Non-intersecting random processes and multi-layer random polymers

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Courant Institute, NYU

April 6, 2017

Papers in this thesis

J. Funk, M. Nica, and M. Noyes.
Stabilization time for a type of evolution on binary strings.
J. Theoretical Probab., 28:848–865, 2015.

M. Nica.
Decorated Young tableaux and the Poissonized Robinson Schensted process.
Stoch. Proc. Appl., 127:449–474, 2017.

I. Corwin and M. Nica.
Intermediate disorder directed polymers and the multi-layer extension of the stochastic heat equation.
Electron. J. Probab., 22:1–49, 2017.

M. Nica.
Intermediate disorder limits for multi-layer semi-discrete directed polymers.
arXiv:1609.00298, September 2016, 46 pages. Submitted.

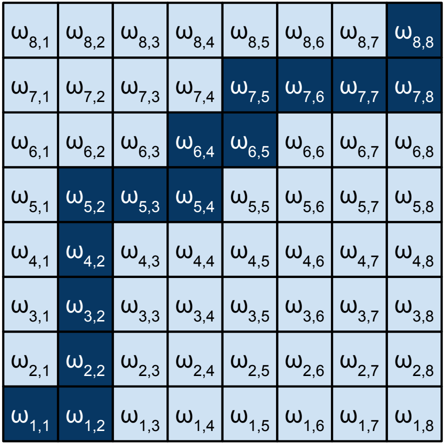
Table of Contents

Introduction - Last Passage Percolation

$\omega_{8,1}$	$\omega_{8,2}$	$\omega_{8,3}$	$\omega_{8,4}$	$\omega_{8,5}$	$\omega_{8,6}$	$\omega_{8,7}$	$\omega_{8,8}$
$\omega_{7,1}$	$\omega_{7,2}$	$\omega_{7,3}$	$\omega_{7,4}$	$\omega_{7,5}$	$\omega_{7,6}$	$\omega_{7,7}$	$\omega_{7,8}$
$\omega_{6,1}$	$\omega_{6,2}$	$\omega_{6,3}$	$\omega_{6,4}$	$\omega_{6,5}$	$\omega_{6,6}$	$\omega_{6,7}$	$\omega_{6,8}$
$\omega_{5,1}$	$\omega_{5,2}$	$\omega_{5,3}$	$\omega_{5,4}$	$\omega_{5,5}$	$\omega_{5,6}$	$\omega_{5,7}$	$\omega_{5,8}$
$\omega_{4,1}$	$\omega_{4,2}$	$\omega_{4,3}$	$\omega_{4,4}$	$\omega_{4,5}$	$\omega_{4,6}$	$\omega_{4,7}$	$\omega_{4,8}$
$\omega_{3,1}$	$\omega_{3,2}$	$\omega_{3,3}$	$\omega_{3,4}$	$\omega_{3,5}$	$\omega_{3,6}$	$\omega_{3,7}$	$\omega_{3,8}$
$\omega_{2,1}$	$\omega_{2,2}$	$\omega_{2,3}$	$\omega_{2,4}$	$\omega_{2,5}$	$\omega_{2,6}$	$\omega_{2,7}$	$\omega_{2,8}$
$\omega_{1,1}$	$\omega_{1,2}$	$\omega_{1,3}$	$\omega_{1,4}$	$\omega_{1,5}$	$\omega_{1,6}$	$\omega_{1,7}$	$\omega_{1,8}$

