{cases}$$

Then setting the Greens functions $G(x,y) = \Phi(y-x) - \phi^x(y)$, we will get (again by greens formula):

$$\begin{aligned} u(x) &= -\int\limits_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\mathrm{d}S(y) - \int\limits_{U} G(x,y)\Delta u(y)\mathrm{d}y \\ &= -\int\limits_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\mathrm{d}S(y) - \int\limits_{U} G(x,y)f(y)\mathrm{d}y \end{aligned}$$

1.2.4.1. *Green's Function for a half-space.* Use the method of images, put a refleced point at the mirror image across the half space. Get:

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$$

This leads to:

$$K(x,y) = \frac{2}{n\alpha(n)}\frac{x_n}{|x-y|^n}$$

In two dimensions:

$$K(x_0,y_0,x,y) = \frac{y}{(x-x_0)^2+(y-y_0)^2}$$

1.2.4.2. *Green's Function for a ball.* Define:

$$\tilde{x} = \frac{1}{|x|}\left(\frac{x}{|x|}\right)$$

To be the inversion in $B(0,1)$ of the point x. Check that for $y \in \partial B(0,1)$ one has:

$$\begin{aligned} |x|^2\,|y-\tilde{x}|^2 &= |x|^2\left(|y|^2 + \frac{1}{|x|^2} - \frac{2y\cdot x}{|x|^2}\right) \\ &= |x|^2 - 2y\cdot x + 1 = |x-y|^2 \end{aligned}$$

I.e.:

$$\frac{|x-y|}{|\tilde{x}-y|} = |x|$$

So, for $n \geq 3$, if we put $\phi^x(y) = |x|^{2-n}\,\Phi\left((y-\tilde{x})\right) = \Phi\left(|x|\,(y-\tilde{x})\right)$ then this will be Harmonic too and will have:

$$\phi^x(y) = \Phi(y-x) \text{ for } y \in \partial B(0,1)$$

(This works since $\Phi(z)$ depends only on $|z|$) Hence the Green's function for n. The resulting Green's function is hence:

$$G(x,y) = \Phi(y-x) - \Phi\left(|x|\,(y-\tilde{x})\right)$$

This leads to the kernal for a ball of radius r:

$$K(x,y) = \frac{r^2-|x|^2}{n\alpha(n)r}\frac{1}{|x-y|^n}$$

In two dimesnions this is:

$$K(x,y,x_0,y_0)=\frac{1}{2\pi}\frac{1-x^2-y^2}{\left(x-x_0\right)^2+(y-y_0)^2}$$

This looks sexy in polar coordinates or in terms of complex variables, where we put $r=\sqrt{x^2+y^2}$ and θ to be the angle between the rays to(x,y) and (x_0,y_0) at the origin:

$$P_r(\theta)=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta)+r^2}=\mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)$$

with the solution being given by:

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}u\left(e^{i\phi}\right)\left(\frac{1-\left|z_0\right|^2}{\left(z_0-e^{i\phi}\right)^2}\right)\mathrm{d}\phi$$

1.2.5. Energy Methods. Energy methods are often a quick way to get answers to come out without having to develope too much theory first.

1.2.5.1. *Uniqueness.*

THEOREM 1.11. *There is at most one solution to $\Delta u=f$ in U and $u=g$ on ∂U*

PROOF. Suffices to check it for 0 boundary data since the PDE is linear. Look at the energy functional

$$\begin{aligned} I[u] &= \int_U \frac{1}{2}\left|\nabla u\right|^2-uf\ \mathrm{d}x \\ &= \int_U \frac{1}{2}\left|\nabla u\right|^2\mathrm{d}x \text{ since we take } f=0 \end{aligned}$$

By integration by parts:

$$\begin{aligned} \int_U \frac{1}{2}\left|\nabla u\right|^2\mathrm{d}x &= -\int_U u\Delta u+\int_{\partial U}u\nabla u \\ &= 0+0 \end{aligned}$$

Since this is zero, we get that ∇u must be zero everywhere! □

1.2.5.2. *Dirichlet's Principle.* Define as above the energy:

$$I[u]=\int_U \frac{1}{2}\left|\nabla u\right|^2-uf\ \mathrm{d}x$$

THEOREM 1.12. *u solves $\Delta u=f$ in U and $u=g$ on ∂U if and only if:*

$$I[u]=\min_{w\in A}I[w]$$

Where $A=\left\{w\in C^2(\overline{U})\,|w=g \text{ on } \partial U\right\}$
I.e. it satisfies the PDE if and only if it minizes the energy.

PROOF. ($\Longrightarrow$) If u solves the PDE, then $0 = \int(-\Delta u - f)(u-w)\,\mathrm{d}x = \int \nabla u \nabla(u-w) - f(u-w)$ for any w. Integration by parts gets us the result after an application of a Cauchy-Schwarz inequaltiy.

($\Longleftarrow$)For any v, look at $i(t) = I[u+tv]$. Since u is a minizer, $i'(0) = 0$. This gives the PDE. □

1.3. Heat Equation

The Heat equation is:

$$u_t - \Delta u = 0$$

And the non-homogenous heat equation is:

$$u_t - \Delta u = f$$

subect to appropraite inital boundary conditions.

1.3.1. Fundemental Solution. Let search for a special solution of the form $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$ (or equivlently, go look for a solution which is invariant under some scaling, $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$, and then the function v will be $v(y) = u(y,1)$.) We will find $\beta = 1$ and $\alpha = n/2$ If we also guess that $v(x) = w(|x|)$ is radial, we get to the ODE for w namely $w' = -\frac{1}{2}rw \implies w = e^{-r^2/4}$ so this give us the fundemental solution:

$$u(x,t) = \frac{C}{t^{n/2}} e^{-|x|^2/4t}$$

This is called the **fundemental solution.**

1.3.1.1. *Initial Value Problem.* Consider the **initial value problem** (aka **Cauchy problem**):

$$\begin{aligned} u_t - \Delta u &= 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

THEOREM 1.13. *For $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ define $u = \Phi * g$, where Φ is the fundemental solution. Then:*

$$\begin{aligned} u &\in C^\infty(\mathbb{R}^n \times (0,\infty)) \\ u_t - \Delta u &= 0 \quad \textit{in } \mathbb{R}^n, t > 0 \\ \lim_{(x,t)\to(x^0,0)} u(x,t) &= g(x^0) \end{aligned}$$

PROOF. Φ is C^∞ for $t > \delta > 0$ so the first two follow by differentiating under the integral sign. The last one works by using that $\int_{|x|>\delta} \Phi(x,t)\mathrm{d}x \to 0$ as $t \to 0$ and by continuity of g at the limit point. □

REMARK 1.4. Notice that the information propogates infinetly fast here, not at finite speed as before.

1.3.1.2. *Nonhomogenous Problem.* If we want $u_t - \Delta u = f$ with 0 boundry condtion, now how do we solve this?

Duhamel's Principle asserts we can solve non-homegenous problems like this in a way similar to variation of parameters. The idea is to try and write the solution as an integral:

$$u(x,t) = \int_0^t u(x,t;s)\mathrm{d}s$$

Where each $u(x,t;s)$ is the solution in times $t \geq s$ with intial data given at $t = s$:

$$\begin{aligned} u_t(x,t;s) - \Delta u(x,t;s) &= 0 \; t > s \\ u(x,t;s) &= f(x,s) \; t = s \end{aligned}$$

By our work before, the solution is $u(x,t;s) = \int \Phi(x-y,t-s) f(y,s) \mathrm{d}y$. So we get:

$$u(x,t) = \int_0^t \int \Phi(x-y,t-s) f(y,s) \mathrm{d}y \mathrm{d}s$$

and indeed this works.

1.3.2. Mean Value Formula. For a fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$ define:

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^{n+1} \middle| s \leq t, \Phi(x-y,t-s) \geq \frac{1}{r^n} \right\}$$

THEOREM 1.14. *(Mean value property) Let $u \in C_1^2(U \times (0,T])$ solve the heat equation. Then:*

$$u(x,t) = \frac{1}{4r^n} \int \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dy ds$$

REMARK 1.5. This is the integral around “the heat ball”, a kind of ellipse for which (x,t) is the top point.

PROOF. Let $\phi(r) = \frac{1}{r^n} \int \int_{E(r)} u(y,x) \frac{|y|^2}{s^2} \mathrm{d}y \mathrm{d}s$ and work to comput ϕ'. □

1.3.3. Properties of Solutions.

1.3.3.1. *Strong Maximum Principle.*

THEOREM 1.15. *(Strong Maximum Principle for heat Equation) Assume that $u \in C_1^2 \left(U \times (0,T] \right) \cap C \left(U \times [0,T] \right)$ solves the heat equation. Then:*

$$\max_{\overline{U}_T} u = \max_{\overline{U}_T - U_T} u$$

i.e. the maximum is achieved either at the edges or the initial condition, but not in the interior or a later time.

PROOF. There is a proof using the Heat Ball. □

REMARK 1.6. In Fritz John, there is a proof using properties of maximma: Suppose first $u_t - \Delta u < 0$. Restrict attention to $\Omega_\epsilon = \omega \times (0, T-\epsilon)$ so that we have derivateive at the top end. $u_t - \Delta u < 0$ here means that interior local maxima are impossible (since these must have either $u_t = 0$ and $\Delta u \leq 0$ which contradicts $u_t - \Delta u < 0$) There can also not be a local maxima at at time $T - \epsilon$ (since these would have $u_t > 0$ and $\Delta u \leq 0$, again contradicting $u_t - \Delta u < 0$). Hence:

$$\max_{\overline{\Omega}_\epsilon} u = \max_{\partial^{(1)} \Omega_\epsilon} u \leq \max_{\partial^{(1)} \Omega} u$$

THEOREM 1.16. *(Uniqueness in bounded domains)*

PROOF. Subtract the solutions, minimum and maximum must be zero. □

1.3.3.2. *Unbounded Domains.*

THEOREM 1.17. *(Maximum principle in undbounded domain with sub-gaussian growth assumptions)*

If $u(x,t) \leq Ae^{a|x|^2}$ *and* u *satisfies the heat equation with initial value* $g(x)$ *, then:*

$$\sup_{\mathbb{R}^n\times[0,T]} u = \sup_{\mathbb{R}^n} g$$

PROOF. Take time small enough so that $\mathbf{E}\left(u(X_t,t)\right)$ makes sense. Fix $y \in \mathbb{R}^n$, $\mu > 0$ and put:

$$v_\mu = u(x,t) - \mu K(ix, it, T+\epsilon - t) \sim u(x,t) - \mu \exp\left((x-y)^2/T\right)$$

Then v will still satisfy the heat equation, and in a bounded region it satisfies the maximum principle, and very far away from y, the big negative exponetial term means that $\sup v \leq \sup g$. Since this works for every μ get the solution. □

THEOREM 1.18. *(Uniqueness for subgaussian tails in unbounded domains)*

PROOF. Subtract the two solutions. □

1.3.3.3. *Regularity.*

THEOREM 1.19. *(Smoothness) If* $u \in C_1^2(U_T)$ *solves the heat equation, then actually* $u \in C^\infty$

PROOF. Evan's makes a big deal of this....I think its not so hard to see by differentiating under the integral sign for the heat kernal? □

1.3.4. Energy Methods.

1.3.4.1. *Uniqueness.*

THEOREM 1.20. *(Uniqueness) There exists at most one solution* $u \in C_1^2\left(\overline{U}_T\right)$ *to the heat equation IVP, on a comapct set.*

PROOF. By subtracting the two solutions, we can suppose that $w_t - \Delta w = 0$ in U_T and $w = 0$ on Γ_T. Define the energy:

$$e(t) = \int_U w^2(x,t)\mathrm{d}x$$

Then:

$$\begin{aligned} \dot{e}(t) &= 2\int ww_t\mathrm{d}x \\ &= 2\int w\Delta w\mathrm{d}x \\ &= -2\int_U |Dw|^2\,\mathrm{d}x \leq 0 \end{aligned}$$

Hence the energy is only decreasing, $0 \leq e(t) \leq e(0) = 0$ means that $e(t) = 0$ for all t and so $w \equiv 0$. □

1.3.4.2. *Backwards Uniquencess.* What happens if we try to solve the heat equation backwards in time?

THEOREM 1.21. *(Uniquenss for the backwards equation) Suppose $u, \tilde{u}$ are two solutions to $u_t - \Delta u = 0$ in U_T and $u = g$ on $\partial U \times [0, T]$. If $u(x, T) = \tilde{u}(x, T)$ agree at the final time, then $u \equiv \tilde{u}$ within $\tilde{U_T}$.*

REMARK 1.7. In other words, if two temperature distributions on U agree at some time $T > 0$ and have the same boudnary conditions, then they must have been identically equal at all earlier times.

PROOF. Write $w = u - \tilde{u}$ again. Again we set $e(t) = \int_U w^2(x,t)\mathrm{d}x$ and we get $\dot{e}(t) = -2\int_U |Dw|^2 \,\mathrm{d}x$, a bit more work gets us $\ddot{e}(t) = 4\int(\Delta w)^2\mathrm{d}x$. Using $\int (Dw)^2 \,\mathrm{d}x = -\int w\Delta w \mathrm{d}x \le \left(\int w^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_U (\Delta w)^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$ we get:

$$\dot{e}(t)^2 \le e(t)\ddot{e}(t)$$

This will force $e(t) = 0$ for all t, for otherwise suppose that $e(t) > 0$ for $t_1 \le t < t_2$ but $e(t_2) = 0$. Set $f(t) = \log e(t)$ in the intervale $t_1 \le t < t_2$ and notice that $\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \ge 0$ which means that f is convex. But then f cannot go to $-\infty$ as $t \to t_2$. ☐

REMARK 1.8. The first thing we saw with the energy method is that its non-increasing (this proved uniqueness for the ordinary problem). The next thing we say, trhough the ODE $\dot{e}(t)^2 \le e(t)\ddot{e}(t)$ is that the energy is log-convex and so the energy can't go from being non-zero to being zero...it can approach zero but never get there.

1.4. Wave Equation

The wave equation is:

$$u_{tt} - \Delta u = 0$$

And the **non-homegneous wave equation** is:

$$u_{tt} - \Delta u = f$$

subject to the appropraite initial and boundary conditions. Here $t > 0$ and $x \in U$ where $U \subset \mathbb{R}^n$ is open. It is common to write $\Box u = u_{tt} - \Delta u$

1.4.1. Solution by Spherical Means.

1.4.1.1. *Solution when $n = 1$, d'Alembert's formula.* In one dimension we can "factor" the PDE as:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0$$

(Equivalently, change variables to the characterstics $\xi = x + t$ and $\eta = x - t$ to get $u_{\xi\eta} = 0$) Put $v = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_\xi$ so then $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v = v_\eta = 0$. This is exactly the transport eqaution for v! The solution is $v(x,t) = a(x - t)$ for some function a. Have then $u_t - u_x = a(x - t)$ is the transport eqaution for u. Solving this gives $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)\mathrm{d}y + b(x+t)$. then use the intial conditions to solve for the functions a, b in terms of g, h. End up with:

$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int\limits_{x-t}^{x+t} h(y)\mathrm{d}y$$

This is called **d'Alembert's formula**.

1.4.1.2. *A reflection method.* You can use d'Alembert's formula to solve certain problems with boundary condtions. For example:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{on } \{x = 0\} \times (0,\infty) \end{cases}$$

If you extend the intial data by odd reflection you get the solution. Draw some pictures with characterstic lines to see what's up! Get a solution:

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y & \text{if } x \geq t \geq 0 \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\mathrm{d}y & \end{cases}$$

1.4.1.3. *Spherical Means.* Suppose $n \geq 2$ now. We will reduce the wave equation in dimension n to dimension 1 by the method of spherical measn. Say u solves:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

We will study the space averages of u over certain spheres. These averages, taken as functions of time t and radius mr turn out to solve the **Euler-Poisson-Darboux equation**, a PDE we can (for **odd** n) convert to the 1d wave equation. Applying d'Alembert's formlula

Definition 1.1. Define the spherical mean as:

$$U(x;r,t) := \frac{1}{nr^{n-1}\omega_n}\int\limits_{\partial B(x,r)} u(y,t)\mathrm{d}S(y)$$

Fix x, and regard $U(x;r,t)$ as a function of r and t. What PDE does it solve?

Lemma 1.1. *(Euler-Poisson-Darboux equation) Fix $x \in \mathbb{R}^n$ and let u satisfy the wave equation. The the spherical means U satisfy:*

$$U_{tt} - U_{rr} - \frac{n-1}{r}U_r = U_{tt} - r^{1-n}\partial_r\left(r^{n-1}\partial_r U\right) = 0$$

Proof. Calculate U_r (this is similar to when we calculated ϕ' for $\phi = \frac{1}{r^n\omega_n}\int_{\partial B(x,r)} u(y)\mathrm{d}S(y)$ for the Laplace equation. I do this in the following lemma_. Get:

$$\begin{aligned} U_r &= \frac{1}{nr^{n-1}\omega_n}\int\limits_{B(x,r)} \Delta u(y,t)\mathrm{d}y \\ &= \frac{1}{nr^{n-1}\omega_n}\int\limits_{B(x,r)} u_{tt}\mathrm{d}y \end{aligned}$$

Hence we have:

$$\begin{aligned} r^{n-1}U_r &= \frac{1}{n\omega_n}\int\limits_{B(x,r)} u_{tt}(y,t)\mathrm{d}y \\ \Longrightarrow \frac{\partial}{\partial r}\left(r^{n-1}U_r\right) &= \frac{1}{n\omega_n}\int\limits_{\partial B(x,r)} u_{tt}(y,t)\mathrm{d}S(y) \\ &= r^{n-1}\left(\frac{1}{nr^{n-1}\omega_n}\int\limits_{\partial B(x,r)} u(y,t)\mathrm{d}S(y)\right)_{tt} \\ &= r^{n-1}U_{tt} \end{aligned}$$

□

LEMMA 1.2. *Let* $\phi(r) = \frac{1}{nr^{n-1}\omega_n}\int_{\partial B(x,r)} u(y)dS(y)$. *Then* $\frac{d}{dr}\phi = \frac{1}{nr^{n-1}\omega_n}\int_{B(x,r)} \Delta u(y)dy$

PROOF. First step: **Rewrite as an integral over** $B(0,1)$: have:

$$\begin{aligned} \phi(r) &= \frac{1}{nr^{n-1}\omega_n}\int\limits_{\partial B(x,r)} u(y)\mathrm{d}S(y) \\ &= \frac{1}{\omega_n}\int\limits_{\partial B(0,1)} u(x+zr)\mathrm{d}S(z) \end{aligned}$$

Then:

$$\phi'(r) = \frac{1}{\omega_n}\int\limits_{\partial B(0,1)} \left(Du(x+zr)\cdot z\right)\mathrm{d}S(z)$$

Now z is the outward facing normal, so we can apply the **Divergence theorem**:

$$\begin{aligned} \phi'(r) &= \frac{1}{\omega_n}\int\limits_{\partial B(0,1)} [\nabla u(x+zr)]\cdot\vec{n}\mathrm{d}S(z) \\ &= \frac{1}{nr^{n-1}\omega_n}\int\limits_{\partial B(x,r)} [\nabla u(y)]\cdot\vec{n}\mathrm{d}S(y) \\ &= \frac{1}{nr^{n-1}\omega_n}\int\limits_{B(0,1)} \nabla\cdot[\nabla u(x+zr)]\,\mathrm{d}(z) \\ &= \frac{r}{n}\fint \Delta u(y)\mathrm{d}y \end{aligned}$$

□

1.4.1.4. $n = 3$. When $n = 3$ the equation is:

$$U_{tt} - U_{rr} - \frac{2}{r}U_r = 0$$

If we change variables $\tilde{U} = rU$ now we get:

$$\begin{aligned}\tilde{U}_{tt} &= rU_{tt} \\ &= rU_{rr} - 2U_r \\ &= \partial_r \left(U + rU_r\right)_r \\ &= \partial_{rr}\left(rU\right)\end{aligned}$$

Because of the multiplication by r, we know $\tilde{U} = 0$ at $r = 0$. So $\tilde{U}$ satisfies a 1-D wave equation with this boundry term:

$$\begin{cases}\tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}^+ \times (0,\infty) \\ \tilde{U} = \tilde{G}, \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}^+ \times \{t = 0\} \\ U \tilde{=} 0 & \text{on } \{r = 0\} \times (0,\infty) \\ \end{cases}$$

We have the solution of this when $0 \le r \le t$ (this is what we need when we take $\lim_{r\to 0}$ which is what will let us recover the solution):

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\mathrm{d}y$$

Taking limits now:

$$\begin{aligned}u(x,t) &= \lim_{r\to 0^+} \frac{\tilde{U}(x;r,t)}{r} \\ &= \lim_{r\to 0^+}\left[\frac{\tilde{G}(r+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\mathrm{d}y\right] \\ &= \tilde{G}'(t) + \tilde{H}(t)\end{aligned}$$

Going back to our original equation, we have:

$$u(x,t) = \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\mathrm{d}S(y)$$

This is called **Kirchoff's Formula** for the solution of the initial value problem.

1.4.1.5. *Method of Descent.* To get the soltuion when $n = 2$, use the method of desent...look for 3D solutions that are constant along one of the space direction.

1.4.1.6. *Solution for other odd/even n.* When $n = 2k+1$ is odd you can make the spherical means work out by putting $\tilde{U}(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)$

For even n, again can use the method of descent.