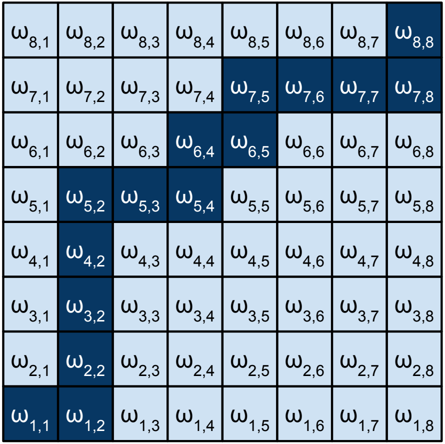
An array of IID random variables

$$\{\omega_{i,j}\}$$



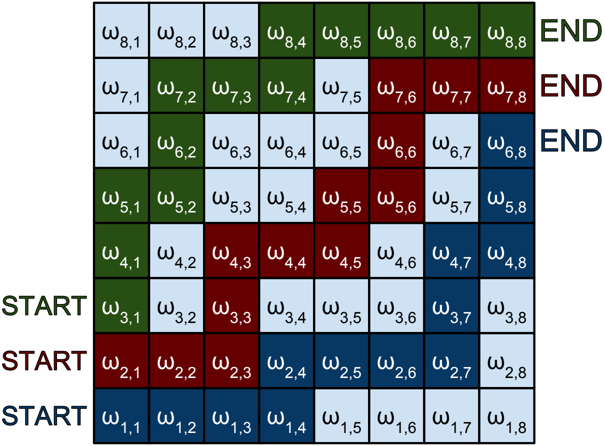
Last Passage Percolation:

$$L = \max_{X \text{ an up-right path}} \left\{ \sum_{t=1}^{2N} \omega_{X(t)} \right\}$$



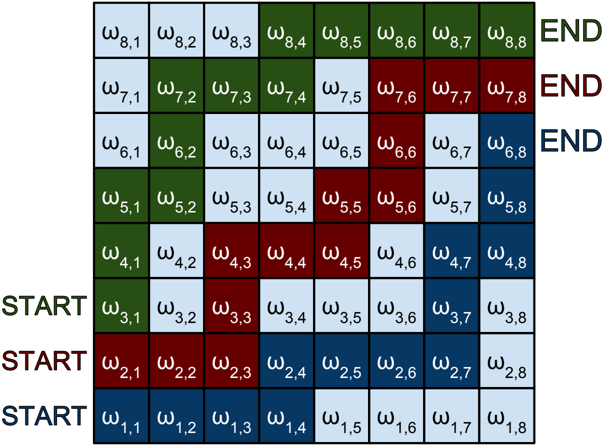
(Generalized) Last Passage Percolation:

$$L_1 = \max_{X \text{ an up-right path}} \left\{ \sum_{t=1}^{2N} \omega_{X(t)} \right\}$$



(Generalized) Last Passage Percolation Problem:

$$L_d = \max_{X_1,\ldots,X_d \text{ non-intersecting paths}} \left\{ \sum_{i=1}^{d} \sum_{t=1}^{2N} \omega_{X_i(t)} \right\}$$



(Generalized) Last Passage Percolation:

$$L_d = \max_{X_1,\ldots,X_d \text{ non-intersecting paths}} \left\{ \sum_{i=1}^{d} \sum_{t=1}^{2N} \omega_{X_i(t)} \right\}, \quad \lambda_d := L_d - L_{d-1}$$

The collection $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_N$ form a Young diagram (definition on board!).

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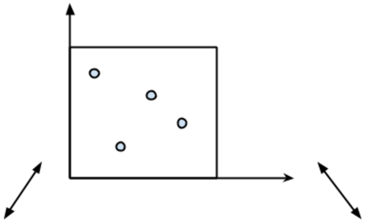
Theorem (Baik-Deift-Johansson '99)

If weights ω_{ij} come from a uniform random permutation $\sigma \in S_N$ as $\omega_{ij} = 1\{\sigma_i = j\}$ then:

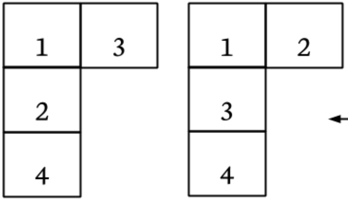
$$\lim_{N\to\infty} \mathbf{P}\left(\frac{\lambda_1 - 2\sqrt{N}}{N^{1/6}} \leq x\right) = F_{GUE}(x)$$

Decorated Young Tableaux and Non-Intersecting Poisson Arches

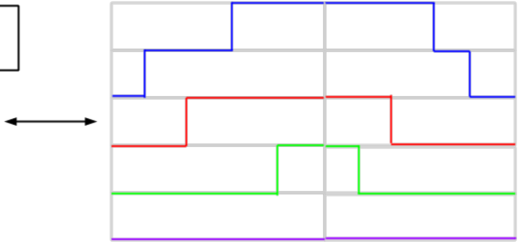
Configurations of points



Pairs of “decorated” Young Tableaux of the same shape

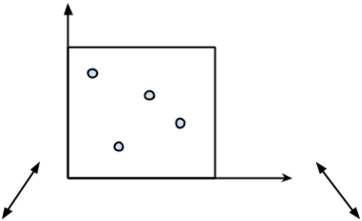


Non-intersecting line ensembles



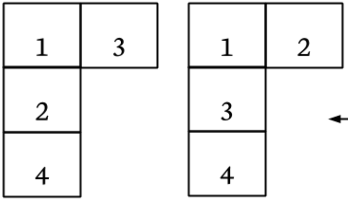
(x_1, x_2, x_3, x_4) (y_1, y_2, y_3, y_4)

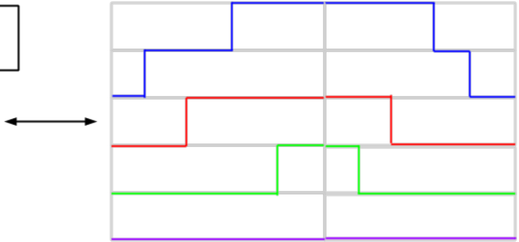
Poisson Point Process



Poissonized Robinson-Schensted Tableaux

Poisson "Arches" (conditioned on non-intersection)

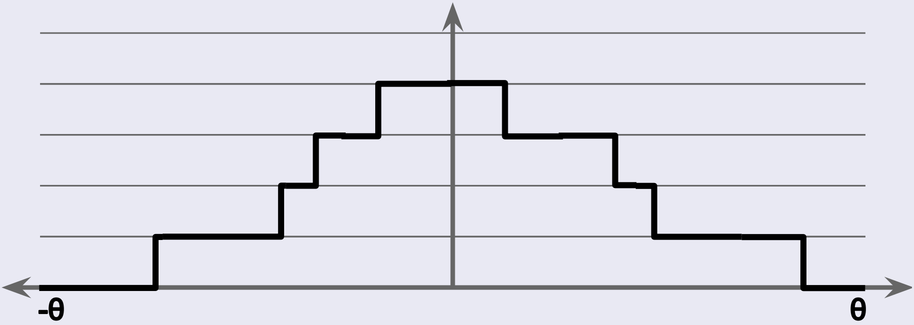




(x_1, x_2, x_3, x_4) (y_1, y_2, y_3, y_4)

Definition

A Poisson arch of parameter $\theta > 0$ is a random process on the interval $[-\theta, \theta]$ in continous time and discrete space:



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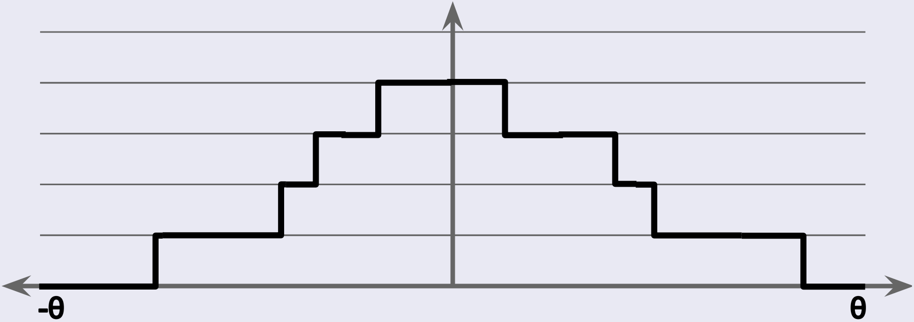