1.4.2. Non-homegeonous problem. Duhamel's principle again. Say you want to solve:

$$\begin{cases}u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0, u_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}\end{cases}$$

First look for solutions to:

$$\begin{cases}u_{tt}(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot;s) = 0, u_t(\cdot;s) = f(\cdot;s) & \end{cases}$$

Then set:

$$u(x,t)=\int_0^t u(x,t;s)\mathrm{d}s$$

1.4.3. Energy Methods.

THEOREM 1.22. *(Uniqueness using energy)*

PROOF. Assume we have 0 boundry conditinons. Define the energy:

$$e(t)=\frac{1}{2}\int_U u_t^2(x,t)+|\nabla u|^2\,\mathrm{d}x$$

Check that $\dot{e}(t)=0$ using integration by parts to make Δ appear:

$$\dot{e}(t)=\int_U uu_t+\nabla u\cdot\nabla u_t\mathrm{d}x$$

Use now$\nabla\cdot(u\nabla v)=\nabla u\cdot\nabla v+u\Delta v$:

$$\begin{aligned}\dot{e}(t) &= \int_U \left(uu_t+\nabla\cdot(u_t\nabla u)-u\nabla u\right)\mathrm{d}x \\ &= \int_U u\left(u_t-\Delta u\right)\mathrm{d}x+\int_{\partial U}(u_t\nabla)\cdot\vec{n}\mathrm{d}S\end{aligned}$$

Both terms are 0 □

1.4.3.1. *Domain of Dependence.* Conisder the cone $C=\{(x,t)\,|0\le t\le t_0,|x-x_0\le t_0-t|\}$

THEOREM 1.23. *(Finite propogation spped) If $u\equiv u_t\equiv 0$ on $B(x_0,t_0)$ then $u\equiv 0$ within the cone C.*

REMARK 1.9. This shows that any distrubances from outside the cone cannot effect the interior of the cone! This is already known from the explicit solutions (in fact when $n=3$, the domain of dependence is the SHELL of a cone, not a cone) but the energy method proof is a much simpler way to prove this face.

PROOF. Define $e(t)$, this time integrating only over the cone:

$$e(t)=\frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2+|\nabla u|^2\,\mathrm{d}x$$

This time $\dot{e}$ will have some boundary terms:=

$$\begin{aligned}\dot{e}(t) &= \int_{B(x_0,t_0-t)} u_tu_{tt}+(\nabla u)\cdot(\nabla u_t)\,\mathrm{d}x-\frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2+|\nabla u|^2\,\mathrm{d}S \\ &= \int_{B(x_0,t_0-t)} u_t\left(u_{tt}-\Delta u\right)\mathrm{d}x+\int_{\partial B(x_0,t_0-t)}(u_t\nabla u)\cdot\vec{n}\mathrm{d}S-\frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2+|\nabla u|^2\,\mathrm{d}S \\ &= \int_{\partial B(x_0,t_0-t)}(u_t\nabla u)\cdot\vec{n}-\frac{1}{2}u_t^2-\frac{1}{2}|\nabla u|^2\,\mathrm{d}S\end{aligned}$$

Now use Cauchy Schwarz, $|u_t\nabla u\cdot\vec{n}|\le|u_t|\,|\nabla u|\le\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2$ so we have $\dot{e}(t)\le 0$ □

Sobolev Spaces

These are notes from Chapter 5 of [**1**].

2.4.4. Weak Derivatives.

DEFINITION 2.2. For a function $u, v \in L^1(U)$ we say that v is the α-th weak partial derivative of u, written $D^\alpha u = v$ if:

$$\int u D^\alpha \phi \mathrm{d}x = (-1)^{|\alpha|} \int\limits_U v \phi \mathrm{d}x$$

for all $\phi \in C_c^\infty(U)$, i.e. infinitely differentiable compactly supported functions.

PROPOSITION 2.2. *Weak derivatives are unique up to a.e.*

PROOF. Since if $v, \tilde{v}$ are weak derivatives of u then:

$$\begin{aligned} \int u D^\alpha \phi \mathrm{d}x &= (-1)^{|\alpha|} \int\limits_U v \phi \mathrm{d}x = (-1)^{|\alpha|} \int\limits_U \tilde{v} \phi \mathrm{d}x \\ \implies \int_U (v - \tilde{v}) \phi \mathrm{d}x &= 0 \ \forall \phi \in C_c^\infty(U) \end{aligned}$$

Since C_c^∞ is dense in L^1, it must be that $v - \tilde{v} = 0$ a.e. □

2.4.5. The Sobolev Space.

DEFINITION 2.3. The Sobolev space:

$$W^{k,p}(U) = \{u \in L^p(U) \ : \ D_{wk}^\alpha u \in L^p(U) \forall |\alpha| \leq k\}$$

i.e. we have k weak derivatvie that are in L^p. We have the norm on this space:

$$\|u\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \int\limits_U |D^\alpha u|^p \, \mathrm{d}x$$

When $p = 2$ this space is a Hilbert space so we write H for "Hilber space":

$$H^k(U) = W^{k,2}(U)$$

with the inner product:

$$\langle u, v \rangle_{H^k} = \sum_{|\alpha| \leq k} \langle D_{wk}^\alpha u, D_{wk}^\alpha v \rangle_{L^2}$$

THEOREM 2.24. *$W^{k,p}$ is a Banach space for every k, p*

PROOF. It's easily verified that its a norm. To see that it is complete, notice that $\|u\|_{W^{k,p}} \leq \|D^\alpha u\|_{L^p}$ for any $|\alpha| \leq k$. Hence for a Cauchy sequence u_m, we know that for each $|\alpha| \leq k$, $\{D^\alpha u_m\}$ is a Cauchy sequence in L^p. Since each L^p is a Banach space, each of these convergese to something, call it u_α so that $D^\alpha u_m \to u_\alpha$ in L^p. In particular $u_0 = \lim_{m\to\infty} u$ exists. We now claim that $u_0 \in W^{k,p}$ and that $D^\alpha u_0 = u_\alpha$. Indeed, just check how they act on test functions. Hence $u_m \to u$ in $W^{k,p}(U)$ as required. □

THEOREM 2.25. *If U is bounded and ∂U is C^1 then if $u \in W^{k,p}$ then there exists $u_m \in C^\infty(\overline{U})$ so that:*

$$u_m \to u \text{ in } W^{k,p}(U)$$

PROOF. The idea is to mollify $v_\epsilon = \eta_\epsilon * u$. The regularity makes sure its ok near the boundary. □

This leads to the alternate definition of the Sobolev space:

$$W^{k,p} = \overline{C^k(U)}_{L^p}$$

The closure of the C^k function in L^p.

THEOREM 2.26. *(Rellich Compactness Theorem)*
$W^{1,p}(U)$ is a compact subset of $L^q(U)$for each $q \leq p^*$ where $\frac{1}{p} + \frac{1}{p^*} = 1$. In particular:

$$H^1 = W^{1,2}(U) \overset{compact}{\subset} L^2(U)$$

PROOF. The proof is basically by arzela ascoli, since any bounded set in $W^{1,2}$will have bounded weak derivatives. You need to work a bit though since the functions themselves are not actually smooth. □

THEOREM 2.27. *(Poincare Inequality). If Ω is a bounded domain, then there is a constant C depending only on Ω so that:*

$$\int_\Omega u^2 \, dx \leq C \int_\Omega |\nabla u|^2 \, dx$$

for every $u \in C_0^1$. By completion the inequality holds for all $u \in H_0^{1,2}$.

REMARK 2.10. For Poincare inequality it is important that we have the zero condition on the boundary.

2.5. Second Order Elliptic Equations

Let L be the operator by:

$$Lu = -\sum_{i,j} a^{ij}(x) u_{x_i x_j} + \sum_i b^i(x) u_{x_i} + c(x) u$$

We call it **uniformly ELLIPTIC** if the prinicple part has

$$\sum a^{ij}(x)\xi_i\xi_j \geq \theta |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and $x \in U$. (in other words the matrix a^{ij} is positive definite at every point $x \in U$, and the poistivie-defintetness is uniform over $x \in U$)

We study the PDE:

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

By integration by parts, we get the following weak form of the solution, for any $v \in C_c^\infty(U)$ we want:

$$\int (Lu)\, v = \int fv$$

$$\int_U \sum_{i,j}^{n} a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} u_{x_i} v + cuv\mathrm{d}x = \int_U fv\mathrm{d}x$$

For this reason it makes sense to look in the space $u \in H_0^1(U)$ (remmeber H stands for Hilbert space - L^2 so this is the space where you have 1 weka derivative whcih is in L^2.)

For this reason it makes sense to look at the bilinear form $B[\ ,\ ]_{H_0^1}$ associated with L on the space H_0^1 defined by:

$$B[u,v]_{H_0^1} = \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_i} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv\mathrm{d}x$$

$u \in H_0^1$ is a weak solutioon of the boundary value problem if:

$$B[u,v]_{H_0^1} = \langle f, v \rangle_{L^2} \text{ for all } v \in H_0^1$$

This is sometimes called the variational formula.

THEOREM 2.28. *(Lax-Milgram Theorem) Let H be a Hilbert space. Assume that $B : H \times H \to \mathbb{R}$ is bilinear mapping for which, $\exists \alpha, \beta > 0$ so that:*

$$|B[u,v]| \le \alpha \|u\| \|v\|$$

and:

$$\beta \|u\|^2 \le B[u,u]$$

Then for any bounded linear functional $f : H \to \mathbb{R}$on H, there exists a unique elemetn $u \in H$ so that:

$$B[u,v] = \langle f, v \rangle$$

for all $v \in H$.

PROOF. By the Riesz representation theorem, we know that:

$$B[u,v] = \langle Au, v \rangle$$

for some bounded linear operator $A : H \to H$ with $\|A\| \le \alpha$. We now claim that A is one-to-one and $range(A)$ is closed in H. Indeed, the lower bound we have $\beta \|u\| \le \|Au\|$ so u is 1-1. Now if Au_n is a Cauchy sequence then u_n is a cauchy sequence too since $\beta \|u_n - u_m\| \le \|A(u_n - u_m)\|$ which shows the range is closed.

We now claim that $R(A) = H$. Otherwise, since $R(A)$ is closed, find a $w \in R(A)^\perp$.Then $\langle Aw, w \rangle = 0$ but we know $\langle Aw, w \rangle = B[w,w] \ge \beta \|w\|^2 \ne 0$.

Hence A is invertible so for any $f \in H$ let $u = A^{-1} f$ and we will have our solution. ∎

We now try to apply this to the bilinear form for $B[u,v]$ we had above. The first inequality $|B[u,v]| \leq \alpha \|u\| \|v\|$ follows as long as the constants that appear a^{ij}, b^j, c are bounded and the lower bound.

Using the uniform ellipticity we can get:

$$\beta \|u\|^2_{H^1_0(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

However if $\gamma \neq 0$ then we can't apply Lax-Milgram directly

EXAMPLE 2.1. For Laplace's equation, the $B[u,v] = \int \nabla u \cdot \nabla v \mathrm{d}x$ so we have $B[u,u] = (a_1 + a_2) \int |\nabla u|^2 \mathrm{d}x \geq \frac{a_1}{C} \int u^2 \mathrm{d}x + a_2 \int |\nabla u|^2 \mathrm{d}x$. If we choose a_1, a_2 so that $a_1/C = a_2$ then this is exactly some constant times $\|u\|_{H^1_0}$ and the Lax Milgram theorem applies.

The Single First-Order Equation

These are notes from Chapter 1 of [**2**].

3.6. Introduction

DEFINITION 3.4. A PDE for a function $u(x, y, \ldots)$ is a relation of the form

$$F(x, y, \ldots, u, u_x, u_y, \ldots) = 0$$

Where F is a given function of the independent variables $x, y, \ldots$ and of the "unknown" function u and a finite number of its partial derivatives. We call u a sol'n if it satisfies the above idenitcally. Unless otherwise stated we assume that u and its derivatives are continuous.

The **order** of a PDE is the order of the highest derivative that occurs. A PDE is **linear** if it is linear in the unknown functuion and its derivatives (the coefficients may depend on $x, y, \ldots$) e.g. $F = a(x, y, \ldots)u + b(x, y, \ldots)u_x + c(x, y, \ldots)u_y + \ldots$ A PDE of order m is called **quasilinear** if it is linear in the derivates of order m with coefficients that depend on x, y and also on the derivatives of order $< m$.

3.7. Examples

Linear equations:
i) The Laplace Equation $\Delta u = u_{x_1 x_1} + \ldots + u_{x_n x_n}$
ii) The wave equation $u_{tt} = c^2 \Delta u$
iii) Maxwell's equations
iv) More complicated wave equations
v) Heat equation $u_t = k\Delta u$
vi) Schrodinger's equation $c_1 \psi_t = -c_2 \Delta\psi + V\psi$
Non-linear equations:
vii) Minimal surface $z = u(x, y)$ has the second-oder quasi-linear equation:

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$$

viii) Velocity potential for a two dimensional flow...
ix) Navier-Stokes equation
x) KdV equation:

$$u_t + cuu_x + u_{xxx} = 0$$

3.8. Analytic Solutions and Aprroximation Methods in a Simple Example

Consider the simplest PDE:

$$u_t + cu_x = 0$$

Along a line of the family:

$$x - ct = const = \xi$$

we have:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} u\left(ct + \xi, t\right) = cu_x + u_t = 0$$

So u is constant along these characteristic lines, and depends only on the parameter ξ which distinguishes the different lines. The general solution then has the form:

$$u(x,t) = f(\xi) = f(x - ct)$$

Notice that:

$$u(x,0) = f(x)$$

And every u of this form is a solution to the PDE provided that f is C^1

Notice that the value of u at the point (x,t) depends only on the intersection of the characterisic line through (x,t) with the initial $x-$axis, namely at the point $x = \xi$. We say in this case that **domain of dependence** of $u(x,t)$ on the initial values is represented by the single point ξ. The **influence** of the initial values at a particular point ξ on the solution $u(x,t)$ is felt only on the points of the characteristic line.

If for each fixed t the function u is represented by its graph in the $xu-$plane, we find that the graph at the time $t = T$ is obtained by translating the graph at the time $t = 0$ parallel to the $x-$axis by the amound cT:

$$u(x,0) = u(x + cT, T) = f(x)$$

The graph of the solution represents a **wave** propagating to the right with velocity c without changing its shape.

We use this example to study the numerical solution of a PDE by **finite differences**. Cover the $x-t$ plane with a mesh of size h in the x-direction and a mesh of size k in the $t-$direction. Then the PDE approximation is:

$$\frac{v(x,t+k) - v(x,t)}{k} + c\frac{v(x+h,t) - v(x,t)}{h} = 0$$

There is a discussion on this finite difference scheme and its stability on pages 6-8

3.9. Quasi-linear equations

Consider a quasi-linear PDE in two dimensions:

$$\begin{aligned} a(x,y,u)u_x + b(x,y,u)u_y &= c(x,y,u) \\ \langle u_x, u_y, -1\rangle \cdot \langle a(x,y,u), b(x,y,u), c(x,y,u)\rangle &= 0 \end{aligned}$$

We represent the function $u(x,y)$ by a surface $z = u(x,y)$ in xyz-space. Surfaces corresponding to solutions of a PDE are called **integral surfaces** of the PDE (i.e. integral surfaces are solutions $u(x,y) = z(x,y)$) The prescribed functions $a(x,y,u), b(x,y,u), c(x,y,u)$ define a field of vectors in xyz-space. $\langle a(x,y,u), b(x,y,u), c(x,y,u)\rangle$ is called the **characteristic direction** for the PDE. Obviously, only the direction of this vector (not its magnitude) matters for the PDE. Since $\langle u_x, u_y, -1\rangle$ constitute direction numbers of the normal of the surface $z = u(x,y)$ we see that the PDE is the condition that the normal of an integral surface at any point is perpendicular to the characteristic direction.

With the field of characteristic directions with direction numbers (a, b, c) we associate the family of **characteristic curves** which at each point are tangent to that direction field. Along a characteristic curve, the relation:

$$\frac{\mathrm{d}x}{a(x,y,z)} = \frac{\mathrm{d}y}{b(x,y,z)} = \frac{\mathrm{d}z}{c(x,y,z)}$$

holds. Refereing the curve to a siutable paratmeter t (or calling the common ration above $\mathrm{d}t$) we get a system of ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(x,y,z), \ \frac{\mathrm{d}y}{\mathrm{d}t} = b(x,y,z), \ \frac{\mathrm{d}z}{\mathrm{d}t} = c(x,y,z)$$

The system is "autonomous" since t does not appear explicitly. Using any other parameter along the curve amounts to replacing a, b, c by proportional quantities, which does not change the characeteristic curve or the PDE.

Assuming that a, b, c are C^1 we know from the theory of ODEs that through each point of Ω there is exactly one characteristic curve. Consequently, there is a 2 parameter family of characteristic curves in xyz-space.

PROPOSITION 3.3. *If a surface S defined by $z = u(x, y)$ is a union of characteristic curves, then S is an integral surface*

PROOF. (This is just a bit of geometry) Take any point P of S. Let Π be the tangent plane through S at P. Since S is a union of char. curves, there is a char. curve γ through P. The tangent to γ has characteristic direction (this is the definition of a char curve). The tangent to γ also lies in the tangent plane Π. The tangent Π is perpendicular to the normal of the surface S, so since γ is in Π, we conclude that the normal to S at P is perpendicular to the characteristic direction, as desired. □

We can also show that every integral surface S is the union of char curves or equivalently that every point of S there passes a char curve contained in S. This is a consequence of the following theorem:

THEOREM 3.29. *Let the point $P = (x_0, y_0, z_0)$ lie on the integral surface $S : z = u(x, y)$. Let γ be the char. curve through P. Then γ lies completely in S.*

PROOF. Suppose γ is parametrized by $\gamma(t) = (x(t), y(t), z(t))$. (i.e. γ is the solution to $\frac{\mathrm{d}x}{\mathrm{d}t} = a(x,y,z)$, $\frac{\mathrm{d}y}{\mathrm{d}t} = b(x,y,z)$, $\frac{\mathrm{d}z}{\mathrm{d}t} = c(x,y,z)$ which passes through $P = (x_0, y_0, z_0)$ at $t = t_0$) From γ and S we form the expression:

$$U = z(t) - u\left(x(t), y(t)\right) = U(t)$$

Clearly $U(t) = 0$ since $P \in S$. And now:

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} &= \frac{\mathrm{d}z}{\mathrm{d}t} - u_x\left(x(t), y(t)\right)\frac{\mathrm{d}x}{\mathrm{d}t} - u_y\left(x(t), y(t)\right)\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= c(x,y,z) - u_x(x,y)a(x,y,z) - u_y(x,y,z)b(x,y,z) \end{aligned}$$

$(x = x(t), y = y(t), z = z(t)$ is tacitly assumed) We have used the fact that γ is a char. curve to know the derivatives $\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}$. Now since, $z = U + u(x, y)$, this is:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = c(x, y, U + u(x,y)) - \ldots - u_y(x, y, U + u(x,y))b(x, y, U + u(x,y))$$

This is a crazy ODE! However, we are lucky since we see that $U(t) \equiv 0$ is a solution by virtue of the fact that S is an integral soluiton i.e. u satisfies the PDE.

Since ODEs have unique solutions we know that $U \equiv 0$ is THE solution. In other words $z(t) = u(x(t), y(t))$ along the curve i.e. γ lies in S. □

3.10. The Cauchy Problem for the Quasi-Linear Equation

We now have a simple description for the general solution u of the Quasi-Linear PDE: namely the integral surface $z = u(x, y)$ is a union of characteristic curves. To get a better insight into the structure of the manifold of solutions, it is desirable to have a definite method of generating solutions in terms of a prescribed set F of functions callsed "data".

A simple way of seeking an individual $u(x, y)$ is to specify a curve Γ in xyz-space which is to be contained in the integral surface $z = u(x, y)$. Let Γ be represented parametrically by:

$$x = f(s), \;\; y = g(s), \;\; z = h(s)$$

We are asking for the solution $u(x, y)$ such that:

$$h(x) = u\left(f(s), g(s)\right)$$

holds. (i.e. the curve $(f(s), g(s), h(s))$ is part of the integral surface) This is known as the **Cauchy Problem** for the PDE.

An example is the initial value problem:

$$u(x, 0) = h(x)$$

So that the curve in question is $x = s, y = 0, z = h(s)$.

THEOREM 3.30. *(Local Existence Theorem for the Cauchy Problem)*

Suppose that $f(s), g(s), h(s)$ are C^1 in a n'h;d of a point s_0 and let $P_0 = (x_0, y_0, z_0) = (f(s_0), g(s_0), h(s_0))$. Assume also that $a(x, y, z), b(x, y, z), c(x, y, z)$ are C^1 in a n'h'd of P_0.

Then if the Jacobian:

$$\begin{vmatrix} f'(s_0) & g'(s_0) \\ a(x_0, y_0, z_0) & b(x_0, y_0, z_0) \end{vmatrix} \neq 0$$

Then there is locally a unique integral surface $\Sigma : z = u(x, y)$ to the PDE.

PROOF. For each s near s_0, consider the ODE:

$$\begin{aligned} \frac{\partial}{\partial t} X(s, t) &= a\left(X(s, t), Y(s, t), Z(s, t)\right) \\ \frac{\partial}{\partial t} Y(s, t) &= b\left(X(s, t), Y(s, t), Z(s, t)\right) \\ \frac{\partial}{\partial t} Z(s, t) &= c\left(X(s, t), Y(s, t), Z(s, t)\right) \end{aligned}$$

With initial condition at $t = 0$:

$$\begin{aligned} X(s, 0) &= f(s) \\ Y(s, 0) &= g(s) \\ Z(s, 0) &= h(s) \end{aligned}$$

From the theory of ODEs, the C^1 condition on the functions tell us that there exists a unique solution AND moreover that the solutions will depend smoothly on the parameter s. i.e. we have C^1 solutions $X(s, t), Y(s, t), Z(s, t)$ in a n'h'd of (s_0, t).

Now when does this represent a surface? It is precisly when we can invert these equations say to get $s = S(x,y), t = T(x,y)$ and then put $z = u(x,y) = Z(S(x,y), T(x,y))$ will solve our PDE.