- Construct a Poisson process (upsteps) at rate 1 for $t \in (-\theta, 0)$

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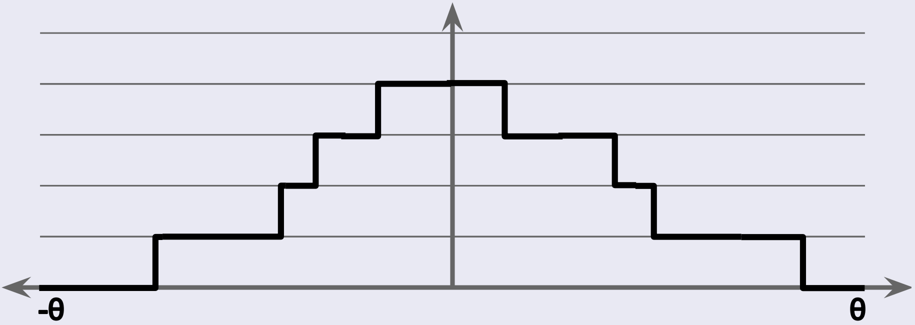
A Poisson arch of parameter $\theta > 0$ is a random process on the interval $[-\theta, \theta]$ in continous time and discrete space:



- Construct a Poisson process (upsteps) at rate 1 for $t \in (-\theta, 0)$
- Construct a Poisson process (downsteps) at rate 1 for $t \in (0, \theta)$

Definition

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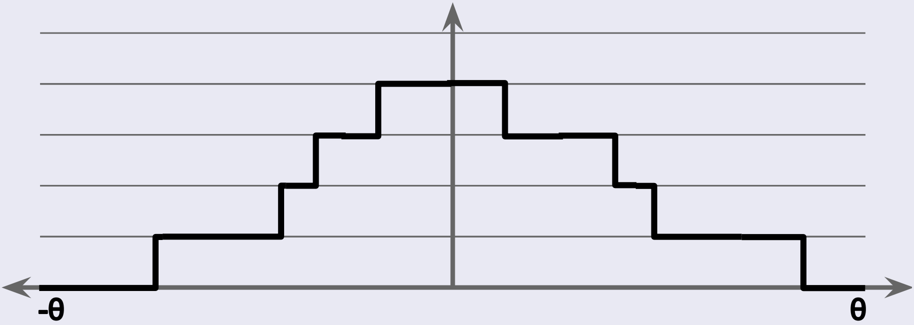
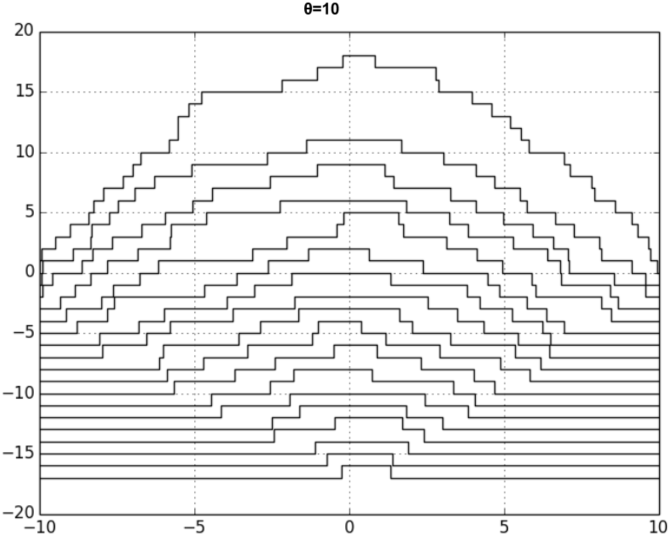
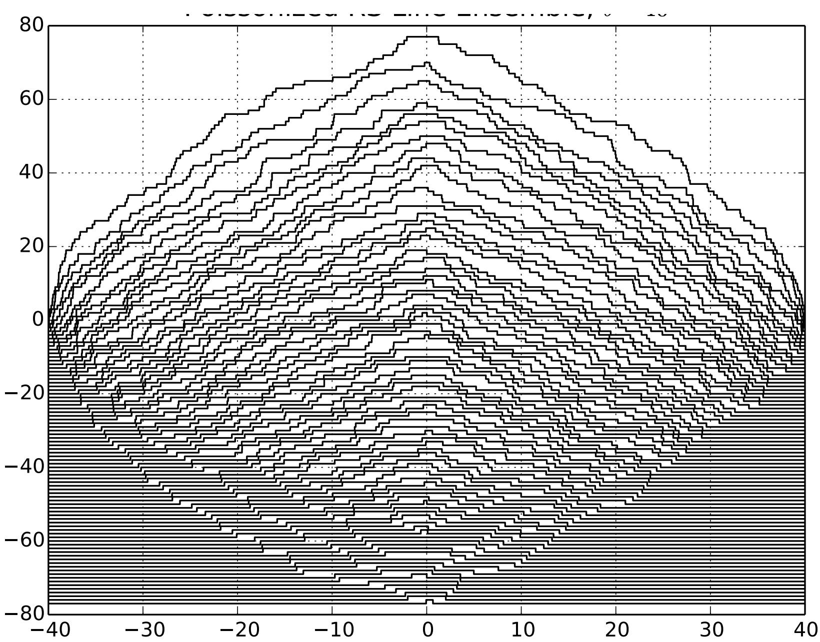