The implicity function theorem guarentees we can do precisly this, as long as we have the Jacobian condition from the hypothesis. □

3.11. Examples

EXAMPLE 3.2. (1)
The PDE:

$$u_y + cu_x = 0$$

With initial condition:

$$u(x,0) = h(x)$$

PROOF. Rewrite the PDE as:

$$\langle u_x, u_y, -1\rangle \cdot \langle c, 1, 0\rangle = 0$$

Put $x = x(t), y = y(t), z = z(t)$ and then a characteristic curve from the intial point $x(0) = s, y(0) = 0, z(0) = h(s)$ has:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= c \\ \frac{\mathrm{d}}{\mathrm{d}t}y(t) &= 1 \\ \frac{\mathrm{d}}{\mathrm{d}t}z(t) &= 0 \\ \Longrightarrow x(t) &= ct + s \\ y(t) &= t \\ z(t) &= h(s) \end{aligned}$$

Now, invert the formula to get s, t as a function of x, y we have $t(x,y) = y$ and $s(x,y) = x - cy$.

[This is where the Jacobian condition comes in....making sure this system is invertable] So finally then:

$$u(x,y) = z(t(x,y), s(x,y)) = h(x - cy)$$

is the solution! □

EXAMPLE 3.3. [Homogeneous Functions]
Suppose:

$$\sum_{k=1}^{n} x_k u_{x_k} = \alpha u$$

With initial condition:

$$u(x_1, \ldots, x_{n-1}, 1) = h(x_1, \ldots, x_{n-1})$$

PROOF. Rewrite the PDE as:

$$\langle x_1, \ldots, x_n, \alpha u\rangle \cdot \langle u_{x_1}, \ldots, u_{x_n}, -1\rangle = 0$$

Find characteristic curves. Letting $x_i = x_i(t)$ and $z(t) = u$, We get the system:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) &= x_i(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}z(t) &= \alpha z(t)\end{aligned}$$

With initial conditions on the specified curve:

$$\begin{aligned}x_i(0) &= s_i \text{ for } 1 \le i \le n-1\\ x_i(0) &= 1 \text{ for } i = n\\ z(0) &= h(s_1, \ldots, s_{n-1})\end{aligned}$$

Solving these we get:

$$\begin{aligned}x_i(t) &= s_i e^t \text{ for } i = 1, \ldots, n-1\\ x_i(t) &= e^t \text{ for } i = n\\ z &= e^{\alpha t} h(s_1, \ldots, s_n)\end{aligned}$$

Inverting this and so on, we get:

$$z = u(x_1, \ldots, x_n) = x_n^\alpha h\left(\frac{x_1}{x_n} \ldots, \frac{x_{n-1}}{x_n}\right)$$

This has the property that:

$$u\left(\lambda x_1, \ldots, \lambda x_n\right) = \lambda^\alpha u(x_1, \ldots, x_n)$$

(The converse also holds: take ∂_λ of both sides and then plug in $\lambda = 1$ to recover the PDE)

This condition also causse some issues if $\alpha < 0$, for then we could take λ very close to zero to get that for x on the unit ball, and taking $\lambda \to 0$ would give $u(0) = \infty u(x)$, so this forces $u(x)$ to be zero everywhere unless we allow the PDE to blow up at 0. □

Example 3.4. (Burger's Equation)

$$u_y + u u_x = 0$$

With initial conditon:

$$u(x, 0) = h(x)$$

Proof. Rewrite the PDE as:

$$\langle u_x, u_y, -1\rangle \cdot \langle u, 1, 0\rangle = 0$$

So the characteristic curves follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= z\\ \frac{\mathrm{d}}{\mathrm{d}t}y(t) &= 1\\ \frac{\mathrm{d}}{\mathrm{d}t}z(t) &= 0\end{aligned}$$

With inital conditions $x(0) = x, y(0) = 0, z(0) = h(s)$. Going through it all we get:

$$u = h(x - uy)$$

This equation has a problem, you can see this from the fact the characteristics INTERSECT unless $h(s)$ is non-decreasing. One way to see this is to take the x derivative of above to get:

$$u_x = \frac{h'(s)}{1 + h'(s)y}$$

Which becomes infinite in finite time if $h'(s) < 0$ (happens at the time $y = -\frac{1}{h'(s)}$)

Another way to see this is to just draw the curves. From $y = 0$ the curves leave the orgin and have slope $h(x)$, so if $h(x_1) > h(x_2)$ with $x_1 < x_2$, then these two curves will meet in finite time. Since the PDE is constant along curves this is a problem as at this point we would have $h(x_1) = h(x_2)$

We can find a weak solution of the Burger's equation as follows. First, rewrite the PDE as:

$$\frac{\partial R(u(x,y))}{\partial y} + \frac{\partial S(u(x,y))}{\partial x} = 0$$

Where $R(u), S(u)$ are any functions for which $S'(u) = uR'(u)$ and for which $R'(u) \neq 0$. (In this case we have $\frac{\partial}{\partial y} R \circ u = R'(u)\frac{\partial u}{\partial y}$ and $\frac{\partial}{\partial x} S \circ u = S'(u)\frac{\partial u}{\partial x} = R'(u)\left(u\frac{\partial u}{\partial x}\right)$ so canceling out $R'(u)$ in the above gives exactly Burger's equation. Integrating the above D.E. gives an integral conservation law:

$$0 = \frac{\mathrm{d}}{\mathrm{d}y}\int_a^b R(u(x,y))\mathrm{d}x + S(u(b,y)) - S(u(a,y))$$

(If you want you can see the above as $\nabla \cdot \vec{(R,S)}(u) = 0$, so we can do a diverengence theorem around any set). We can take this as our defintion for weak solutions: namely we say u is a solution if it satisfies the above integral equation. u does not have to be differentiable for this to make sense.

We can use this to examine the shocks that develop from the burger's equation PDE: □

PROPOSITION 3.4. *Let $x = \xi(y)$ be the curve of the shock. Let u^+ denote the value just to the left and just to the right of the shock by: $u^-(x) := \lim_{y\to\xi(x)^+} u(x,y)$, $u^+(x) := \lim_{y\to\xi(x)^-} u(x,y)$. We have the following shock condition:*

$$\frac{d\xi}{dy}(x) = \frac{S(u^+(x)) - S(u^-(x))}{R(u^+(x)) - R(u^-(x))}$$

PROOF. For a weak solution u, we have the defining integral equation that u satisfies. Our integral equation this over a region $a < \xi(y) < b$ is (we have to do a differentiate-the-integral-chain-rule):

$$\begin{aligned}
0 &= S(u(b,y)) - S(u(a,y)) + \frac{\mathrm{d}}{\mathrm{d}y}\left(\int_a^\xi R(u)\mathrm{d}x + \int_\xi^b R(u)\mathrm{d}x\right) \\
&= S(u(b,y)) - S(u(a,y)) + \left(\frac{\mathrm{d}\xi}{\mathrm{d}y}(x)R(u^-) + \int_a^\xi \frac{\partial R(u)}{\partial y}\mathrm{d}x\right) + \left(-\frac{\mathrm{d}\xi}{\mathrm{d}y}(x)R(u^+) + \int_\xi^b \frac{\partial R(u)}{\partial y}\mathrm{d}x\right) \\
&= S(u(b,y)) - S(u(a,y)) + \frac{\mathrm{d}\xi}{\mathrm{d}y}(x)R(u^-) - \frac{\mathrm{d}\xi}{\mathrm{d}y}(x)R(u^+) + \int_a^\xi \frac{\partial S(u)}{\partial x}\mathrm{d}x + \int_\xi^b \frac{\partial S(u)}{\partial x}\mathrm{d}x \\
&= S(u(b,y)) - S(u(a,y)) + \frac{\mathrm{d}\xi}{\mathrm{d}y}(x)R(u^-) - \frac{\mathrm{d}\xi}{\mathrm{d}y}(x)R(u^+) + S(u^+) - S(u(a,y)) + S(u(b,y)) - S(u^+) \\
&= \frac{\mathrm{d}\xi}{\mathrm{d}y}(x)\left(R(u^-) - R(u^+)\right) + \left(S(u^+) - S(u^-)\right)
\end{aligned}$$

And the condition follows. □

3.12. The General First-Order Equation for a Function of Two Variables

3.12.1. The Monge cone. The general (i.e. not quasilienar) first-order partial differntial equation for a function $z = u(x,y)$ has the form:

$$F(x,y,z,p,q) = 0$$

Where $p = u_x, q = u_y$. There can be fully non-linear dependence on u_x and u_y in this framework. We assume that F is C^2 with respect to x,y,z,p,q. We will again find a geometrical interpretation in terms of integral surfaces that will allow us to solve things.

Think of the PDE $F(x,y,z,p,q) = 0$ as a relation between the coordinates (x,y,z) and the normal to the integral surface $(p,q,-1)$. An integral surface passing through a point $P_0 = (x_0,y_0,z_0)$ must have a tangent plane given by:

$$z - z_0 = p(x-x_0) + q(y-y_0)$$

and, since it is an integral surface, it must satisfy:

$$F(x_0,y_0,z_0,p,q) = 0$$

F is ONE restricting equation where in general we could have had TWO free parameters p and q to play with. Thus the differential equation restricts the possible *tangent planes* of an integral surface at P_0 to a one-parameter family. In general this one parameter family of planes through P_0 can be expected to envelop a cone with vertex P_0 called the **Monge cone** at P_0. (The envelpe made by a one parameter family of planes is a cone!) Each possible tangent plane touches the Monge cone along a certain generator. In this framework the PDE given by $F = 0$ defines a **field of cones**. A surface z is an integral surface if at each of its points P_0 it "touches" a cone with vertex P_0 (in the sense that the cone is tangent to the plane....its lying on the plane)

[Side Remark: in the quasilinear case F is a LINEAR function of p,q when x,y,z are fixed. In this case since F can be written $F = (p,q,-1)\cdot(a,b,c)$ there is a specific direction that is in included in EVERY possible tangent plane (the direction is (a, b, c)....no matter what the value of p, q (a, b, c) is always perpendicular to the normal to the plane and so is always in the tangent plane). Since every possible plane includes this direction, the envelope of all the possible tangent planes is degenerate; it is not a cone as in the general case instead it is just an "axis" in this special direction. This is called the Monge axis.]

A surface z is an integral surface if at each of its points P_0 it "touches" a cone with vertex P_0. In that case, the generator along which the tangent plane tocuhes the cone defines a direction on the surface. These "characteristic" directions are the key to the whole theory of integration for the PDE. The Monge cone at P_0 degenerates into the line with direction (a, b, c) through P_0.

3.12.2. Envelopes. The central notion here is that of the *envelope* of a family of surfaces S_λ. Suppose we have a family of surfaces indexed by a parameter λ given to us the form:

$$S_\lambda : \quad z = G(x, y, \lambda)$$

Suppose also that:

$$0 = G_\lambda(x, y, \lambda)$$

Then for fixed λ, these two equation together determine a curve γ_λ (The first equation gives a surface (two real parameters ... its convenient to use x, y), and the second is one equation that the two real parameters must satisfy $\implies$ one parameter family of solutions i.e. a curve).

The **envelope** of the surfaces $G(x, y, \lambda)$ is defined to be the union $\cup_\lambda G(x, y, \lambda)$.

If one can invert $0 = G_\lambda(x, y, \lambda)$ to get $\lambda = g(x, y)$, then the resulting envelope surface is:

$$E : \quad z = G(x, y, g(x, y))$$

The envelope E touches the surface $S_\lambda$along the curve γ_λ. For in a point (x, y, z) of γ_λ we have that $g(x, y) = \lambda$, so we must be in both E and in S_λ (check the equations that define $S_\lambda$and E). For any surface $z = F(x, y)$, the normal to the surface through a point (x_0, y_0, z_0) is given by $\left(\frac{\mathrm{d}}{\mathrm{d}x}F(x_0, y_0), \frac{\mathrm{d}}{\mathrm{d}y}F(x_0, y_0), -1\right)$. We hence compute using the chain rule that the normal to E is the direction $(G_x + G_\lambda g_x, G_y + G_\lambda g_y, -1) = (G_x, G_y, -1)$ since $G_\lambda = 0$ on the curves γ_λ. In differntial notation what we have discovered is that along curves γ_λ we have:

$$\begin{aligned} \mathrm{d}z &= G_x \mathrm{d}x + G_y \mathrm{d}y \\ 0 &= G_{\lambda x} \mathrm{d}x + G_{\lambda y} \mathrm{d}y \end{aligned}$$

The first holds since $(G_x, G_y, -1)$ is normal to E and the second is just the differential of $0 = G_\lambda(x, y, \lambda)$ along the curve.

3.12.3. Characteristic Equations. Back to our PDE: For fixed $P_0 = (x_0, y_0, z_0)$ we have a one parameter family of surfaces through P_0 given by:

$$\begin{aligned} z - z_0 &= p(x - x_0) + q(y - y_0) \\ F(x_0, y_0, z_0, p, q) &= 0 \end{aligned}$$

(There are a-priori two parameters p, q but the relation F reduces us to one effective parameter). The envelope of this family of surfaces is the Monge cone we were discussing earlier. If we think of p as the parameter, and $q = q(p)$ depending on p, taking differentials above we have:

$$\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y \quad 0 = \mathrm{d}x + \frac{\mathrm{d}q}{\mathrm{d}p}\mathrm{d}y$$

By the relation $F(x_0, y_0, z_0, p, q) = 0$ we know that $F_p + \frac{\mathrm{d}q}{\mathrm{d}p}F_q = 0$. So rearranging a bit and then plugging this in we have that the direction of the generator is characterized by:

$$\mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y, \quad \frac{\mathrm{d}x}{F_p} = \frac{\mathrm{d}y}{F_q}$$

If we are given an integral surface $S: \ z = u(x, y)$, these differential equations define a direction field since $F_p(x, y, u, u_x, u_y)$ and $F_q(x, y, u, u_x, u_y)$ are then known functions of x, y. Hence we can define *characteristic curves* belonging to the integral surface S as those fitting the direction field data. If we paraterize the above with a parameter t we get the following system

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= F_p\left(x, y, z, p, q\right) \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= F_q\left(x, y, z, p, q\right) \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q)
\end{aligned}$$

Where we identify:

$$z = u(x, y) \quad p = u_x(x, y) \quad q = u_y(x, y)$$

[Side remark: Let's compare what we just did to the quasilinear case. In the quasilinear case, suppose $F = (p, q, -1) \cdot (a, b, c)$ then Monge cone degenerated into a Monge axis, so there was only one direction to follow at each point...namely the direction (a, b, c) itself! This led us to the system $\frac{\mathrm{d}x}{\mathrm{d}t} = a$, $\frac{\mathrm{d}y}{\mathrm{d}t} = b$, $\frac{\mathrm{d}z}{\mathrm{d}t} = c$ that we solved to get the soltion. In the fully nonlinear case we instead had a whole Monge cone of possibilities at each point. However, if the surface S is specified, then only one possibility remains and we can follow along that path as we did in the quasilinear case.]

Notice that we have three ODEs but 5 variables here. However, we know that z, p, q are related in that we $z = u(x, y) \quad p = u_x(x, y) \quad q = u_y(x, y)$ so this will save us here. We will use these relations to construct some explicit ODEs for $\frac{\mathrm{d}p}{\mathrm{d}t}$ and $\frac{\mathrm{d}q}{\mathrm{d}t}$ to make everyting tidy. We start with the relation that must hold on an integral surface:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$$

Differentiating ("total" derivatives) with respect to x once and separly with respect to y once, and then plugging in $\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(u_x(x, y)\right) = u_{xx}\frac{\mathrm{d}x}{\mathrm{d}t} + u_{xy}\frac{\mathrm{d}y}{\mathrm{d}t} = u_{xx}F_p + u_{xy}F_q = -F_x - u_xF_z$ we will arive at [This last bit is where we used the relations between z, p, q from their relationship to u]:

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= -F_x\left(x, y, z, p, q\right) - pF_z\left(x, y, z, p, q\right) \\
\frac{\mathrm{d}q}{\mathrm{d}t} &= -F_y(x, yz, p, q) - qF_z(x, y, z, p, q)
\end{aligned}$$

So we have a system of 5 ODEs in 5 unknown variables, that does NOT require knowledge of the integral surface $z = u(x, y)$ to work out. One can explicitly check that $\frac{\mathrm{d}F}{\mathrm{d}t}(x,y,z,p,q) = \ldots = 0$ by these ODEs. We refer to this system of 5 ODES along with $F(x,y,z,p,q) = 0$ as the **characteristic equation**.

Here is a brief recap of how we derived them:

-Start with the equation of a plane through x_0, y_0, z_0 and the PDE condition at that point (this defines a one parameter family of surfaces...the Monge cone)

$$\begin{aligned} z - z_0 &= p(x - x_0) + q(y - y_0) \\ F(x_0, y_0, z_0, p, q) &= 0 \end{aligned}$$

-Think of p as a parameter and differentiate:

$$\implies \quad \mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y \quad 0 = \mathrm{d}x + \frac{\mathrm{d}q}{\mathrm{d}p}\mathrm{d}y$$

-Now use $F(x_0, y_0, z_0, p, q) = 0$ to get $F_p + \frac{\mathrm{d}q}{\mathrm{d}p}F_q = 0$ so we can eliminate the $\frac{\mathrm{d}q}{\mathrm{d}p}$ from above. "If we are given an integral surface $S: \ z = u(x,y)$, these differential equations define a direction field since $F_p(x, y, u, u_x, u_y)$ and $F_q(x, y, u, u_x, u_y)$ are then known functions of x, y. Hence we can define *characteristic curves* belonging to the integral surface S as those fitting the direction field data." If we parametrize the above with a parameter t we get the first three equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= F_p\left(x,y,z,p,q\right) \\ \implies \quad \frac{\mathrm{d}y}{\mathrm{d}t} &= F_q\left(x,y,z,p,q\right) \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= pF_p(x,y,z,p,q) + qF_q(x,y,z,p,q) \\ \text{with} \quad & z = u(x,y) \quad p = u_x(x,y) \quad q = u_y(x,y) \end{aligned}$$

Then use $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$ and $z = u(x, y) \ \ p = u_x(x, y) \ \ q = u_y(x, y)$ to get the last two by differentiating with respect to x, y and w.r.t. t to get the last two equations:

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= -F_x\left(x,y,z,p,q\right) - pF_z\left(x,y,z,p,q\right) \\ \frac{\mathrm{d}q}{\mathrm{d}t} &= -F_y(x,yz,p,q) - qF_z(x,y,z,p,q) \end{aligned}$$

(In summary...if you ever get stuck, go back to $F(x, y, z, p, q)$ and take creative derivatives all over. It's also kind of helpful to imagine you already have the integral surface u worked out and then to later eliminate the need for it.)

3.12.4. Characteristic Strips. A solution of the characteristic equations is a set of five functions $(x(t), y(t), z(t), p(t), q(t))$. Generally, we call a quintuple (x, y, z, p, q) a **plane element** and we interpret it geometrically as consisting of a point (x, y, z) combined with a plane through the point with the equation:

$$\zeta - z = p\left(\xi - x\right) + q\left(\eta - y\right)$$

I.e. the plane element (x, y, z, p, q) is thought of as the point (x, y, z) and the plane with normal $(p, q, -1)$ passing thought that point. A plane element is called **characteristic** if it satisfies $F(x, y, z, p, q) = 0$. (For example, with this jargon, the Monge cone is the envelope of the characteristic plane elements) A one-parameter family of elements $(x(t), y(t), z(t), p(t), q(t))$ is called a **strip** if the plane elements are tangent to the curve $(x(t), y(t), z(t))$ at every point t. In this case the curve $(x(t), y(t), z(t))$ is called the **support** of the strip. For this to be the case, the increment $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ must be perp. to the normal to the plane, so we get the so called **strip condition**:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = p\frac{\mathrm{d}x}{\mathrm{d}t} + q\frac{\mathrm{d}y}{\mathrm{d}t}$$

A solution to the characteristic equations will be called a **characteristic strip.** (Notice that solutions of the characteristic equations automatically satisfy the strip condition....just look at it!)

Any surface $z = u(x, y)$ parametrized by two parameters (s, t) can be thought of as a two parameter family of elemtns $(x(s, t), y(s, t), z(s, t), p(s, t), q(s, t))$ formed by the points of the surface and the corresponding tangent planes. Again, in order to actually be a surface, we need to satisfy the strip condition in the s and t directions. (Indeed, for fixed s or for fixed t the resulting thing must be a strip.)

Since the characteristic strip is the soltion to the ODE system (the char. eqns.), by existence/uniqueness for ODEs the whole strip is determined by any one of the plane elements in it.

In the next little section, the book reparametrises everything in terms of z instead of t...

3.13. The Cauchy Problem

The Cauchy problem for $F(x, y, u, u_x, u_y) = 0$ is the problem of finding the integral surface through a prescibed intial curve Γ. Γ is given parametrically by:

$$x = f(s), \;\; y = g(s), \;\; z = h(s)$$

To do this, we will pass suitable characteristic strips through Γ. If f, g, h are C^1 for s near a value s_0 corresponding to a point $P_0 = (x_0, y_0, z_0) = (f(s_0), g(s_0), h(s_0))$. The specified curve Γ is NOT a strip...it is just a curve. (Recall a strip is a set of plane elements while a curve is just a set of point). So the first thing we do is augment Γ to a strip using the strip equations. Say $p = \phi(s)$ and $q = \phi(s)$ so that (x, y, z, p, q) make a strip. We must have:

$$\begin{aligned} h'(s) &= \phi(s)f'(s) + \psi(s)g'(s) \\ F\left(f(s), g(s), h(s), \phi(s), \psi(s)\right) &= 0 \end{aligned}$$

Since F is non-linear, there might be more than one solution for ϕ and ψ. If we have a starting plane element $(x_0, y_0, z_0, p_0, q_0)$, and if the genertor of the Monge cone at P_0 is not in the same direction as Γ, then we can use the implicit function theorem to get unique function ϕ and ψ near s_0.

Through each points $(f(s), g(s), h(s), \phi(s), \psi(s))$ we treat these as the intial conditions at $t = 0$ and we solve for a charactersitc strip starting here. In this way we will get 5 solution functions:

$$\begin{aligned} x &= X(s, t) \\ y &= Y(s, t) \\ z &= Z(s, t) \\ p &= P(s, t) \\ q &= Q(s, t) \end{aligned}$$

This gives us a parametric equation in terms of the parameters (s,t) for the integral surface S. If we can invert this system, we get a solution to the Cauchy problem (we can do this incversion at least locally if the Jacobian $\frac{\partial(x,y)}{\partial(s,t)} \neq 0$. This turns into an equation involving f', g', F_p, F_q here by all the relations we know x, y satisfy)

3.13.1. Higher order equations. The same kind of game works in higher dimensions. The characteristic equations are:

$$\begin{aligned}\frac{\mathrm{d}x_i}{\mathrm{d}t} &= F_{p_i}\\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= -F_{x_i} - F_z p_i\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \sum_{i=1}^{n} p_i F_{p_i}\end{aligned}$$

3.13.2. An example. There is an example here from geometric optics. I'm going to come back to it in the morning.

3.14. Solutions Generated as Envelopes

PROPOSITION 3.5. *If we have a one parameter family of* integral surfaces S_λ *with equation:*

$$S_\lambda : \ z = G(x, y, \lambda)$$

The the enevelope of these integral surfaces is again an integral surface.

PROOF. Every tangent plane of the envelope is a tangent plane to one of the surfaces S_λ. Hence, since $S_\lambda$is an integral surface, that tangent plane is a characterstic plane element. This works at every point, so the surface is indeed an integral surface. □

REMARK 3.11. The particular value of λ for which the envelope surface S touches $S_\lambda$is the one for which $0 = G_\lambda(x, y, \lambda)$. Along with the equations $p = G_x(x, y, \lambda)$ and $q = G_y(x, y, \lambda)$ this defines a characteristic strip! We can solve the family of equations:

$$\begin{aligned}0 &= G_\lambda(x, y, \lambda)\\ p &= G_x(x, y, \lambda)\\ q &= G_y(x, y, \lambda)\end{aligned}$$

....I'm going to jsut rememeber the general idea here and not go in to so much detail...

Equations for Functions of Two Independent Variables

These are notes from Chapter 2 of [**2**].

4.15. Characteristics for Linear and Quasi-Linear Second-Order Equations

We will examine the general quasi-linear second-order equation for a function $u(x, y)$ namely:

$$au_{xx} + 2bu_{xy} + cu_{xx} = d$$

where $a, b, c, d,$ are functions of x, y, u, u_x, u_y. The *Cauchy problem* consists of finding a solution u of the PDE with specified data u, u_x, u_y on a curve γ in the $xy-$plane. If γ is given by:

$$\gamma : \quad x = f(s) \ y = g(s)$$

Then the cauchy data is given by some functions:

$$u = h(s) \ u_x = \phi(s) \ u_y = \psi(s)$$

The value of the function and its derivatives must be related by the "strip condition" (essentially chain rule along the curve), this gives:

$$h'(s) = \phi(s)f'(s) + \psi(s)g'(s)$$

Thus no more than 2 of the three parameters u, u_x, u_y may be prescribed along the curve.

We can similarly get relations for higher derivatives that must be satisfied:

$$\frac{\mathrm{d}u_x}{\mathrm{d}s} = u_{xx}f'(s) + u_{xy}g'(s) \quad \frac{\mathrm{d}u_y}{\mathrm{d}s} = u_{xy}f'(s) + u_{yy}g'(s)$$

Similar relations are valid for u_{xx}, u_{xy}, u_{yy} etc. If u is a solition of the PDE with specified data along a curve, then we have a system with 3 equations:

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} &= d \\ fu_{xx} + g'u_{xy} &= \phi' \\ f'u_{xy} + g'u_{yy} &= \psi' \end{aligned}$$

These determine u_{xx}, u_{yy} and u_{xy} uniquely along γ as long as the system is non-degenerate. I.e. we have the determinant

$$\begin{aligned} \Delta &= \left| \det \begin{pmatrix} f' & g' & \\ & f' & g' \\ a & 2b & c \end{pmatrix} \right| \\ &= a\left(g'\right)^2 - 2bf'g' + c\left(f'\right)^2 \end{aligned}$$

is non-zero.

We call the initial curve γ a *characteristic* if this is zero (i.e. the system is degenarte) and a non-characterstic if $\Delta \neq 0$ along γ. Along a non-characteristic cure, we can hence solve the system and get u_{xx}, u_{xy}, u_{yy} at these points. By repeated use of this,m we can get all mixed derivatives for u. We might hope to use this to construct a power series solution...we will handle this in more detail in the next chapter under the Cauchy-Kowalevski theorem.

To find the characteristic curves at a point, we write $g' = \frac{\mathrm{d}y}{\mathrm{d}s}$ and $f' = \frac{\mathrm{d}x}{\mathrm{d}s}$ and then $\Delta = 0$ becomes:

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 - 2b\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)(\frac{\mathrm{d}y}{\mathrm{d}s}) + c\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 = 0$$

Solving implictly gives us:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

Notice there are two characteristic directions if $b^2 - ac > 0$, there is one characeristsic equation if $b^2 - ac = 0$ and NO characteristic equations if $b^2 - ac < 0$. We call these hyperbolic, parabolic and elliptic equations respectively.

Type	Condition	Number of characteristics at a point
Hyperbolic	$b^2 - ac > 0$	2
Parabolic	$b^2 - ac = 0$	1
Elliptic	$b^2 - ac = 0$	0

Notice that since in general a, b, c depend on x, yu, u_x, u_y this type classificiation can depend on the location, or which solution. For *linear* equations, where the thing only depends on x, y then this depends only on which region of the plane one is in.

4.16. Propagation of Singularities

Suppose we wish to think of solutions to u which are not C^2 but instead satisfy the PDE in some weaker sense. Suppose that the solution is C^2 in a region I and C^2 in a region II and that the 2 regions meet along some curve γ. If γ was a non-characteristic curve, then the 2nd derivatives would have to be equal along γ since they are uniquely determined by the first derivatives along non-charactersitic curves. Hence this can only happen if γ is a characterstic curve. One can derive relations for the jump across a characeteristic curve.

....

4.17. The Linear Second-Order Equation

Let us analyze in more detail the *linear second order equation:*

$$au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y + fu = 0$$

where a, b, c, d, e, f depend only on x, y. We do a change of variable now by introducing variables ξ and η defined by:

$$\xi = \phi(x, y), \quad \eta = \psi(x, y)$$

Then the equation becomes:

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + 2Du_\xi + 2Eu_\eta + Fu = 0$$

Where:

$$\begin{aligned} A &= a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 \\ B &= a\phi_x\phi_y + 2\left(\phi_x\psi_y + \phi_y\psi_x\right) + c\phi_y\psi_y \\ C &= a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 \\ &\quad etc. \end{aligned}$$

4.17.1. Hyperbolic Case. In the hyperbolic case, it is helpful to find the characteristic curves and put:

$$\xi = \phi(x,y) = cons't \quad \eta = \psi(x,y) = cons't$$

So that for every $c$the set $\phi(x,y) = c$are characteristic curves. If this is a charactersitic curve, we know that $a\left(\mathrm{d}y\right)^2 - 2b\left(\mathrm{d}x\right)\left(\mathrm{d}y\right) + c\left(\mathrm{d}x\right)^2 = 0$ so putting in the relation $\phi_x \mathrm{d}x + \phi_y \mathrm{d}y = 0$ along the curve, we get:

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0$$

This is exactly saying that the "coefficient" $A = 0$! In the same way, choosin η to be a characteristic eliminates C. We then divide out by B everywhere to remain with:

$$u_{\xi\eta} + 2Du_\xi + 2Eu_\eta + Fu = 0$$

And this will have $\xi = cons't$ and $\eta = cons't$ as its characteristic equations.

Sometimes it is nicer to put:

$$x' = \xi + \eta, \quad y' = \xi - \eta$$

So that the equation looks like the wave equation:

$$u_{y'y'} - u_{x'x'} + \ldots = 0$$

4.17.2. Elliptic Case. In the elliptic case $b^2 - ac < 0$ there are no real characterstics. We instead try to force $A = C$ and $B = 0$ to get an equation of the form:

$$u_{\xi\xi} + u_{\eta\eta} + 2Du_\xi + 2Eu_\eta + Fu = 0$$

This leads to solving:

$$\phi_x = \frac{b\psi_x + c\psi_y}{\sqrt{ac - b^2}}, \quad \phi_y = -\frac{a\psi_x + b\psi_y}{\sqrt{ac - b^2}}$$

Eliminating ϕ from here leads to the so called "Beltrami Equation":

$$\left(\frac{b\psi_x + c\psi_y}{\sqrt{ac - b^2}}\right)_y + \left(\frac{a\psi_x + b\psi_y}{\sqrt{ac - b^2}}\right)_x = 0$$

EXAMPLE 4.5. (The Tricomi Equation)

Consider:

$$u_{yy} - yu_{xx} = 0$$

This has $ac - b^2 = -y$ so this is hyperbolic for $y > 0$, parabolic at $y = 0$ and elliptic at $y < 0$.The characteristic equation is $a\left(\mathrm{d}y\right)^2 - 2b\left(\mathrm{d}x\right)\left(\mathrm{d}y\right) + c\left(\mathrm{d}x\right)^2 = 0$ is $-y\left(\mathrm{d}y\right)^2 + \mathrm{d}x^2 = 0$. This gives:

$$\mathrm{d}x \pm \sqrt{y}\mathrm{d}y = 0 \text{ for } y > 0$$

The characeteristic curves are therfore:

$$x \pm \frac{2}{3}y^{3/2} = cons't$$

Hence if we put $\xi = 3x - 2y^{3/2}$ and $\eta = 3x + 2y^{3/2}$ we get to:

$$u_{\xi\eta} - \frac{1}{6}\frac{u_\xi - u_\eta}{\xi - \eta} = 0$$

4.18. The One-Dimensional Wave Equation

The simplets hyperbolic PDE is the one-dimensional wave equation:

$$Lu = u_{tt} - c^2 u_{xx} = 0$$

The characteristic curves are given by $1\left(\mathrm{d}x^2\right) - 2\cdot 0\cdot(\mathrm{d}x)(\mathrm{d}t) - c^2\left(\mathrm{d}t^2\right) = 0 \implies \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = c$ so $x \pm ct = cons't$. Introducing $\xi = x + ct$ and $\eta = x - ct$ as new coordinates, we get to:

$$u_{\xi\eta} = 0$$

Assume that we are dealing with a convex region Ω. (Not 100% sure why this is relevant....I think it might be so that if you integrate along lines connecting two points you never leave the region) Integrating with respect to $u_{\xi\eta} = 0 \iff (u_\xi)_\eta = 0 \implies u_\xi = f(\xi)$ does not depend on η. Integrating with respect to η now gives, $u = \int f(\xi)\mathrm{d}\xi + G(\eta) = F(\xi) + G(\eta)$ in the original variables this is:

$$u = F(x + ct) + G(x - ct)$$

Here $u \in C^2$ iff $F, G \in C^2$.Thus the general solution of the PDE is obtained from superposition of the solution $F(x + ct) = v$ of $v_t - cv_x = 0$ and a solution $G(x - ct) = w$ of $w_t + cw_x = 0$. This correponds to the factoring:

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial^2 x} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

Thus the graph of $u(x,t)$ in the $x - u$ plane consits of two waves propagating without change of shape with velocity c in opposite directions along the x axis.

If we impose initial conditions:

$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

Then putting $t = 0$ into $u = F(x - ct) + G(x + ct)$ we get:

$$\begin{aligned} f(x) &= F(x) + G(x) \\ g(x) &= cF'(x) - cG'(x) \end{aligned}$$

Differentiating the first equation and then solving for F' and G' we get:

$$\begin{aligned} F'(x) &= \frac{cf'(x) + g(x)}{2c} \\ G'(x) &= \frac{cf'(x) - g(x)}{2c} \end{aligned}$$

And then integrating gives:

$$\begin{aligned} F(x) &= \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\xi)\mathrm{d}\xi \\ G(x) &= \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\xi)\mathrm{d}\xi \end{aligned}$$

So the final solution is:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int\limits_{x-ct}^{x+Ct} g(\xi)\mathrm{d}\xi$$

From this we see that u is determined uniquel by the vales of the intial functions in the interval $[x-ct, x+ct]$ whose end points are cut out by the characteristic curves from (x,t). This is called the **domain of dependence** for the solution at the point (x,t). Conversly a point at time $t = 0$ only effects points in the (x,t) plane that lie in a cone bounded by the two characterstics from that point.

4.18.1. Generalized Solution. One can take the solution $u = F(x+ct) + G(x-ct)$ where F, G are not nessisarly C^2 as the weak form or generalized form of soltuions for the PDE. Geometrically one can check that $u(x_A, t_A) + u(x_C, t_C) = u(x_B, t_B) + u(x_D, t_D)$ for any points A, B, C, D that form a parallelogram in the (x,t) plane.

4.18.2. Wave Equation with Boundary Conditions. Consider now the vertical strip $0 < x < L$ and $t > 0$. Suppose we initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

at the bottom and we have boundary conditions on the boundaries of the strip

$$u(0,t) = \alpha(t) \quad u(L,t) = \beta(t)$$

One way to solve this is take the Fourier series in the x-direction. (Seperation of variables)

Alternativly, you can divide the strip like region into subregions where you can solve it by the parallelolgram sum rule. For example, the soltuion in the region $\{(x,t):\ x - ct > 0 - c\cdot 0 = 0 \text{ and } x + ct < L + c\cdot 0 = L\}$ boudned by the characterstics from the end of the strip has an ordinary solution in the same way the non-boundary condition PDE does. Once we solve in this region, we can solve in the two triangular shapped regions on the edge etc.

4.19. Systems of First-Order Equations

Skip for now

4.20. A Quasi-linear System and Simple Waves

If you have the PDE:

$$v_t + B(v)v_x = 0$$

You might hope to find a solution of the form $v = F(\theta(x,t))$. By plugging in and noting that F must be an eigenvector of the B matrix, we get to the PDE for θ:

$$\theta_t + c\left(\theta\right)\theta_x = 0$$

Where $c(\theta)$ is an eigenvalue of matrix B in some way.

Characteristic Manifolds and the Cauchy Problem

These are notes from Chapter 3 of [**2**].

5.21. Notation of Laurent Schwartz

For vectors $x = (x_1, \ldots, x_n)$ and a *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ whose components are non-negative integers α_k we define:

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} \ldots x_n^{\alpha_n} \\ |\alpha| &= \alpha_1 + \ldots + \alpha_n \\ \alpha! &= \alpha_1! \cdot \ldots \cdot \alpha_n! \end{aligned}$$

We also define the differential operator

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n}}$$

5.22. The Cauchy Problem

The most general m-th order linear differential equation for a function $u(x) = u(x_1, \ldots, x_n)$ takes the form:

$$Lu = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha u = B(x)$$

The sme formula describes the general mth order system of N differentail equaions in N unknowns if you think of u and B as column vectors and A_α as $N \times N$ matrices.

The general $m-$th order quasi-linear equation is:

$$Lu = \sum_{|\alpha| = m} A_\alpha D^\alpha u + C = 0$$

where now A_α's and C are functions of x and of the lower order derivatives $D^\beta u$ for any $|\beta| \leq m - 1$. These can be reduced to large systems of first order by treating $y_\beta := D^\beta u$ as a new independent variable (this adds a lot of new independent variabels) and then also introducting the "compatability conditions" in the form of equations like $y_\beta - D^\beta u = 0$.

Nonlinear equations or systems are given by:

$$F(x, D^\alpha u) = 0$$

These can be formally reduced to first order quasi-linear systems by applying a first order differential operator to this. (?)

5.22.1. Definition of the Cauchy Problem.

Definition 5.5. The **Cauchy Problem** consists of trying to find a solution of u to $Lu = 0$ having been prescribed **Cauchy data** on a hypersurface $S \subset \mathbb{R}^n$ given by:

$$S: \; \phi(x_1, \ldots, x_n) = 0$$

Here ϕ should be smooth enough so that it has m continuous derivatives adn the surface should be regular in the sense that:

$$D\phi = (D_1\phi, \ldots, D_n\phi) = \left(\frac{\partial}{\partial x_1}\phi, \ldots, \frac{\partial}{\partial x_n}\phi\right) \neq 0$$

everywhere on the surface S. (this makes sure the surface has no "pinch" points or other wrinkles).

The **Cauchy data on** S for an m-th order equations consists of the derivatives of u of orders less than or eqaul to $m-1$. As we have seen already, these cannot be specified arbitrarily...they must satisfy compatibility conditions (we called these strip conditions before....for example if $u = f$, $u_x = g$, $u_y = h$ on a curve $x = \phi(s), y = \psi(s)$ then $\frac{\mathrm{d}}{\mathrm{d}s}u(x(s), y(x)) = u_x\phi' + u_y\psi'$ is a compatability relation)

(One thing that CAN be specified is the *normal* derivatives to the surface...indeed all the compatiability relations come from differentiating along the surface so these are ok)

We are now going to find a solution of u near S which has this given Cauchy data on the hypersurface S. We call a surface S **non-characteristic** if we can get all of $D^\alpha u$ for $|\alpha| = m$ on S from the linear albegraic system of equations consistinf of the compatability relations for the data and the PDE together. Otherwise we call the hypersurface S a **characteristic**.

We have seen before that if a generalized solution of class C^{m-1} has jump discontinuities in the m-th order derivative, they must occur along characteristic surfaces.

5.22.2. The compatability conditions. Lets suppose for now that the surface S we are interested in is the hyper-surface $x_n = 0$ and that the data is specified here by a family of functions ψ_k that specify the derivatives on this hyper surfaces in the normal direction $\vec{x_n}$:

$$D^k_{x_n} u = \frac{\partial^k}{\partial\left(x_n\right)^k} u = \psi_k(x_1, \ldots, x_{n-1}) \; 1 \leq k \leq m-1$$

By using the fact that the surfarace is $x_n = 0$, we can get compatibility conditions that give all the other derivatives:

$$\begin{aligned} D^\beta u &= D^{\beta_1}_{x_1} \ldots D^{\beta_{n-1}}_{x_{n-1}} D^{\beta_n}_{x_n} u \\ &= D^{\beta_1}_{x_1} \ldots D^{\beta_{n-1}}_{x_{n-1}} \psi_{\beta_n} \end{aligned}$$

This express all of the derivatives as long as $\beta_n \leq m-1$. To extend to the derivatives with $\beta_n \geq m$ we must use the PDE. Let $\alpha^* = (0, 0, \ldots, 0, m)$ be the index for the derivatives of this top order that are not expressable just from the Cauchy data. We can hope to solve for $D^{\alpha^*} u$ in terms of the other deratives by using the PDE:

$$\begin{aligned} 0 = Lu &= \sum_{|\alpha|=m} A_\alpha D^\alpha u + C \\ &= A_{\alpha^*} D^{\alpha^*} u + \sum_{|\alpha|\leq m-1} A_\alpha D^\alpha u + C \end{aligned}$$

So we will be able to solve for D^{α^*} if the matrix $A_{\alpha^*}$is invertible i.e. $\det(A_{\alpha^*}) \neq 0$. In the linear case, this is just a question of the function A_{α^*} but in the quasi-linear case A_{α^*}(and whether or not it is differentiable) may depend on the given Cauchy data.

This condition depends only on the coefficients of A_{α^*}.Define the **principle part of the PDE** to be the operator:

$$L_{pr} = \sum_{|\alpha|=m} A_\alpha D^\alpha$$

Define the **symbol** of this operator by the matrix Λ an $N \times N$ matrix form (i.e. a multilinear thing that acts of vecotrs) defined by:

$$\Lambda(\zeta) := \sum_{|\alpha|=m} A_\alpha \zeta^\alpha$$

Where ζ is a vector $\zeta = (\zeta_1, \ldots, \zeta_n)$

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5.23. Real Analytic Functions and Cauchy-Kowalevski Theorem

5.23.1. Multiple Infintie sereis. We say $\sum_\alpha C_\alpha$converges if it it converges absolutly $\sum_\alpha |C_\alpha|$ and in this case the order of summation does not matter.

5.23.2. Real Analytic Functions.

DEFINITION 5.6. We say f is real analytic at a point y if it has a power series in a n'h'd of y:

$$f(x) = \sum_\alpha c_\alpha (x-y)^\alpha \, \forall x \in B_\epsilon(y)$$

We call the space of such functions C^w. We say a vector is C^w if each of its components are real analytic.

THEOREM 5.31. *If $f = (f_1, \ldots, f_m) \in C^w$ then $f \in C^\infty$ is infitently differentiable and for all $y \in \Omega$ there exists a n'h'd N of y and postive numbers M, r such that for all $x \in N$ we have:*

$$f(x) = \sum_\alpha \frac{1}{\alpha!} \left(D^\alpha f(y)\right) (x-y)^\alpha$$

and we have that

$$\left|D^\beta f_k(x)\right| \leq M \, |\beta|! r^{-|\beta|} \text{ for all } \beta \in \mathbb{Z}^m \text{ and all } k$$

PROOF. Suppose $f(x) = \sum_\alpha c_\alpha x^\alpha$ converges in some radius $|x| < r_0$. Since the covergence is uniform, we claim that $D^\beta f = \sum_\alpha D^\beta (c_\alpha x^\alpha)$ exists and again converges uniformly inside $|x| < r_0$ with geometric estimates strictly inside the radius of convergence.