- Construct a Poisson process (upsteps) at rate 1 for $t \in (-\theta, 0)$
- Construct a Poisson process (downsteps) at rate 1 for $t \in (0, \theta)$
- Condition on the number of jumps in $(-\theta, 0)$ and $(0, \theta)$ to be equal.

Non-intersecting Poisson Arches ($\theta = 10$):

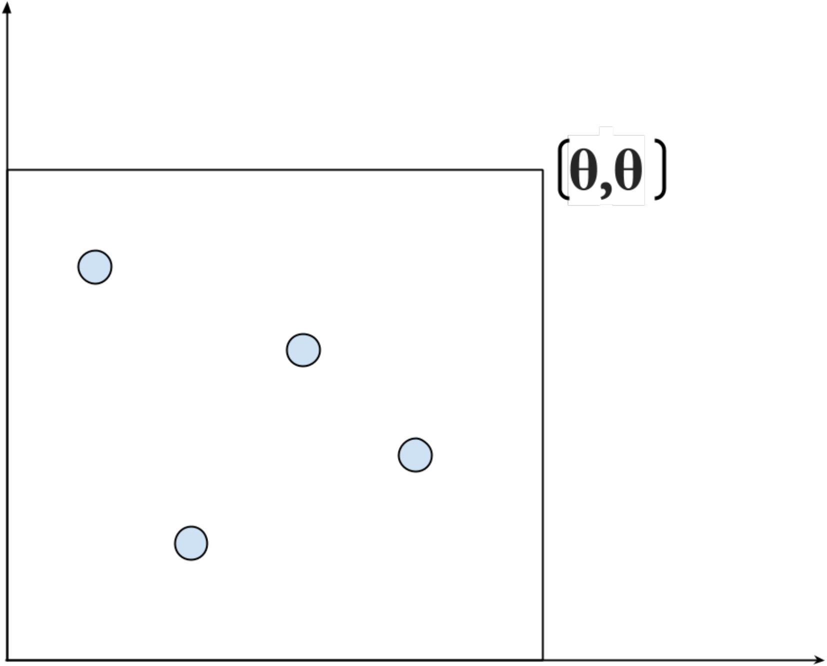