(Here is a sketch of this: let $f_N = \sum_{|\alpha|\leq N} c_\alpha x^\alpha$ so that $f_N \to f$ uniformly. Define $g_N = \sum_{|\alpha|\leq N} D^\beta(c_\alpha x^\alpha)$ and $g = \lim_{N\to\infty} g_N$. We can verify that the series has the same radius of convergence and $g_N \to g$ uniformly here. We know for each N that $D^\beta f_N = g_N$ here since its a finite sum. Hence we can write $\Delta_h^\beta f_N(x) = \int_0^1 g_N(x + v(h))\mathrm{d}x$ where Δ_h^β is a finite difference operator so that $\Delta_h^\beta f \to D^\beta f$ as $h \to 0$ and $\int$ is a suitable integral over g (For example: $\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\int_x^{x+h} g(s)\mathrm{d}s = \int_0^1 g(x+sh)\mathrm{d}s$.) Now taking the limit $h \to 0$ we have that the LHS $\Delta_h^\beta f_N(x) \to D^\beta f$ while the RHS $\to \int_0^1 g(x)\mathrm{d}s = g(x)$ by the LDCT. Hence $D^\beta f = g$ as desired) □

5.23.3. Statement of the C-K theorem.

THEOREM 5.32. *Suppose that S is a non-characterstic surface. Suppose we have real analytic data at $x^0 \in S$ (i.e. locally a power series). Suppose that the coefficeitns that appear in the PDE are real analytic functions of their arguments.*

Then there is locally a solution that is real analytic in some n'h'd of x_0. This is the unique real analytic solution here.

PROOF. On a non-characterstic curve, we can find all the derivatives D^α, so we can then construct the candidate power series about x_0. From there, you must show that the power series converges in some n'h'd.

The idea to prove this is by majorisation: you create a new problem where the coefficents are bigger and for which you can explicitly show the series converges.

Usual one reduces to the case of the hypersurface $x_n = 0$ first. □

5.23.4. Counterxamples. Consider the heat equation $u_t = u_{xx}$ with data specified at the initial time $u(0, x) = g(x)$.

The $x(s) = s, t(s) = 0$ is a characterstic curve for the PDE. Notice the principle part of the PDE is just $u_{xx} = 0$ which has characteristics $1(\mathrm{d}t)^2 = 0$ so $\mathrm{d}t = 0$ or $t =$cons't.

Here is an explicit example where you can check that the power series does NOT converge:

$$u_t = u_{xx} \quad u(x, 0) = \frac{1}{1+x^2}$$

Write:

$$u(t, x) = \sum a_{m,n} \frac{t^m}{m!}\frac{x^n}{n!}$$

Then the PDE implies:

$$a_{m+1,n} = a_{m,n+2}$$

The initial conditions imply:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \ldots = u(x, 0) = \sum a_{0,n}\frac{x^n}{n!}$$

$$a_{0,2n+1} = 0 \quad a_{0,2n} = (-1)^n (2n!)$$

Hence we get:

$$a_{m,2n} = a_{0,2n+2m} = (-1)^{n+m} (2(n+m))!$$

But now:

$$\frac{a_{m,2n}}{m!(2n)!} = \frac{2(n+m)!}{m!(2n)!}$$

Along the diagonal $m = n$ this is:

$$\frac{(4n)!}{n!(2n!)} \sim \frac{(4n)^{4n}}{n^n(2n)^{2n}} \sim n^n$$

which blows up so fast that the series cannot coverge in any radius.
This also reflects the fact that the equation is ill posed for backwards time.

5.24. The Lagrange-Green Identity

Apply the divergence theroem to $f = uD_k v$ to get an integration by parts formula:

$$\int_\Omega v^T D_k u \mathrm{d}x = \int_{\partial\Omega} v^T u \cdot \vec{n_k} dS - \int_\Omega (D_k v)^t u \mathrm{d}x$$

Or more generally:

$$\int v^T \left(\sum_{\alpha \le m} a_\alpha(x) D^\alpha u \right) \mathrm{d}x = \int_\Omega \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha \left(v^T a_\alpha(x) \right) u \mathrm{d}x + \int_{\partial\Omega} M(v, u, \vec{n}) \mathrm{d}S$$

where M is some surface intgral that is linear in $\vec{n}$ with coefficients that are bilinear in the derivatives u, v. This expression is not determined uniquely, but depends on the order in which we apply integration by parts.

5.25. The Uniqueness Theorem of Holmgren

Skip!

5.26. Distribution Solutions

The integration by parts buisness lets us define weak solutions. We define a **distribution** to be a linear functional from the compactly supported infintely differnetiable test functions ϕ denoted by $f[\phi]$. We require that the f are continuous in the sense that:

$$\forall \alpha, \lim_{k\to\infty} D^\alpha \phi_k(x) = 0 \implies \lim_{k\to\infty} f[\phi_k] = 0$$

Of course and continuous functions f creates a distribution by $f[\phi] := \int f\phi \mathrm{d}x$. The dirac delta function $\delta_{x_0}[\phi] = \phi(x_0)$ is another distribution.

We can define derivatives by $D_{x_k} f :=$ the unique distribution so that :

$$D_{x_k} f[\phi] := -f\left[D_{x_k}\phi\right]$$

In this way any differential operator on functions is a differential operator on distributions too. We call a solution u a **fundemental solution** if it has (as a distribution):

$$Lu = \delta_{x_0}$$

We call u a weak solution to $Lu = w$ if:

$$\int \phi w \mathrm{d}x = \int \left(\tilde{L}\phi\right) u \mathrm{d}x$$

For all test functions w. If $Lu(x_0, x) = \delta_{x_0}(x)$ then $u = \int u(x_0, x) w(x_0) \mathrm{d}x$ will be a weak solution of the $Lu = w$.

Laplace Equation

These are notes from Chapter 4 of [**2**].

6.27. Green's Identity, Fundamental Solutions and Poisson's Equation

6.27.1. Green's Identity, Energy Identity, Uniqueness. The **Laplace operator** acting on a function $u(x) = u(x_1, \ldots, x_n)$ of class C^2 in a region Ω defined by:

$$\Delta = \sum_{k=1}^{n} D_k^2$$

The Laplace equation is the equation $\Delta u = 0$ inside some region Ω. The Green's idenity for two functions $u, v \in C^2\left(\overline{\Omega}\right)$ is that:

$$\begin{aligned}\int_{\Omega} v \Delta u \mathrm{d}x &= -\int_{\Omega} \sum_i v_{x_i} u_{x_i} \mathrm{d}x + \int_{\partial\Omega} v \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} \mathrm{d}S \\ \int_{\Omega} v \Delta u \mathrm{d}x &= \int_{\Omega} u \Delta v \mathrm{d}x + \int_{\partial\Omega} \left(v \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u \frac{\mathrm{d}v}{\mathrm{d}\vec{n}} \right) \mathrm{d}S\end{aligned}$$

Where here $\frac{\mathrm{d}}{\mathrm{d}\vec{n}}$ indicates the derivative in the normal outward direction of the surface, $\frac{\mathrm{d}y}{\mathrm{d}\vec{n}} = \nabla y \cdot \vec{n}$. (To get these you can apply the divergence theorem $\int_{\Omega} \nabla \cdot F \mathrm{d}V = \int_{\partial\Omega} (F \cdot \vec{n}) \mathrm{d}S$ to $F = \phi \nabla \psi$ and using the idenity "$\Delta\psi = \nabla \cdot \nabla \psi$)) When $v = 1$ is a cons't function, the above gives:

$$\int_{\Omega} \Delta u \mathrm{d}x = \int_{\partial\Omega} \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} \mathrm{d}S$$

Another special case is when $v = u$ in which case we get the **energy identity:**

$$\int_{\Omega} \sum_i u_{x_i}^2 \mathbf{d}x + \int_{\Omega} u \Delta u = \int_{\partial\Omega} u \frac{\mathbf{d}u}{\mathbf{d}\vec{n}} \mathbf{d}S$$

This energy idenity is enough to show us **uniqueness** for the Laplace equation $\Delta u = 0$ inside a region Ω when u or $\frac{\mathrm{d}u}{\mathrm{d}\vec{n}}$ is specified on the boundary $\partial\Omega$.

THEOREM 6.33. *Suppose we specify either the boundary data $u\,|_{\partial\Omega}$ (This is referred to as a* ***Dirchlet problem****) or we specify $\frac{du}{d\vec{n}}\,|_{\partial\Omega}$ (This is reffered to as the* ***Neumann problem****). Then there is* ***at most*** *one solution u for the Dirchlet problem, and at most one solution for the Neummann problem up to an additive constant.*

PROOF. Since the Laplace equation is linear, by letting $u = u_1 - u_2$ for any two solutions u_1, u_2, it suffices to check that if $u\,|_{\partial\Omega} \equiv 0$ or $\frac{\mathrm{d}u}{\mathrm{d}\vec{n}}\,|_{\partial\Omega} \equiv 0$ then $u \equiv 0$ for the Dirchlet problem or $u \equiv cons't$ for the Neumann problem. Indeed, by the energy equation we have that:

$$\int_\Omega \sum_i u_{x_i}^2 \mathrm{d}x = 0$$

And since the integrand is non-negative everywhere inside Ω, it must be the case that it is identically 0. Hence $u_{x_i} \equiv 0$ everywhere, which shows $u \equiv cons't$. For the Neumann problem, the boudnary data gives us that $u \equiv 0$ since it is 0 on the boundary. □

6.27.2. Fundemental Spherically Symmetric Sol'n. One of the principle features of the Laplace equation is that it is spherically symmetric i.e. if $\psi(x)$ is a sol'n then $\psi(Ox)$ where O is an orthogonal transformation is a sol'n too. It is natrual to then look for a spherically symmetric sol'n, i.e. one for which $\psi(x) = \psi(Ox)$ for every x. In polar coordinates, such a function must be a function only of the radius r. Since $r^2 = \sum x_i^2$, we have that $2\left(\partial_{x_i} r\right) r = 2x_i \implies \partial_{x_i} r = \frac{x_i}{r}$ and we then get for any function $\psi(r)$ which is function only of the radius r that:

$$\begin{aligned}
\Delta\psi(r) &= \sum \partial_{x_i}\partial_{x_i}\psi(r) \\
&= \sum \partial_{x_i}\left(\psi'(r)\left(\partial_{x_i} r\right)\right) \\
&= \sum \partial_{x_i}\left(\psi'(r)\frac{x_i}{r}\right) \\
&= \sum \psi''(r)\left(\partial_{x_i} r\right)\left(\frac{x_i}{r}\right) + \psi'(r)\left(\partial_{x_i}\frac{x_i}{r}\right) \\
&= \sum \psi''(r)\left(\frac{x_i}{r}\right)^2 + \psi'(r)\left(\frac{1 \cdot r - \left(\frac{x_i}{r}\right)x_i}{r^2}\right) \\
&= \psi''(r) + \psi'(r)\frac{n-1}{r} \\
&= r^{1-n}\left(r^{n-1}\psi'(r)\right)'
\end{aligned}$$

So if $\Delta\psi(r) = 0$ we get the ODE $\left(r^{n-1}\psi'(r)\right)' = 0 \implies \psi'(r) = Cr^{1-n}$ which has the sol'n:

$$\psi(r) = \begin{cases} C\frac{r^{2-n}}{2-n} & \text{for } n > 2 \\ C\log r & \text{for } n = 2 \end{cases}$$

It is clear that $\Delta\psi = 0$ for $r > 0$ but notice that ψ is singular at $r = 0$. The following result gives some information about what's going on at $r = 0$:

THEOREM 6.34. *The solution $\psi(r)$ defined as above with the choice of constant $C = \frac{1}{\omega_n}$ (where ω_n is the SURFACE AREA CONSTANT of the n-sphere. E.g. $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, $\omega_n = 2\sqrt{\pi^n}/\Gamma(\frac{1}{2}n)$) is a* ***fundemental sol'n,*** *in the sense that $\Delta\psi = \delta_{\vec{0}}$ in the sense of distributions.*

PROOF. Say Ω is a region containg $\vec{0}$, and take any function $u \in C^2\left(\overline{\Omega}\right)$. For some small $\rho > 0$, take $B_\rho(0)$ the ball of radius ρ centered at 0. Assume that ρ is so small so that this is contained in Ω. Let $\Omega_\rho = \Omega - B_\rho(0)$ be the region with this ball removed. Apply Green's identity and the fact that $\Delta\psi = 0$ inside Ω_ρ to get:

$$\int_{\Omega_\rho} \psi \Delta u \mathrm{d}x = \int_{\partial\Omega} \left(\psi \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u \frac{\mathrm{d}\psi}{\mathrm{d}\vec{n}} \right) \mathrm{d}S + \int_{\partial B_\rho(0)} \left(\psi \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u \frac{\mathrm{d}\psi}{\mathrm{d}\vec{n}} \right) \mathrm{d}S$$

The normal points INTO the ball $B_\rho(0)$, so $\frac{\mathrm{d}\psi}{\mathrm{d}\vec{n}} = -\psi'(\rho) = -C\rho^{1-n}$ is constant on $\partial B_\rho(0)$ Also, $\psi = \psi(\rho)$ is constant on $\partial B_\rho(0)$ so this can be factored out of the integral. These two facts let us simplify the last term on the right hand side:

$$\begin{aligned} \int_{\partial B_\rho(0)} \left(\psi \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u \frac{\mathrm{d}\psi}{\mathrm{d}\vec{n}} \right) \mathrm{d}S &= \psi(\rho) \int_{\partial B_\rho(0)} \left(\frac{\mathrm{d}u}{\mathrm{d}\vec{n}} \right) \mathrm{d}S + \psi'(\rho) \int_{\partial B_\rho(0)} u \mathrm{d}S \\ &= -\psi(\rho) \int_{B_\rho(0)} \Delta u \mathrm{d}x + \psi'(\rho) \int_{\partial B_\rho(0)} u \mathrm{d}S \end{aligned}$$

As $\rho \to 0$, we know by continuity that $\int_{B_\rho(0)} \Delta u \mathrm{d}x \sim \Delta u(0) \cdot \rho^n$ up to a constant. Since $\psi(\rho)$ has $\rho^n \psi(\rho) = o(1)$ as $\rho \to 0$ the first term completely dies as $\rho \to 0$. In the second term, we have $\psi'(\rho) = C\rho^{1-n}$ and $\int_{\partial B_\rho(0)} u \mathrm{d}S \sim \omega_n u(0) \rho^{n-1}$ where ω_n is the SURFACE AREA CONSTANT of the n-sphere. E.g. $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, $\omega_n = 2\sqrt{\pi^n}/\Gamma(\frac{1}{2}n)$. Hence $\psi'(\rho) \int_{\partial B_\rho(0)} u \mathrm{d}S \to C\omega_n u(0)$ as $\rho \to 0$ and we get:

$$\int_{\Omega} \psi \Delta u \mathrm{d}x = \lim_{\rho \to 0} \int_{\Omega_\rho} \psi \Delta u \mathrm{d}x = \int_{\partial\Omega} \left(\psi \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u \frac{\mathrm{d}\psi}{\mathrm{d}\vec{n}} \right) \mathrm{d}S + C\omega_n u(0)$$

Have then:

$$\int_{\Omega} u\left((\Delta\psi)\right) \mathrm{d}x := \int_{\Omega} \psi \Delta u \mathrm{d}x - \int_{\partial\Omega} \left(\psi \frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u \frac{\mathrm{d}\psi}{\mathrm{d}\vec{n}} \right) \mathrm{d}S = C\omega_n u(0)$$

(You can also use that for test function supported on compact subsets of Ω, all the stuff on $\partial\Omega$ vanishes).

So indeed $\Delta\psi(0) = C\omega_n \delta_{\vec{0}}$ in the distributional-integration-by-parts sense. □

6.27.3. Integral Representation Formula. Let us define now a kernal $K(x,\xi) := \psi(|x-\xi|)$ to be this fundamental sol'n centered at some arbitrary point ξ. By what we did above $\Delta K(x,\xi) = \delta_\xi$. If $\Delta u = 0$ then the identity developed above gives a very important identity:

PROPOSITION 6.6. *Integral Representation for solutions to Laplace's equation:*

$$u(\xi) = -\int_{\partial\Omega} \left(K(x,\xi) \frac{\mathrm{d}u}{\mathrm{d}\vec{n}}(x) - u(x) \frac{\mathrm{d}K(x,\xi)}{\mathrm{d}\vec{n}} \right) \mathrm{d}S(x)$$

This shows that the solution u depends only on the data on the boundary. This also lets us see very clearly that one CANNOT prescribe both $u\,|_{\partial\Omega}$ and $\frac{\mathrm{d}u}{\mathrm{d}\vec{n}}\,|_{\partial\Omega}$.

COROLLARY 6.1. *Let u be a sol'n to the Dirchlet problem with $u\,|_{\partial\Omega} = f$ prescribed. Then there are NO soln's for the problem $v\,|_{\partial\Omega} = f$ and $\frac{dv}{d\vec{n}}\,|_{\partial\Omega} = g$ except for the very exceptional case that $g = \frac{du}{d\vec{n}}\,|_{\partial\Omega}$ matches the solution u we had found earlier.*

PROOF. This follows directly by uniquness (since any such sol'n v must be $v = u$) or you can see it from the above integral formula. □

This integral representation also shows that u is infinetly differentiable: we can take any number of ξ derivatives by differentiating under the integral sign and using the differentiability of

COROLLARY 6.2. *Any sol'n u is infinetly differentiable $u \in C^{\infty}(\Omega)$*

PROOF. Just use the integral representation and integrate under the integral sign. Since $K(x,\xi)$ is differentiable w.r.t. ξ away from $\xi = x$ (i.e. away from $r = 0$). For points $\xi \in \Omega$, since Ω is an open set, there is no issue with these "problem points" where $x = \xi$. □

We can actually show a slightly stronger result: that sol'ns u are real analytic...i.e. u is locally equal to a power series.

COROLLARY 6.3. *Any sol'n u is real analytic.*

PROOF. It suffices to show that any sol'n u has an analytic extension. Again, observe $K(x,\xi)$ is real analytic away from $x = \xi$. ??? □

EXAMPLE 6.6. (The Cauchy problem for the Laplace equation is generally unsovable.)

Suppose we look at the half space $B^+ = \{x_n > 0\}$ (equally well will be a compact "slice" from this half space) with data prescribed on the boundary $x_n = 0$:

$$u = 0,\ u_{x_n} = g(x_1, \ldots, x_{n-1})$$

We will show that if g is not real analytic, then there cannot be any solution.

Any solution u here can be extended to the whole space by "Schwarz reflection principle" $u(x_1, \ldots, x_{n-1}, x_n) = -u\left(x_1, \ldots, x_n, -x_n\right)$. Since u is real analytic it must be that this extended function has $u\,|_{x_n=0}$ is real analytic. But this is exactly g, so g must be real analytic.

6.27.4. Law of averages. Notice that if $w(x) = w(x_1, \ldots x_n)$ is any sol'n of $\Delta w = 0$ then:

$$G(x,\xi) = K(x,\xi) + w(x)$$

is again a fundemental sol'n of the Laplace equation with a pole at ξ. The advantage of doing this is that by adding this w, we can manipulate the behaiviour of G on the boundary $\partial\Omega$. The integral identity still holds:

$$u(\xi) = \int_{\Omega} G(x,\xi)\Delta u \mathrm{d}x - \int_{\partial\Omega} \left(G(x,\xi)\frac{\mathrm{d}u(x)}{\mathrm{d}\vec{n}_x} - u(x)\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x} \right) \mathrm{d}S_x$$

For example, in the case $\Omega = B_{r_0}(x_0)$ adding on a CONSTANT function $w(x) = -\psi\left(r_0\right)$. Notice then that:

$$G(x,\xi) = K(x,\xi) - \psi\left(r_0\right) = \psi\left(|x-\xi|\right) - \psi\left(r_0\right) \text{ by choice of } \psi$$

From this it is clear that $G(x, x_0) = 0$ for any $x \in \partial\Omega$, since $|x - x_0| = r_0$ for such x. We can also explictly compute

$$\frac{\mathrm{d}G(x,x_0)}{\mathrm{d}\vec{n}_x} = \frac{\mathrm{d}K(x,x_0)}{\mathrm{d}\vec{n}_x} = \psi'(r_0) = \frac{1}{\omega_n} r_0^{1-n}$$

So our integral formula becomes:

$$u(x_0) = \int_{\Omega} G(x,x_0)\Delta u(x)\mathrm{d}x + \frac{1}{\omega_n r_0^{n-1}} \int_{\partial\Omega} u(x)\mathrm{d}S_x$$

When $\Delta u = 0$ this gives **Gauss's Law of the Arithmetic Mean:**

THEOREM 6.35. *For $\Delta u = 0$, we have:*

$$u(x_0) = \frac{1}{\omega_n r_0^{n-1}} \int_{|x-x_0|=r_0} u(x)\,dS_x$$

PROOF. See above formula. □

REMARK 6.12. Notice that ωr_0^{n-1} is the surface area of the ball $|x - x_0| = r_0$, so this says that the value at the center of the ball is equal to the average value on the surface of the ball.

A function is called **subharmonic** if it satifies the inequality:

$$u(x_0) \leq \frac{1}{\omega_n r_0^{n-1}} \int_{|x-x_0|=r_0} u(x)\mathrm{d}S_x$$

The above identity, along with the monotonicity of ψ shows that functions with $\Delta u \geq 0$ are subharmonic.

6.27.5. Poisson's formula. From our proof that the Kernal $K(x,\xi)$ has $\Delta K(x,\xi) = \delta_\xi$, we had the integral formula that for any compactly supported test function u that:

$$\begin{aligned} u(\xi) &= \int (\Delta_x K(x,\xi))\, u(x)\mathrm{d}x \\ &= \int (\Delta_\xi K(x,\xi))\, u(x)\mathrm{d}x \\ &= \Delta_\xi \left(\int K(x,\xi)u(x)\mathrm{d}x \right) \end{aligned}$$

If we put $w(\xi) = \int K(x,\xi)u(x)\mathrm{d}x$ then this is saying that u satisfies the differential equation:

$$\Delta_\xi w(\xi) = u(\xi)$$

This actually works even for functions u which are $C^2(\overline{\Omega})$...i.e. they don't have to be compactly supprted.

PROPOSITION 6.7. *Suppose we are given a function u and we seek a solution to the DE:*

$$\Delta_\xi w(\xi) = u(\xi)$$

Then $w(\xi) = \int K(x,\xi)u(x)dx$ is a solution!

PROOF. To prove this a bit more rigorously, one can use a partition of unity to approximate $C^2(\overline{\Omega})$ solutions by compactly supported ones. □

6.27.6. Relationship to Holomorphic functions in dimension $n = 2$. Come back to this. Also go over how to map solutions via conformal maps.

6.28. The Maximum Principle

6.28.1. Without using the averging principle; analyze potential maxima.

THEOREM 6.36. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and suppose $\Delta u > 0$ in Ω. Then:*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

PROOF. Suppose by contradiction that x_0 is an internal maxima. Since x_0 a local extremum, we have $\nabla u(x_0) = 0$. At x_0 we have $\Delta u(x_0) = \sum u_{x_i x_i}(x_0) > 0 \implies \exists i_0$ so $u_{x_{i_0} x_{i_0}}(x_0) > 0$. But then by traveling in the x_{i_0}direction, we get an even greater value!. □

THEOREM 6.37. *Same theorem as before, but now assume only that $\Delta u \geq 0$. Then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$*

PROOF. Define a helper function $v = |x|^2$. For any $\epsilon > 0$ we have that $u + \epsilon v$ satisfies the conditions of the previous theorem. Hence:

$$\max_{\overline{\Omega}} (u + \epsilon v) = \max_{\partial\Omega} (u + \epsilon v)$$

Hence we have for every $\epsilon > 0$ the inequality:

$$\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} (u + \epsilon v) \leq \max_{\partial\Omega} (u + \epsilon v) = \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} v$$

Taking $\epsilon \to 0$ in this inequality gives the desired result. □

In the special case that u has $\Delta u = 0$, we have both a maximum principle AND a "minimum principle"...just apply the maximum principle to $-u$. Combining the max and min principle gives:

THEOREM 6.38. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and suppose $\Delta u = 0$ in Ω. Then:*

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$$

COROLLARY 6.4. *(Uniqueness) A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is determined uniquely by the values of Δu in Ω and of u on $\partial\Omega$.*

PROOF. Subtract any two such solutions. The difference is harmonic and is 0 on the boundary, so by the maximum principle it must be 0 everywhere. Hence the two solutions are the same! □

REMARK 6.13. This method of proving the maximum principle works for arbitary elliptic operators:

$$au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y = 0$$

by exploiting the fact that at local maxima we must have that $\det Hess = u_{xx}u_{yy} - u_{xy}^2 \geq 0$ and $u_{xx} \leq 0$ and $u_{yy} \leq 0$

6.28.2. Using the averaging principle. The averaging principle $u(x_0) = \frac{1}{\omega_n r_0^{n-1}} \int_{|x-x_0|=r_0} u(x)\mathrm{d}S_x$ we had from the last section is a powerful way to prove maximum principles.

THEOREM 6.39. *Suppose* $u \in C^2(\Omega)$ *and* $\Delta u \geq 0$ *in* Ω. *Then either* u *is a constant or:*

$$u(\xi) < \sup_{\Omega} u \text{ for all } \xi \in \Omega$$

REMARK 6.14. The result actually holds for any u that is subharmonic in Ω.

PROOF. We use a "connectedness" argument. Take $M = \sup_\Omega u$ and then let $\Omega_1 = \{\xi \in \Omega : u(\xi) = M\}$ and let $\Omega_2 = \{\xi \in \Omega : u(\xi) < M\}$. Ω_2 is an open set by continuity of u. Ω_1 can also seen to be open by the averaging principle for subharmonic functions. Take $x_0 \in \Omega_1$ and consider that for any r_0 that:

$$\begin{aligned} 0 &\leq \int\limits_{|x-x_0|=r_0} (u(x) - u(x_0))\, \mathrm{d}S_x \\ &= \int\limits_{|x-x_0|=r_0} (u(x) - M)\, \mathrm{d}S_x \\ &\leq 0 \end{aligned}$$

So we have an equality sandwhich and everything must be equal. Consequently $u(x) = M$ in a n'h'd of x_0. Hence by the usual connectedness argument, we can conclude that Ω_1 must be empty and that $u < M$ everyewhere on the interior of Ω. □

THEOREM 6.40. *Suppose* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *and* $\Delta u \geq 0$ *in* Ω. *Then either* u *is a constant or:*

$$u(\xi) < \max_{\partial\Omega} u \text{ for all } \xi \in \Omega$$

PROOF. u must achieve its maximum on the compact set $\overline{\Omega}$ somewhere. By the previous theorem, it cannot be on interior. Hence it is attained on the boundary and $\max_{\partial\Omega} u = \max_{\overline{\Omega}} u$. □

6.29. The Dirichlet Problem, Green's Function, and Possion's Formula

Recall that if we have a fundemental solution $G(x,\xi)$ with $\Delta_x G(x,\xi) = \delta_x$, then we had the integral representation:

$$u(\xi) = \int\limits_{\Omega} G(x,\xi)\Delta u \mathrm{d}x - \int\limits_{\partial\Omega} \left(G(x,\xi)\frac{\mathrm{d}u(x)}{\mathrm{d}\vec{n}_x} - u(x)\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x} \right) \mathrm{d}S_x$$

DEFINITION 6.7. We call a fundemental sol'n $G(x,\xi)$ with pole ξ a **Green's function** for the Dirichlet problem for the Laplace equation in the domain Ω if:

$$G(x,\xi) = K(x,\xi) + v(x,\xi)$$

for $x \in \overline{\Omega}$, $\xi \in \Omega$ and $x \neq \xi$ with $K(x,\xi) = \psi(|x-\xi|)$ our spherically symmetric solution and $v(x,\xi)$ is a solution to $\Delta_x v = 0$ inside Ω of class $C^2(\overline{\Omega})$. We also require that:

$$G(x,\xi) = 0 \text{ for } x \in \partial\Omega, \xi \in \Omega$$

(This is the part that makes it the Green's function *for the Dirichlet problem*)

REMARK 6.15. Physically, $K(\cdot,\xi)$ is the potential function caused by a point charge at the point $\xi \in \Omega$. We are looking for a potential function $v(\cdot,\xi)$, which depends on ξ, that will work out to *exactly cancel* any voltage contribution on the boundary $\partial\Omega$ that $K(\cdot,\xi)$. We want to do this without adding any charges inside Ω....this is the condition that $\Delta_x v = 0$ inside Ω. (We will see that adding point charge in a smart way *outside* of Ω is actually a very smart way to go about doing this)

If we have a Green's function, notice that the terms with $\frac{\mathrm{d}u}{\mathrm{d}\vec{n}}$ at the boundary $\partial\Omega$ vanish, so we will have an integral formula for any u that has $\Delta u = 0$, namely:

$$u(\xi) = \int_{\partial\Omega} u(x)\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x}\mathrm{d}S_x$$

Notice that to construct such a G amounts to solving a Dirchlet problem: you have to find a harmonic $v = -K$ on $\partial\Omega$. In some cases G can be produced expliticlty. We will show how this can be done for halfspaces and for balls.

When $n = 2$, the Laplace equation is conformally invariant under conformal mappings, so if we can solve it here and we are good at conformal mappings, we can solve it in a number of regions.

6.29.1. Poisson Integral formula, Method of Images. Lets try to find such a Green's function in the region $\Omega = B_a(0)$, a ball of radius a centered at the origin. Recall the definition of inversion of a point through a sphere, namely $\xi \to \xi^* = \frac{a^2}{|\xi|^2}\xi$. Sphere inversion has the property that for every x on the sphere, we have:

$$\frac{|x-\xi^*|}{|x-\xi|} = \frac{a}{|\xi|} = const$$

This is handy because the function K depends only on these radii, so if put $G(x,\xi) = K(x,\xi) + v(x,\xi)$ and we choose $v(x,\xi)$ to be something like $-K(x,\xi^*)$ we can make sure the Green's funtion vanishes on the boundary.

Physically this is known as the **method of images**. In the physics picture, finding $G(x,\xi)$ is trying to find the voltage function for a point charge at the point ξ where the boundary of the sphere is held at fixed voltage $V = 0$..imagine it is a conducting shell made of metal. One way to ensure that the boundary condition $V = 0$ is met is to imagine that there is a *mirror image* particle of opposite charge placed in exactly the right spot to cancel any contributions from the first charge. (Physically: the electrons in the metal shell will rearrange themselves to mimic such a charge distribution in order to keep their voltage at 0)

Notice that by the relationship that $\frac{|x-\xi^*|}{|x-\xi|} = \frac{a}{|\xi|} = const$, we know that:

$$K(x,\xi) = \frac{1}{(2-n)\omega_n}\left|x-\xi\right|^{2-n}, \quad K(x,\xi^*) = \frac{1}{(2-n)\omega_n}\left|x-\xi^*\right|^{2-n}$$

(This is for the case $n > 2$) So for $x \in \partial\Omega$ the two quantities are equal up to the factor $\left(\frac{a}{|\xi|}\right)^{2-n}$. Hence the following Green's function works:

$$G(x,\xi) = K(x,\xi) - \left(\frac{|\xi|}{a}\right)^{2-n} K(x,\xi^*)$$

vanishes for $x \in \partial\Omega$ exactly as desired! The second term, $\left(\frac{|\xi|}{a}\right)^{2-n} K(x,\xi^*)$, is singular only outside of the sphere Ω so this is a valid solution with $\Delta_x G(x,\xi) = \delta_\xi$ inside Ω.

We can now do a calculation to see compute the following:

PROPOSITION 6.8. *Have:*

$$\frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x} = \frac{1}{a\omega_n}\frac{a^2 - |\xi|^2}{|x-\xi|^n}$$

PROOF. Since $K(x,\xi) = \frac{1}{(2-n)\omega_n}|x-\xi|^{2-n}$, we have:

$$\begin{aligned} \frac{\mathrm{d}K(x,\xi)}{\mathrm{d}\vec{n}_x} &= \nabla K(x,\xi)\cdot\left(\frac{x}{|x|}\right) \\ &= \frac{1}{\omega_n}|x-\xi|^{1-n}\left(\frac{x-\xi}{|x-\xi|}\right)\cdot\left(\frac{x}{|x|}\right) \\ &= \frac{1}{a\omega_n}|x-\xi|^{-n}(x-\xi)\cdot(x) \\ &= \frac{1}{a\omega_n}|x-\xi|^{-n}\left(a^2 - \xi\cdot x\right) \end{aligned}$$

(Since $x\cdot x = |x|^2 = a^2$ here) So we have then:

$$\begin{aligned} \frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x} &= \frac{\mathrm{d}K(x,\xi)}{\mathrm{d}\vec{n}_x} - \left(\frac{|\xi|}{a}\right)^{2-n}\frac{\mathrm{d}K(x,\xi^*)}{\mathrm{d}\vec{n}_x} \\ &= \frac{1}{a\omega_n}\left[|x-\xi|^{-n}\left(a^2-\xi\cdot x\right) - \left(\frac{|\xi|}{a}\right)^{2-n}|x-\xi^*|^{-n}\left(a^2-\xi^*\cdot x\right)\right] \\ &= \frac{1}{a\omega_n}\left[|x-\xi|^{-n}\left(a^2-\xi\cdot x\right) - \left(\frac{|\xi|}{a}|x-\xi^*|\right)^{-n}\left(\left(\frac{|\xi|}{a}\right)^2 a^2 - \left(\frac{|\xi|}{a}\right)^2\xi^*\cdot x\right)\right] \end{aligned}$$

We now use the fact that $|x-\xi| = \frac{|\xi|}{a}|x-\xi^*|$ and $\xi^* = \left(\frac{a}{|\xi|}\right)^2\xi$ to get to:

$$\begin{aligned} \frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x} &= \frac{1}{a\omega_n}\left[|x-\xi|^{-n}\left(a^2-\xi\cdot x\right) - (|x-\xi|)^{-n}\left(|\xi|^2 - \xi\cdot x\right)\right] \\ &= \frac{a^2-|\xi|^2}{a\omega_n|x-\xi|^n} \end{aligned}$$

□

REMARK 6.16. Since the solution only depends on ∇K here, this actually works for $n=2$ as well. (Since in any dimension $\psi'(r) = \frac{1}{\omega_n}r^{1-n}$)

PROPOSITION 6.9. *(Poisson Integral formula). If $u(\xi)$ is a solution to the Dirichelet problem in the sphere of radius a, then we have an integral formula for u in terms of its boundary data:*

$$u(\xi) = \int\limits_{|x|=a} H(x,\xi)u(x)dS_x$$

The kernal is given by:

$$H(x,\xi) := \frac{dG(x,\xi)}{d\vec{n}_x} = \frac{a^2 - |\xi|^2}{a\omega_n |x-\xi|^n}$$

*The kernal is called the **Poisson Kernal**.*

REMARK 6.17. Note that if someone specifies some arbitary boudnary data f, there might not be a solution u with this as its boudnary data; we need some kind of smoothness criteria. The following properties of H make this kind of thing possible to prove however:

LEMMA 6.3. *The Poisson Kernal H has the following properties:*
i) $H(x,\xi) \in C^\infty$ for $|x| \leq a$. $|\xi| < a$ and $x \neq \xi$
ii) $\Delta_\xi H(x,\xi) = 0$ for $|\xi| < a$ and $|x| = a$
iii)$\int_{|x|=a} H(x,\xi) dS_x = 1$ for all $|\xi| < a$
iv) $H(x,\xi) > 0$ for $|x| = a$, $|\xi| < a$
v) If $|x_0| = a$ then:

$$\lim_{\xi \to x_0} H(x,\xi) = 0$$

uniformly in x for $|x - \xi_0| > \delta > 0$

PROOF. These are not too hard to check. □

PROPOSITION 6.10. *Suppose f is continuous on $|x| = a$. Then the function $u(\xi) = \int_{|x|=a} H(x,\xi) f(x) dS_x$ for $|\xi| < a$ and $u(\xi) = f(\xi)$ for $|\xi| = a$ is a continuous solution to the Dirichlet problem and is C^∞ inside the interior $|\xi| < a$*

PROOF. By differentiating under the integral sign, it is clear that $u(\xi)$ defined in this way is harmonic and C^∞. It remains to check that u is continuous on the boundary and that it equal to f there.Consider any point $x_0 \in \partial\Omega$ on the boundary and think of a limit $\xi \to x_0$. By properties of the kernal H we write:

$$\begin{aligned} u(\xi) - f(x_0) &= \int\limits_{|x|=a} H(x,\xi)\left(f(x) - f(x_0)\right) \mathrm{d}S_x \\ &= \int\limits_{|x|=a, |x-x_0|<\delta} + \int\limits_{|x|=a, |x-x_0|>\delta} \ldots \end{aligned}$$

The integral over nearby points $|x - x_0| < \delta$ will $\to 0$ by the continuity of f since we can choose δ so small so that $|f(x) - f(x_0)|$ is small there. The integral over "far away" poitns $|x - x_0| > \delta$ tends to 0 by the property v) from the last lemma □

6.29.2. Estimates from the integral/compactness results. Differentiating under the integral in the formula $u(\xi) = \int_{\partial\Omega} H(x,\xi) u(x) \mathrm{d}S_x$ gives (Recall that $H(x,\xi) := \frac{\mathrm{d}G(x,\xi)}{\mathrm{d}\vec{n}_x} = \frac{a^2 - |\xi|^2}{a\omega_n |x-\xi|^n}$):

$$u_{\xi_i}(0) = \frac{n}{a^{n+1}\omega_n} \int\limits_{|x|=a} x_i u(x) \mathrm{d}S_x$$

and hence:

$$|u_{\xi_i}(0)| \leq \frac{n}{a} \max_{|x|=a} |u(x)|$$

More generally if if we have a point ξ in the domain Ω, and $d(\xi)$ is the distance from ξ to the boundary of Ω, we can apply the above result in a sphere centered at ξ and of radius a for any $a < d(\xi)$. As $a \to d(\xi)$ we get the estimate:

$$\left|u_{\xi_i}\right|(0) \le \frac{n}{d(\xi)} \sup_{\Omega} |u|$$

Similar inequalites hold for higher derivatives too.

These inequalites lead to completness and compactness properties for the set of harmonic functions. For example, suppose we have a sequence $u_k \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfing $\Delta u_k = 0$ in Ω and suppose that on the boundary we have u_k converging uniformly on $\partial\Omega$ to a fixed function f. Notice that if the sequence u_k is uniformly Cauchy on the boundary, then it is uniformly Cauchy on the interior Ω by the maximum principle. Hence the sequence u_k must converge to some function u. The above estimates let us see that all the derivatives are congerging too! Hence the limit point u must also be harmonic.

If we have only that u_k is a BOUNDED set $|u_k| \le M$ on Ω, the estimates for the derivatives for the family u_k we have above actually know the family is equicontinuous. Hence, by the Arzela Ascoli theorem, we know that there is a uniformly convergence subsequence. As before the limit point myst be harmonic too.

6.30. Proof of Existence of Solutions for the Dirichlet Problem Using Subharmonic Functions ("Perron's Method")

We can prove existence of solutions to the Laplace equation by using the fact that harmonic functions are the pointwise-largest functions that are subharmonic (recall that subharmonic was defined in terms of an inequality in Gauss' averaging principle)

THEOREM 6.41. *Let $\sigma(\Omega)$ denote the set of subharmonic functions, namely those which satisfy at every point ξ:*

$$u(\xi) \le \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(\xi)} u(x) dS_x \text{ for sufficiently small } \rho$$

Suppose we have some boundary data f on $\partial\Omega$. Let $\sigma_f(\Omega)$ denote the set of subharmonic functon $u \in \sigma(\Omega)$ that also have $u \le f$ on $\partial\Omega$. Then the function:

$$w_f(x) := \sup_{u \in \sigma_f(\Omega)} u(x)$$

This function w_f is harmonic in the region Ω.

REMARK 6.18. This does not show that $w_f = f$ on $\partial\Omega$....to get this additional information you need some information about the regularity of the boundary $\partial\Omega$. We will see this in the barrier functions we look at later.

We divide the proof into a number of sub-lemmas

LEMMA 6.4. *For $u \in \sigma(\Omega) \cap C^0(\overline{\Omega})$ we have:*

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u$$

PROOF. This was the maximum principle for subharmonic functions! □

DEFINITION 6.8. For $\xi \in \Omega$ and ρ a radius such that $B_\rho(\xi) \subset \Omega$, and $u \in C^0(\Omega)$ define the function $u_{\xi,\rho}$ by replacing the value of u inside $B_\rho(\xi)$ with a harmonic function there. I.e. $u_{\xi,\rho}$ is characterized by:

$$\begin{aligned} u_{\xi,\rho}(x) &= u(x) \text{ for all } x \notin B_\rho(\xi) \\ u_{\xi,\rho} &\text{ is } \text{harminic for } x \in B_\rho(\xi) \end{aligned}$$

This is ok since we know how to explicitly construct (via Poisson integral) solutions to the Laplace equation inside a sphere.

LEMMA 6.5. *For $u \in \sigma(\Omega)$ and $B_\rho(\xi) \subset \Omega$ we have:*

$$u(x) \leq u_{\xi,\rho}(x) \text{ for all } x \in \Omega$$

and moreover $u_{\xi,\rho}$ is subharmonic, i.e.:

$$u_{\xi,\rho} \in \sigma(\Omega)$$

PROOF. The inequality $u(x) \leq u_{\xi,\rho}(x)$ is clear outside of the ball $B_\rho(\xi)$. Inside the ball, we will apply the maximum principle to get the result we need. We know that $u - u_{\xi,\rho}$ is a subharmonic function (since $u_{\xi,\rho}$ is harmonic here and u is subharmonic). Hence by the maximum principle, $\max_{B_\xi(\rho)} u - u_{\xi,\rho} \leq \max_{\partial B_\xi(\rho)} u - u_{\xi,\rho} = 0$ since the two agree on the boundary.

To see that $u_{\xi,\rho}$ is subharmonic we must verify that $u_{\xi,\rho}(x) \leq \frac{1}{\omega_n r^{n-1}} \int u(x) \mathrm{d}S$ at each point x for r small enough. This is clear if $x \in \Omega \backslash \overline{B_\rho}(\xi)$ since $u_{\xi,\rho} \equiv u$ here and u is subharmonic, and it is clear if $x \in B_\rho(\xi)$ since $u_{\xi,\rho}$ is harmonic inside the ball. For $x \in \partial B_\rho(\xi)$ on the boundary of the ball, we use the inequality $u(x) \leq u_{\xi,\rho}(x)$:

$$u_{\xi,\rho}(x) = u(x) \leq \frac{1}{\omega_n r^{n-1}} \int u(x) \mathrm{d}S \leq \frac{1}{\omega_n r^{n-1}} \int u_{\xi,\rho}(x) \mathrm{d}S$$

□

LEMMA 6.6. *We defined "subharmonic" to mean that $\forall x_0 \in \Omega \exists r$ so that $u(x_0) \leq \frac{1}{\omega_n r^{n-1}} \int u(x) dS$. In fact, if u is subharmonic then this inequality holds for ANY choice of r, as long as $B(x_0, r) \subset \Omega$*

PROOF. The proof goes by comparing with $u_{\xi,\rho}$. Have:

$$u(x_0) \leq u_{x_0,r}(x_0) \leq \frac{1}{\omega_n r^{n-1}} \int u_{x_0,\rho}(x) \mathrm{d}S = \frac{1}{\omega_n r^{n-1}} \int u(x) \mathrm{d}S$$

□

LEMMA 6.7. *A function u is harmonic if and only if both $u \in \sigma(\Omega)$ is subharmonic and $-u \in \sigma(\Omega)$ is subharmonic.*

PROOF. $\implies$ is clear since $\Delta u \geq 0$ and $\Delta(-u) \geq 0$.

To see the converse, we will show that $u(x) = u_{\xi,\rho}(x)$ at every point x. Indeed, $u(x) \leq u_{\xi,\rho}(x)$ for every point x since $u \in \sigma(\Omega)$ and $-u(x) \leq -u_{\xi,\rho}(x)$ since $-u \in \sigma(\Omega)$. Hence they are equal! □

LEMMA 6.8. *If for every x_0 , there is r small enough so that $u(x_0) = \frac{1}{r^{n-1}\omega_n} \int u(x) dx$, then u is harmonic.*

PROOF. The equality means that both u and $-u$ are subharmonic. □

LEMMA 6.9. *If $u_1, \ldots, u_k \in \sigma_f(\Omega)$ then $v = \max(u_1, \ldots, u_k) \in \sigma_f(\bar{\Omega})$ too.*

PROOF. For any r sufficiently small,

$$\begin{aligned} v(x_0) &= \max\left(u_1(x_0),\ldots,u_k(x_0)\right) \\ &\leq \max\left(\frac{1}{r^{n-1}\omega_n}\int u_1(x)\mathrm{d}x,\ldots,\frac{1}{r^{n-1}\omega_n}\int u_k(x)\mathrm{d}x\right) \\ &\leq \max\left(\frac{1}{r^{n-1}\omega_n}\int v(x)\mathrm{d}x,\ldots,\frac{1}{r^{n-1}\omega_n}\int v(x)\mathrm{d}x\right) \\ &= \frac{1}{r^{n-1}\omega_n}\int v(x)\mathrm{d}x \end{aligned}$$

□

LEMMA 6.10. *w_f is harmonic in Ω.*

PROOF. Fix an arbitrary n'hd $B_\rho(\xi)$. We will show that w_f is harmonic here. Let $x_1,\ldots$ be a dense subset of $B_\rho(\xi)$. Then, for each x_i since $w_f=\sup_{u\in\sigma_f(\Omega)}u$ find a sequence $u_{i,j}$ of functions in $\sigma_f(\Omega)$ so that $\lim_{j\to\infty}u_{i,j}(x_i)\uparrow w_f(x_i)$ for each x_i. Define $u_j:=\max\left(u_{1,j},u_{2,j},\ldots u_{j,j}\right)$. By the lemma, $u_j\in\sigma_f(\Omega)$ too. We have now that $\lim_{j\to\infty}u_j(x_i)\geq\lim_{j\to\infty}u_{i,j}(x_i)=w_f(x_i)$ and since this is a sup, we actually have an equality. Moreover, since $u_{j,(\xi,\rho)}\geq u_j$ for each j and is still in $\sigma_f(\Omega)$ we have $\lim_{j\to\infty}u_{j,(\xi,\rho)}(x_i)=w_f(x_i)$ for each i.

We can suppose WOLOG that the u_j are bounded from below by $m=\inf_{\partial\Omega}f$ by replacing $u_j$by $\max(u_j,m)$ if nessasary. The u_j are automatically bounded from above by $M=\sup_{\partial\Omega}f$ by the maximum principle. Now since u_j is a bounded set of harmonic functions, by the derivative estimates we had in the last section and the Arzela Ascoli theorem, we know there is a unifromly convergent subsequence $u_{j_k,(\xi,\rho)}\to W$ for some harmonic W.

It must be the case that $W=w$ however, since the two functions agree on the dense set $\{x_i\}$ and the functions are continuous: $W(x_i)=\lim_{k\to\infty}u_{j_k,(\xi,\rho)}(x_i)=w(x_i)$. □

6.31. Solution by Hilbert Space methods

Convert the Dirichlet problem into:

$$\begin{aligned} \Delta u &= w \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Define an inner product for $u,v\in C^1(\Omega)$ with $u=0$ on $\partial\Omega$ by:

$$\langle u,v\rangle=\int_\Omega\nabla u\cdot\nabla v\mathrm{d}x$$

And take the completion w.r.t to this inner product to get a Hilbert space H_0^1 with this norm (Turns out it contains all the functions with weak derivatives in L^2)

Define now the functional:

$$\phi(u)=\int uw\mathrm{d}x$$

We check that this is a bounded linear operator on H_0^1 by using the Poincare Inequatiy:

PROPOSITION 6.11. *(Poincare) There exists C so that*

$$\int_\Omega u^2 dx \leq C \int_\Omega |\nabla u|^2 \, dx$$

PROOF. First prove it for $u \in C^1(\Omega)$. Find a cube that contains Ω. Then integrate by parts in the x_1 direction:

$$\begin{aligned} \left| \int_\Omega u^2 \mathrm{d}x \right|^2 &= \left| -2 \int_\Omega x_1 u \frac{\partial u}{\partial x_1} \mathrm{d}x \right|^2 \\ &\leq 2a \int_\Omega |u| \left| \frac{\partial u}{\partial x_1} \right|^2 \\ &\leq 2a \left(\int u^2 \mathrm{d}x \right) \left(\int |\nabla u|^2 \, \mathrm{d}x \right) \end{aligned}$$

So dividing through by $\int_\Omega u^2 \mathrm{d}x$ gives the result:

$$\int_\Omega u^2 \mathrm{d}x \leq C \int_\Omega |\nabla u|^2 \, \mathrm{d} \text{ for } u \in C^1$$

Now to prove it for arbitrary $u \in H_0^1$, take a sequence $u_j \to u$ in the sense of H_0^1 so that each $u_j \in C^1$. But then u_j is H_0^1-Cauchy. By the above, it is also $\|\cdot\|_{L^2}$Cauchy. Hence both sides of teh inequality are converging, and we get it for general $u \in H_0^1$. □

PROPOSITION 6.12. *The functional $\phi(u) = \int uw dx$ is continuous*

PROOF. Have:

$$\begin{aligned} |\phi(u)|^2 &= \left| \int uw \mathrm{d}x \right|^2 \\ &\leq \left| \int u^2 \mathrm{d}x \right| \left| \int w^2 \mathrm{d}x \right| \\ &\leq C \left| \int |\nabla u|^2 \, \mathrm{d}x \right| \left| \int w^2 \mathrm{d}x \right| \\ &\leq C \left| \int w^2 \mathrm{d}x \right| \|u\|^2 \end{aligned}$$

□

PROPOSITION 6.13. *By the Riesz Representation theorem, there exists a weak solution to:*

$$\Delta u = w \text{ in } \Omega u = 0 \text{ on } \partial\Omega$$

PROOF. By Riesz, find v so that for every $u \in H^1$:

$$\int_\Omega uw \mathrm{d}x = \phi(u) = \langle v, u \rangle = \int \nabla u \cdot \nabla v \mathrm{d}x$$

And then by integration by parts we have:

$$\begin{aligned}\int uw\mathrm{d}x &= \int \nabla u\cdot\nabla v\mathrm{d}x\\ \int uw\mathrm{d}x &= \int u\Delta v\mathrm{d}x\\ \implies \left(\int u\left(w-\Delta v\right)\right) &= 0\end{aligned}$$

So v solves it in the weak sense. □

6.32. Energy Methods

THEOREM 6.42. *There is at most one soltuion to $\Delta u = -f$ in U and $u = g$ on ∂U.*

PROOF. Let $w = u - \tilde{u}$ be the difference of any two such solutions. Then $\Delta w = 0$ and $w = 0$ on ∂U. But:

$$0 = -\int_U w\Delta w\mathrm{d}x = \int_U \left|\nabla w\right|^2 \mathrm{d}x$$

Hence $\nabla w \equiv 0$ i.e. w is a constant. Since $w = 0$ on the boundary, must have $w \equiv 0$. □

PROOF. dfhs □

THEOREM 6.43. *(Dirichlet's Principle) Assume u solves $\Delta u = -f$ in U and $u = g$ on ∂U. Then:*

$$I[u] = \min_{w\in\mathcal{A}} I[w]$$

where $\mathcal{A} = \left\{w \in C^2 : w = g \text{ on } \partial U\right\}$

REMARK 6.19. This is saying the solution to the PDE is the one that minimizes the energy.

PROOF. Let u be a solution. For $w \in \mathcal{A}$, we know that:

$$0 = \int_U \left(-\Delta u - f\right)\left(u - w\right)\mathrm{d}x$$

Integration by parts gives:

$$0 = \int_U \nabla u\cdot\nabla\left(u-w\right) - f(u-w)\mathrm{d}x$$

(There is no boundary term). Hence:

$$\begin{aligned}
\int_U |\nabla u|^2 - uf\mathrm{d}x &= \int_U \nabla u \cdot \nabla w - wf\mathrm{d}x \\
&\le \int |\nabla u|\,|\nabla v| - \int_U \nabla w - wf\mathrm{d}x \\
&\le \frac{1}{2}\int |\nabla u|^2 \mathrm{d}x + \left(\int_U \frac{1}{2}|\nabla w|^2 - wf\mathrm{d}x\right) \\
\implies I[u] &\le I[w]
\end{aligned}$$

□

THEOREM 6.44. *Conversly, if* $I[u] = \min_{w\in\mathcal{A}} I[w]$ *then* u *satisfies the equation.*

PROOF. This is a calculus of variations proof. For any test function v notice that $I[u+\epsilon v] \ge I[u]$ so the function $f(\epsilon) = I[u+\epsilon v]$ must have an extremum at $\epsilon = 0$. Hence $f'(0) = 0$. On the other hand:

$$f'(\epsilon)\,|_{\epsilon=0} = \int Du \cdot Dv - vf\mathrm{d}x = \int_U (-\Delta u - f)\, v\mathrm{d}x$$

Since this is zero for every v it must be that $-\Delta u - f = 0$ inside U. □

Hyperbolic Equations in Higher Dimensions

These are notes from Chapter 5 of [**2**].

7.33. The Wave Equation in n-dimensional space

Definition 7.9. The wave equation for a function $u(x_1,\ldots,x_n,t)=u(x,t)$ of n space variables $x_1,\ldots,x_n$ and for time t is given by:

$$\Box u = u_{tt} - c^2\Delta u = 0$$

The operator $\Box$ is called the **D'Alembertian**. The initial value problem asks for a solution of this PDE in the $(n+1)$-dimensional halfspace $t>0$ given Cauchy data at $t=0$:

$$u=f(x),\ u_t=g(x) \text{ at } t=0$$

7.33.1. Method of spherical means. We will show how we can reduce this problem in a way to the 2 dimensional case that we have already studied. For any function $h(x)=h(x_1,\ldots,x_n)$in $\mathbb{R}^n$ we define its average $M_h(x,r)$ on a sphere of radius r centered at x:

$$\begin{aligned} M_h(x,r) &= \frac{1}{\omega_n r^{n-1}} \int\limits_{|y-x|=r} h(y)\mathrm{d}S_y \\ &= \frac{1}{\omega_n}\int\limits_{|\xi|=1} h\left(x+r\xi\right)\mathrm{d}S_\xi \end{aligned}$$

This is defined only for $r>0$, but we can extend it evenly to all of $r<0$ (the last formula also works for $r<0$ essentially amounts to a change of variable $\xi\to-\xi$)

This integral formula shows that $M_h(x,r)\in C^s(\mathbb{R}^{n+1})$ whenever $h\in C^s(\mathbb{R})$ since we can differentiate under the integral sign. For $h\in C^2\left(\mathbb{R}^n\right)$if use the divergence theorem on M_h we get:

Proposition 7.14. *M_h satifies **Darboux's Equation**:*

$$\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)M_h(x,r)=\Delta_x M_h\left(x,r\right)$$

Remark 7.20. Rememeber that the Laplacian in spherical coordiantes for a function that has only radial dependence is (See 4.1.2) :

$$\Delta\psi(r)=\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)\psi(r)=r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\psi(r)\right)$$

So you can think of this operator as like Δ_r or something, and then Darboux's equation is:

$$\Delta_r M_h(x,r)=\Delta_x M_h(x,r)$$

PROOF. The general idea is that this follows from the DIVERGENCE THEOREM, which will relate the radial derivative $\frac{\partial}{\partial r}$ on the boundary of the sphere of radisu r to Δ_x inside the region of radius r.

$$\begin{aligned}
\frac{\partial}{\partial r}M_h(x,r) &= \frac{1}{\omega_n}\int\limits_{|\xi|=1}\sum_{i=1}^{n} h_{x_i}(x+r\xi)\,\xi_i \mathrm{d}S_\xi \\
&= \frac{r}{\omega_n}\int\limits_{|\xi|<1}\Delta_x h(x+r\xi)\mathrm{d}S_\xi \text{ by the Divergence Theorem} \\
&= \frac{r^{1-n}}{\omega_n}\Delta_x\left(\int\limits_{|y-x|<r} h(y)\mathrm{d}y\right) \\
&= \frac{r^{1-n}}{\omega_n}\Delta_x\left(\int\limits_0^r\left(\int\limits_{|y-x|=\rho} h(y)\mathrm{d}y\right)\mathrm{d}\rho\right) \\
&= r^{1-n}\Delta_x\left(\int\limits_0^r \rho^{n-1}M_h(x,\rho)\mathrm{d}\rho\right)
\end{aligned}$$

Now mupliply by $r^{n-1}$and take another r derivative:

$$\begin{aligned}
\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}M_h(x,r)\right) &= \frac{\partial}{\partial r}\left(\Delta_x\left(\int\limits_0^r \rho^{n-1}M_h(x,\rho)\mathrm{d}\rho\right)\right) \\
&= \Delta_x\left(\frac{\partial}{\partial r}\left(\int\limits_0^r \rho^{n-1}M_h(x,\rho)\mathrm{d}\rho\right)\right) \\
&= \Delta_x\left(r^{n-1}M_h(x,r)\right) \\
&= r^{n-1}\Delta\left(M_h(x,r)\right)
\end{aligned}$$

And the claim follows. □

Suppose now that $u(x,t)$ is a solution to the wave equation:

$$\Box u = u_{tt} - c^2\Delta u$$

Consider $M_u(x,r,t)$, its spherical average. (We can always recover u by setting $r=0$, so knowing M_u is the same as knowing u). By The Darboux equation, we have:

$$\Delta_x M_u = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u$$

On the other hand:

$$\begin{aligned}
\Delta_x M_u &= \Delta_x\left(\frac{1}{\omega_n}\int\limits_{|\xi|=1} u(x+r\xi,t)\mathrm{d}S_\xi\right)\\
&= \frac{1}{\omega_n}\int\limits_{|\xi|=1} \Delta_x u(x+r\xi,t)\mathrm{d}S_\xi\\
&= \frac{1}{\omega_n}\int\limits_{|\xi|=1} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x+r\xi,t)\mathrm{d}S_\xi\\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}M_u
\end{aligned}$$

Hence, when u is a solution to the multi-dimensional wave equation $\Box u = 0$, the spherical averages satisfy the so called ***Euler-Poissson-Darboux Equation***:

$$\frac{\partial^2}{\partial t^2}M_u = c^2\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}\right)M_u = c^2 r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}M_u\right)$$

7.33.2. In three dimensions $n = 3$: In three dimensions the equation takes on a particularly nice form for the function $rM_u(x,r,t)$:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}M_u &= c^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)M_u\\
\Longrightarrow \frac{\partial^2}{\partial t^2}(rM_u) &= c^2\left(r\frac{\partial^2}{\partial r^2}+2\frac{\partial}{\partial r}\right)M_u\\
&= c^2\frac{\partial^2}{\partial r^2}(rM_u)
\end{aligned}$$

If we think of x as fixed here, this is just the 1-d wave equation for $rM_u(x,r,t)$. The initial condiitons are:

$$u(x,0)=f(x),\ u_t(x,0)=g(x) \implies rM_u(r,x,0)=rM_f(x,r)\ \frac{\partial}{\partial t}(rM_u(r,x,0))=rM_g(r,x)$$

By our solution for the $1-d$ wave equation $v_{tt}=c^2v_{xx}$ with $v(x,0)=f(x),\ v_t(x,0)=g(x)$ is $v(x,t)=\frac{1}{2}(f(x-ct)+f(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\xi)\mathrm{d}\xi$. So our problem for M_u has the solution:

$$\begin{aligned}
rM_u(x,r,t) &= \frac{1}{2}\left[(r+ct)\,M_f(x,r+ct)+(r-ct)\,M_f(x,r-ct)\right]\\
&\quad +\frac{1}{2c}\int\limits_{r-ct}^{r+ct}\xi M_g(x,\xi)\mathrm{d}\xi
\end{aligned}$$

To recover u we need to put $r = 0$. This involves a $0/0$ indeterminant, which we sort out using L'Hopital's rule:

$$\begin{aligned}
M_u(x,0,t) &= \lim_{r\to 0} M_u(x,r,t) \\
&= \lim_{r\to 0}\frac{1}{2}\left[\left(1+\frac{ct}{r}\right)M_f(x,r+ct)+\left(1-\frac{ct}{r}\right)M_f(x,r-ct)\right] \\
&\quad +\frac{1}{2cr}\int\limits_{r-ct}^{r+ct}\xi M_g(x,\xi)\mathrm{d}\xi \\
&= \frac{1}{2}\left[M_f(x,r+0)+M_f(x,r-0)\right]+\frac{1}{2}ct\left[\frac{\frac{\mathrm{d}}{\mathrm{d}r}\left(M_f(x,r+ct)-M_f(x,ct-r)\right)}{\frac{\mathrm{d}}{\mathrm{d}r}r}\right] \\
&\quad +\frac{\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{r-ct}^{r+ct}\xi M_g(x,\xi)\mathrm{d}\xi\right)}{\frac{\mathrm{d}}{\mathrm{d}r}(2cr)} \\
&= M_f(x,r)+\frac{1}{2}ct\left[\frac{\partial}{\partial t}M_f(x,0+ct)-(-1)\frac{\partial}{\partial t}M_f(x,ct-0)\right] \\
&\quad +\frac{((0+ct)M_g(x,0+ct)-(0-ct)M_g(x,0-ct))}{2c} \\
&= M_f(x,r)+ct\left(\frac{\partial}{\partial t}M_f(x,ct)\right)+tM_g(x,ct) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(tM_f(x,ct)\right)+tM_g(x,ct)
\end{aligned}$$

This can be written out soley in terms of the original functions u, f, g:

$$\begin{aligned}
u(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4\pi c^2 t}\int\limits_{|y-x|=ct} f(y)\mathrm{d}S_y\right)+t\frac{1}{4\pi\,(ct)^2}\int\limits_{|y-x|=ct} g(y)\mathrm{d}S_y \\
&= \frac{1}{4\pi c^2 t^2}\int\limits_{|y-x|=ct}\left(tg(y)+f(y)+\sum_i f_{y_i}(y)\,(y_i-x_i)\right)\mathrm{d}S_y
\end{aligned}$$

Any solution u of the inital-value problem of class C^2 for $t \geq 0$ in $n = 3$ dimensions is given by this integral formula and is hence **unique.**

Conversly, every f, g can be integrated to give such a solution u by this integral formula. Notice that we need to take a derivative of f in the formula. This means that if we want u to be a C^2 solution, we need $f \in C^3$ and $g \in C^2$. The fact that f needs to be smoother than u is the effect of **focusing**: it is possible for bad initial data to focus themselves in space and get even worse (i.e. discontinuous 3rd derivatives in f can focus to cause discontinuous 2nd derivatives for u at some later time.)

7.33.3. Energy. The pointwise behaviour of f is subject to things like focusing. However the L^2 average of the thing remains ok for all time. Consider the

energy norm of u:

$$\begin{aligned} E(t) &= \frac{1}{2}\int\int\int\left(u_t^2(x,t)+c^2\sum_i u_{x_i}^2(x,t)\right)\mathrm{d}x \\ &= \frac{1}{2}\int\left(u_t^2+c^2\nabla u\cdot\nabla u\right)\mathrm{d}x \\ &= \frac{1}{2}\left\|u_t\right\|_{L^2}+\frac{c^2}{2}\left\|\nabla_x u\right\|_{L^2} \end{aligned}$$

This is conserved if $\Box u = u_{tt} - c^2\Delta u = 0$ satifies the wave equation and is compactly supported by the following calculation:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \int\left(u_t u_{tt}+c^2\sum u_{x_i}u_{x_i t}\right)\mathrm{d}x \\ &= \int\left(u_t u_{tt}+c^2\nabla u\cdot\nabla u_t\right)\mathrm{d}x \\ &= \int\left(u_t u_{tt}-u_t\left(c^2\Delta u\right)+c^2\nabla u\cdot\nabla u_t+u_t\left(c^2\Delta u\right)\right)\mathrm{d}x \\ &= \int\left(u_t\Box u+\nabla\cdot\left(u_t\nabla u\right)\right)\mathrm{d}x \\ &= \int\left(u_t\Box u+c^2\sum\left(u_t u_{x_i}\right)_{x_i}\right)\mathrm{d}x \\ &= \int\left(0+\nabla\cdot\left(u_t\nabla u\right)\right)\mathrm{d}x \\ &= \int\left(u_t\nabla u\right)\cdot\vec{n}\,\mathrm{d}S_x \text{ by the Divergence theorem} \end{aligned}$$

If we assume that u is compactly supported, we can restrict ourselves to this region and apply the divergence theorem on the last line. This will be 0 since we can choose the region so that $u = 0$ on the boundary.

7.33.4. Domain of Dependence. According to the integral formula for u, the value $u(x,t)$ depends on the values of g and of f and its first derivatives on the sphere $S(x,ct)$ of center x and radius ct. Thus the **domain of dependence** is the **surface of the sphere** centered at x and of radius ct.

The fact that this is a **surface** and not the filled in sphere is called **Huygen's principle in the strong form**. For most hyperbolic equations, this does not hold: in general the domain of dependence will be the filled in region. (When the dimension $n = 1$, the solution depended on the filled in region). It is sometimes true that disturbance travel with finite speed.

Conversely, if we think of an inital point y at $t = 0$, this point will only influence points x in the future if $|x-y| = ct$. This makes a **cone** of points that are effected by this point.

This means that if we start with a compactly supported inital data, the solution at time t is compactly supported: the support of u is contained in the region $\Omega_{final} \subset \cup_{x\in\Omega_{intial}} B_{ct}(x)$ where $B_{ct}(x)$ is the ball of radius ct centered at x.

Example: If the inital data is given in a region $\Omega_{initial}$ in a ball of radius ρ centered at some x_0 $\Omega_{initial} = B_\rho(x_0)$ then the region in space-time that sees the inital condition is contained in the region (x,t) such that $ct-\rho < |x-x_0| < ct+\rho$

(Since $\exists y$ s.t. $y \in B_\rho(x_0)$ implies $|x-y| = ct$ if and only if $|x-x_0| < |x-y| + |y-x_0| < ct+\rho$ and $|x-x_0| > |x-y| - |y-x_0| > ct-\rho$)

7.33.5. Decay. While the support of u is always increasing in time, the height of the function $u(x,t)$ is decaying to zero as $t \to 0$. Indeed, look at:

$$u(x,t) \quad = \quad \frac{1}{4\pi c^2 t^2} \int\limits_{|y-x|=ct} \left(tg(y) + f(y) + \sum_i f_{y_i}(y)\,(y_i - x_i) \right) \mathrm{d}S_y$$

If $\Omega_{intiial} \subset B_r(x_0)$, then the contibutions to $u(x,t)$ come only from the intersection $S(x,ct) \cap B_r(x_0)$ so the volume of the region that effect the point (x,t) is no more than $4\pi\rho^2$. If $f \in C^1$ is bounded with bounded derivative and g is bounded, then the above integral is bounded by $\frac{4\pi\rho^2}{4\pi c^2 t^2}\left(t\left\|g\right\|_\infty + \left\|f\right\|_\infty + \left\|f'\right\|_\infty ct\right)$ which shows that $u(x,t) = O(\frac{1}{t})$ as $t \to \infty$.

7.33.6. Two dimensions: Hadamard's method of descent. You can solve in dimension $n = 2$ by thinking of solutions $u(x_1,x_2,t)$ as solutions in $n=3$ $u(x_1,x_2,x_3,t)$ that happen to have the following property: $u(x_1,x_2,x_3,t) = u(x_1,x_2,\tilde{x_3},t)$ i.e. it doesn't depend on x_3 at all.

This means we can look at $u(x_1,x_2,0,t)$ as our solution to the 2D problem.

Take the intial data:

$$g(y) = g(y_1,y_2)\ f(y) = f(y_1,y_2)$$

We think of these as functions of 3 variables, with the 3rd variable not playing any role.

We now do a bit of work to show how we can eliminate the 3rd direction from showing up explicitly (we hide its dependence with change of variables etc). On the sphere $S(x_1,x_2,0,ct)$ we can write the surface element in terms of $y_1$and y_2 alone:

$$\begin{aligned} \mathrm{d}S_y &= \sqrt{1 + \left(\frac{\partial y_3}{\partial y_1}\right)^2 + \left(\frac{\partial y_3}{\partial y_1}\right)^2}\mathrm{d}y_1\mathrm{d}y_2 \\ &= \frac{ct}{|y_3|}\mathrm{d}y_1\mathrm{d}y_2 \\ &= \frac{ct}{\sqrt{(ct)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\mathrm{d}y_1\mathrm{d}y_2 \text{ since } (x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2 = (ct)^2 \end{aligned}$$

We also put in a factor of 2 to account for the fact that (y_1,y_2,y_3) has the same contribution as $(y_1,y_2,-y_3)$. Putting $r = \sqrt{(y_1-x_1)^2 - (y_2-x_2)^2}$ as a shorthand, we have:

$$\begin{aligned} u(x_1.x_2,0,t) &= \frac{1}{4\pi c^2t^2}\int\int\int_{|y-x|=ct}\left(tg(y_1,y_2,y_3)+f(y_1,y_2,y_3)+\sum_i f_{y_i}(y_1,y_2,y_3)\,(y_i-x_i)\right)\mathrm{d}S_y \\ &= 2\frac{1}{4\pi c^2t^2}\int\int_{|(y_1-x_1,y_2-x_2)|}\left(tg(y_1,y_2,0)+f(y_1,y_2,0)+\sum_{i=1,2} f_{y_i}(y_1,y_2,0)\,(y_i-x_i)\right)\frac{ct}{|y_3|}\Big|_{|y_3|=\sqrt{}} \\ &= \frac{1}{2\pi c}\int\int_{r<ct}\frac{g(y_1,y_2)}{\sqrt{(ct)^2-r^2}}\mathrm{d}y_1\mathrm{d}y_2+\frac{1}{2\pi ct}\int\int_{r<ct}\frac{f(y_1,y_2)}{\sqrt{(ct)^2-r^2}}\mathrm{d}y_1\mathrm{d}y_2+\frac{1}{2\pi ct}\int\int_{r<ct}\frac{f_{y_1}(y_1,y_2)\,(y}{} \\ &= \frac{1}{2\pi c}\int\int_{r<ct}\frac{g(y_1,y_2)}{\sqrt{(ct)^2-r^2}}\mathrm{d}y_1\mathrm{d}y_2+\frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int\int_{r<c}\frac{f(y_1,y_2)}{\sqrt{(ct)^2-r^2}}\mathrm{d}y_1\mathrm{d}y_2\right) \end{aligned}$$

Notice that the **domain of dependence is now the solid disk** of radius r. i.e. it is filled in. This is the **weak form of Huygen's principle**. This means that if you start with a region $B_\rho(x_0)$ of initial disturbance, the disturbance will continue forevoer (in the 3d case, we saw that the disturbance was isolated to a shell at radius $r=ct$ and of width 2ρ)

This is called the **Method of Descent:** we descend from the solution with $n=3$ to the solution for $n=2$.

7.33.7. Other tricky methods of descents. In the above section we used the special class of functions $\{u(x_1,x_2,x_3,t):\ u \text{ does not depend on } x_3\}$ to allow us to solve the 2d problem. In other words, we turned a 2D function $v(x_1,x_2,t)$ into a 3D function $u(x_1,x_2,x_3,t)$ with this special property by:

$$u(x_1,x_2,x_3,t):=v(x_1,x_2,t)\ \forall x_3$$

By considering other special classes of functions, we can solve other PDEs in lower dimesnions

EXAMPLE 7.7. How does one solve:

$$v_{tt}=c^2\left(v_{x_1x_1}+v_{x_2x_2}\right)-\lambda^2c^2v$$

Consider the special class of functions:

$$u(x_1,x_2,x_3,t)=e^{i\lambda x_3}v(x_1,x_2,t)$$

The $\Box u-c^2\Delta u=0 \iff v_{tt}=c^2\left(v_{x_1x_1}+v_{x_2x_2}\right)-\lambda^2c^2v$. So by solving u using our 3D solution, we get the solution for v. We have to translate the initial conditions:

$$u(x_1,x_2,x_3,0)=e^{i\lambda x_3}v\left(x_1,x_2,0\right)\ \ u_t(x_1,x_2,x_3,0)=e^{i\lambda x_3}v_t(x_1,x_2,t)$$

Parabolic Equations

These are notes from Chapter 7 of [**2**].

8.34. The Heat Equation

8.34.1. The initial-value problem. The heat equation is:

$$u_t = k\Delta u$$

with a positive constant k that is the conductivity coefficient. The characterstic surfaces $\phi(x,t) = t - \psi(x) = 0$ must satisfy:

$$\sum_{k=1}^{n} \psi_{x_i}^2 = 0$$

And hence the characteristic surfaces are $t = 0$.

Notice that the PDE is preserved under maps $x \to ax$ and $t \to a^2 t$, i.e. anything that leaves $|x|^2/t$ invariant.

Notice if u is a solution of the form:

$$u(x,t) = e^{i(\lambda t + c\cdot\xi)}$$

for some fixed ξ then, if plug into the PDE, we get that for this to solve the heat equation we must have $i\lambda = -|\xi|^2$ so the solution is:

$$u(x,t) = e^{ix\cdot\xi - |\xi|^2 t}$$

If we now are asked to solve the equation with initial condition:

$$\begin{aligned} u_t - \Delta u &= 0 \\ u(x,0) &= f(x) \end{aligned}$$

We can do so by superposition of the special solutions from before using the Fourier transform. Write:

$$f(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \hat{f}(\xi) \mathrm{d}\xi$$

So then we expect a solution of the form:

$$\begin{aligned} u(x,t) &= (2\pi)^{-n/2} \int e^{ix\cdot\xi - |\xi|^2} \hat{f(\xi)} \mathrm{d}\xi \\ &= (2\pi)^{-n/2} \int e^{ix\cdot\xi - |\xi|^2} \int (2\pi)^{-n/2} e^{-iy\cdot\xi} f(y) \mathrm{d}y \mathrm{d}\xi \\ &= \int K(x,y,t) f(y) \mathrm{d}y \end{aligned}$$

Where:

$$K(x,y,t) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi-|\xi|^2 t}\mathrm{d}\xi$$

This is the Fourier transform of a Gaussian random variable, and is easily evaluated by completing the square. We get:

$$K(x,y,t) = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t}\right)$$

PROPOSITION 8.15. *The kernal K above has:*
a) $K(x,y,t)\in C^\infty$
b) $\left(\frac{\partial}{\partial t}-\Delta_x\right)K(x,y,t)=0$ for $t>0$
c) $K(x,y,t)>0$ for $t>0$
d) $\int K(x,y,t)dy = 1$ for all $x\in\mathbb{R}^n$ and $t>0$
e) For any $\delta>0$ we have:

$$\lim_{t\to 0^+}\int\limits_{|y-x|>\delta} K(x,y,t)dy = 0 \text{ uniformly in } x$$

REMARK 8.21. The whole thing can be easily thought about if you notice that:

$$\int K(x,y,t)f(y)\mathrm{d}y = \mathbf{E}\left(f(X)\right)$$

where $X\sim N(x,\sqrt{2}t)$. (There is a factor of $\sqrt{2}$ that pops in snice the equation is $u_t = u_{xx}$ rather than $u_t = \frac{1}{2}u_{xx}$ which is more natural for Brownian motion/probability)

PROPOSITION 8.16. *Let $f(x)$ be continuous and bounded for $x\in\mathbb{R}^n$. Define for $t>0$. $u(x,t)=\int K(x,y,t)f(y)dy$. Then $u\in C^\infty$ for $t>0$ and satisfies $u_t=\Delta u$ for $t>0$. Moreover, $\lim_{t\to 0,x\to\xi}u(\xi,t)=f(x)$. This means that we can extended u continuously to the space $\{t=0\}\cup\{t>0\}$ and u solves the initial value problem for the heat equation there.*

PROOF. The only that needs checking is the fact that $\lim_{t\to 0,x\to\xi}u(\xi,t)=f(x)$. For any $\epsilon>0$ choose δ so small so that $|f(x)-f(y)|<\epsilon$ for all $|x-y|<\delta$ and then we have:

$$\begin{aligned}
|u(x,t)-f(\xi)| &= \left|\int K(x,y,t)\left(f(y)-f(\xi)\right)\mathrm{d}y\right| \\
&= \mathbf{E}\left[f(X)-f(\xi)\right] \text{ where } X\sim N(x,\sqrt{2}t) \\
&= \mathbf{E}\left[f(X)-f(\xi);|X-\xi|>\delta\right]+\mathbf{E}\left[f(X)-f(\xi);|X-\xi|\le\delta\right] \\
&\le 2\left\|f\right\|_\infty\mathbf{P}\left(|X-\xi|>\delta\right)+\epsilon \\
&\to 0+\epsilon \text{ as } x\to\xi, t\to 0
\end{aligned}$$

And the result follows. ☐

REMARK 8.22. The same thing works if $|f(x)|\le Me^{|x|^\beta}$ if $\beta<2$. (Divide up $\mathbf{E}(\cdot;|X-\xi|>\delta)$ into $\mathbf{E}(\ ;M>|X-\xi|>\delta)+\mathbf{E}(\ ;|X-\xi|>M)$. Since $\mathbf{E}(f(X))<\infty$ the second term $\to 0$ as $M\to\infty$, so can choose M large enough so that its $<\epsilon$. The first term then $\to 0$ as $x\to\xi$ and $t\to 0$.) Similarly if $|f(x)|\le Me^{a|x|^2}$ then the solution exists at least up to time $t=\frac{1}{4a}$ (Basically as long as $\mathbf{E}(|f(X)|)<\infty$ its ok!)

REMARK 8.23. Notice that information travels with **infinite speed** here, and depends on the value of f at all points. It also satisfies a maximum principle. You can check that is actually analytic, not just smooth.

REMARK 8.24. Notice that the solution above doesn't make sense if $\mathbf{E}(|f(X)|)$ is not finite. Indeed there is no uniqueness for the intial value problem with some additional assumption like this. We will see later on that in fact f non-negative is enough to make it work.

EXAMPLE 8.8. Choose $g(t)$ so that all derivatives of $g(t)$ are 0 at $t = 0$. For example $g(t) = \exp\left[-t^{-2}\right]$. Then define the power series $u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}$. This has $u(x, 0) = 0$ by the choice of g. Then check that actually this power series converges uniformly.

8.34.2. Maximum Principle, Uniqueness and Regularity. Let ω be an open bounded set in $\mathbb{R}^n$. For a fixed $T > 0$, we form the cylinder $\Omega \in \mathbb{R}^{n+1}$ with base ω and height T:

$$\Omega = \omega \times (0, T)$$

We divide the boundary into two pieces; the "walls" and the "end piece":

$$\begin{aligned} \partial^{(1)}\Omega : &= (\omega \times \{0\}) \cup (\partial\omega \times [0, T]) \\ \partial^{(2)}\Omega &:= \omega \times \{T\} \end{aligned}$$

THEOREM 8.45. *Let u be continuous in $\overline{\Omega}$ and $u_t, u_{x_i x_k}$ exist and be continuos in Ω and satisfy $u_t - \Delta u \leq 0$. Then:*

$$\max_{\overline{\Omega}} u = \max_{\partial^{(1)}\Omega} u$$

i.e. the maximum always occurs on the walls.

PROOF. Suppose first $u_t - \Delta u < 0$. Restrict attention to $\Omega_\epsilon = \omega \times (0, T - \epsilon)$ so that we have derivateive at the top end. $u_t - \Delta u < 0$ here means that interior local maxima are impossible (since these must have either $u_t = 0$ and $\Delta u \leq 0$ which contradicts $u_t - \Delta u < 0$) There can also not be a local maxima at at time $T - \epsilon$ (since these would have $u_t > 0$ and $\Delta u \leq 0$, again contradicting $u_t - \Delta u < 0$). Hence:

$$\max_{\overline{\Omega}_\epsilon} u = \max_{\partial^{(1)}\Omega_\epsilon} u \leq \max_{\partial^{(1)}\Omega} u$$

Since this hold for every ϵ and since u is continuous, we get $\max_{\overline{\Omega}} u = \lim_{\epsilon \to 0} \max_{\overline{\Omega}_\epsilon} u = u \leq \max_{\partial^{(1)}\Omega} u$ and the other inequality holds since $\partial^{(1)}\Omega \subset \overline{\Omega}$.

Now if $u_t - \Delta u \leq 0$ instead, introduce $v(x,t) = u(x,t) - kt$ so that $v_t - \Delta v = u_t - \Delta u - k < 0$. Then the maximum principle on k holds, and taking $k \to 0$ recovers our result. □

THEOREM 8.46. *(Uniqueness) If u is continuous in $\overline{\Omega}$ and u_t, $u_{x_i x_k}$ exist and are continuous in Ω, then u is uniquely detrmined in $\overline{\Omega}$ by the value of $u_t - \Delta u$ in Ω and the boundary $\partial^{(1)}\Omega$.*

PROOF. By linearity it suffices to show that the only solution with $u_t - \Delta u = 0$ and $u = 0$ on $\partial^{(1)}\Omega$ is the the $u \equiv 0$. By the maximum principle $\max u \leq 0$ and since $-u$ also has $u_t - \Delta u \leq 0$ we have $\min u \leq 0$ so indeed $u \equiv 0$. □

THEOREM 8.47. *Suppose that:*

$$\begin{aligned} u(x,t) &\leq Me^{a\left|x^2\right|} \\ u(x,0) &= f(x) \\ u_t - \Delta u &\leq 0 \end{aligned}$$

Then $u(x,t) \leq \sup_z f(z)$.

PROOF. Restrict attention to a ball of size ρ. Let $v_\mu = u(x,t) - \mu K(ix, it, T + \epsilon - t) \sim u(x,t) - \mu \exp\left((x-y)^2/T\right)$ this still satisfies the heat equation. On a ball of size ρ, $v_\mu$satisfies the maximum principle on the interior, and the exponential term makes v_μ small outside of the ball of radius ρ. Hence v_μ satisfies the maximum principle. Taking $\mu \to 0$ we get the maximum principle for u too. □

8.34.3. Boundary value problems. Use the reflection principle to solve these ones...go over your PDE for finance notes.

8.34.4. Non-negative solutions.

THEOREM 8.48. *For:*

$$\begin{aligned} u_t - u_{xx} &= 0 \\ u(x,0) &= f(x) \\ u(x,t) &\geq 0 \end{aligned}$$

The unique solution is given by:

$$u(x,t) = \int K(x,y,t) f(y) dy$$

PROOF. Again truncate to a bounded region by letting $\zeta^a(x) = 1$ for $|x| \leq a-1$ and $\zeta^a(x) = a - |x|$ for $a-1 < |x| < a$. Then define $v^a(x,t) = \int K(x,y,t)\zeta^a(y) f(y) \mathrm{d}y$ which will be our approximation to the solution u. Fix an a. Then take ρ large enough so that $v^a(x,t) < \epsilon \leq \epsilon + u(x,t)$ on $|x| = \rho$ and $v^a(x,0) \leq f(x) \leq \epsilon + u(x,0)$ for $|x| \leq \rho$. By the maximum principle $v^a(x,t) \leq \epsilon + u(x,t)$ everywhere in here. Then by taking $\rho \to \infty$ this works on all of $\mathbb{R}$.

Then define $v(x,t) = \lim_{a\to\infty} v^a(x,t) = \int K(x,y,t) f(y) \mathrm{d}y$ (ok by MCT) will have $v(x,t) \leq u(x,t)$ and then show $v - w \equiv 0$. □

8.35. Heat Equation (From Evans PDE book)

8.35.1. Duhamel's Principle. Suppose you want to solve the non-homogeonous heat equation:

$$\begin{aligned} u_t - \Delta u &= f(x,s) \\ u(x,0) &= 0 \end{aligned}$$

To do this put $u(x,t;s)$ to be the solution of $u_t - \Delta u = 0$ and initial value $u(x,s;s) = f(x,s)$ at time s. Then define:

$$u(x,t) = \int_0^t u(x,t;s) \mathrm{d}s$$

So that $u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\mathrm{d}s = f(x,t) + \int_0^t \Delta_x u(x,t;s)\mathrm{d}s = f(x,t) + \Delta_x\left(u(x,t)\right)$ so indeed it works. This is a sort of variation of parameteres for PDEs.

8.35.2. Mean-Value formula. For fixed $r > 0$ define:

$$E(x,t;r) := \left\{(y,s) \in \mathbb{R}^n \times [0,T] : \ s \leq t, K(x-y,t-s) \geq \frac{1}{r^n}\right\}$$

This is a set in space time whose boundary is the level set $K(x-y,t-s) = \frac{1}{r^n}$. It looks like an ellipse of sorts....for small time the set has points very near x (since $K(\cdot,t-s)$ is peaked sharply) then it expands for intermediate time, and then shrinks back down to nothing for large time (since $K(\cdot,t-s)$ will become so wide that its maximum is $\leq \frac{1}{r^n}$)

THEOREM 8.49. *(Mean value property for the Heat Equation) If u solves the heat equation then:*

$$u(x,t) = \frac{1}{4r^n} \int \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dy ds$$

PROOF. Define $\phi(r)$ to be the integral on the right, then check that $\phi'(r) = 0$ and $\lim_{r\to 0} \phi(r) = u(x,t)$. □

THEOREM 8.50. *Assume that u is C_1 and satisfies the heat eqaution in $U_T := U \times (0,T]$ and $\partial^{(1)} U_T$ is the parabolic boundary as defined in Fritz John. Then:*

$$\max_{\overline{U}_T} u = \max_{\partial^{(1)} U_T} u$$

and furthermore if there is an interior maximum (x_0,t_0) so $u(x_0,t_0) = \max_{\overline{U}_T} u$ then u is constant in $U \times (0,t_0)$.

PROOF. Suppose (x_0,t_0) is an interior maximum with $u(x_0,t_0) = \max_{\overline{U}_T} u =: M$. By the mean-value property we have:

$$\begin{aligned} M = u(x_0,t_0) &= \frac{1}{4r^n} \int \int_{E(x_0,t_0;r)} u(y,s) \frac{|x_0-y|^2}{(t_0-s)^2} \mathrm{d}y\mathrm{d}s \\ &\leq \frac{1}{4r^n} \int \int_{E(x_0,t_0;r)} M \frac{|x_0-y|^2}{(t_0-s)^2} \mathrm{d}y\mathrm{d}s \\ &= M \end{aligned}$$

Hence the inequality above is actually an equality so $u(y,s) = M$ a.e. □

THEOREM 8.51. *(Uniqueness in a bounded domain)*
By the maximum property $u_t - \Delta u = f$ in U_T, $u = g$ on $\partial^{(1)} U_T$ has a unique solution.

8.35.3. Energy Methods. Assume $u(x,t) = 0$ on the boundary $\partial^{(1)} U_T$. Define the energy of the solution at time t in a bounded domain by:

$$e(t) = \int_U u(x,t)^2 \mathrm{d}x$$

Check that:

$$\begin{aligned}
\frac{\partial}{\partial t}e(t) &= 2\int_U uu_t \mathrm{d}x \\
&= 2\int_U u\Delta u \mathrm{d}x \\
&= 2\int_U \nabla\cdot(u\nabla u) - |\nabla u|^2 \,\mathrm{d}x \\
&= 0 - 2\int_U |\nabla u|^2 \,\mathrm{d}x \text{ by div. thm and since } u = 0 \text{ on } \partial^{(1)}U_T \\
&\leq 0
\end{aligned}$$

So the energy is decreasing. Hence for the initial condition $u(x, 0) = 0$ the unique solution is 0.

8.35.4. Backward Heat Equation. More playing around with the energy functional can show that there is a unique solution to the backward heat equation too. However, the problem is not well posed since tiny perturbations in the inital data lead to large perturbations later (look at the Fourier series)

Bibliography

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