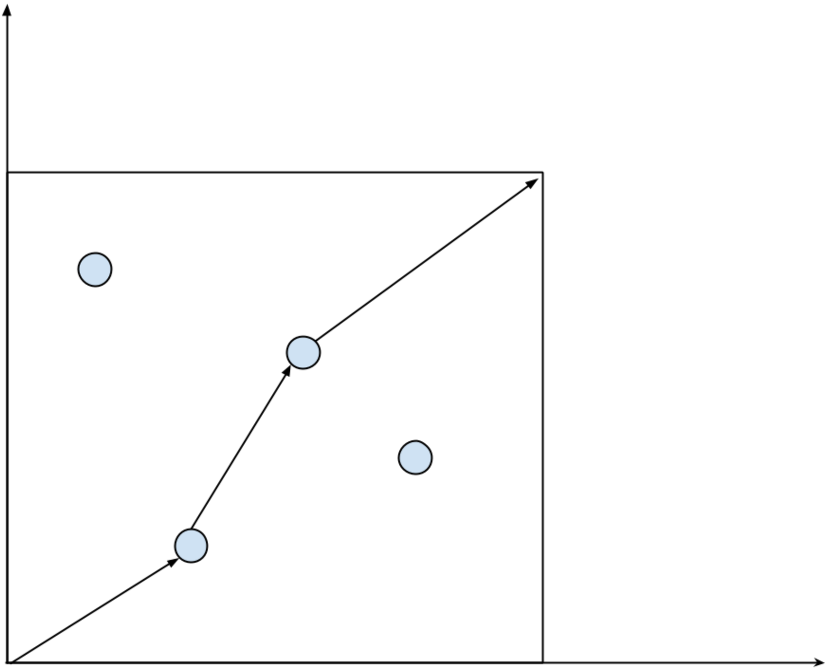
Non-intersecting Poisson Arches ($\theta = 40$):

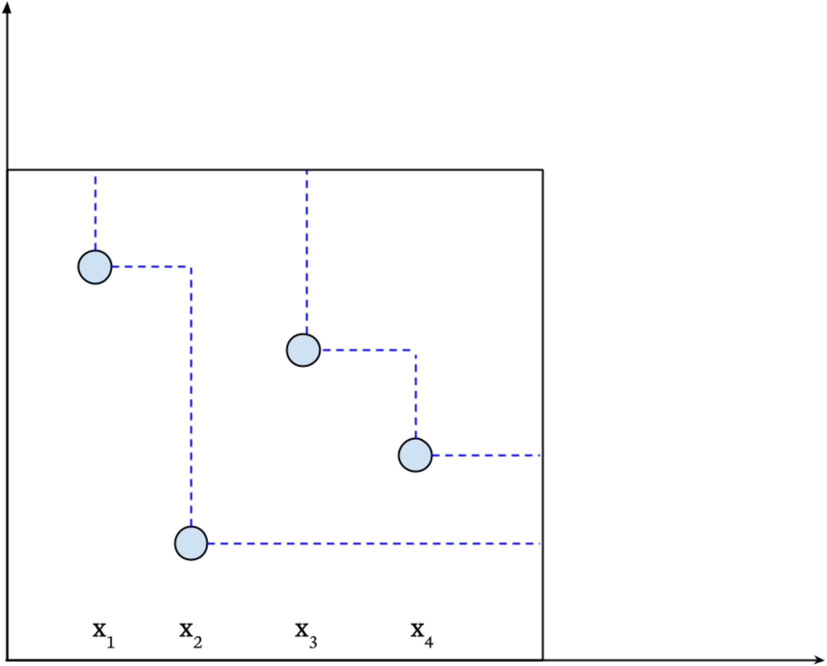


How is this related to Last Passage Percolation?

How is this related to Last Passage Percolation? I present the "shadow line" graphical construction.


(θ,θ)




$\mathbf{x}_1$
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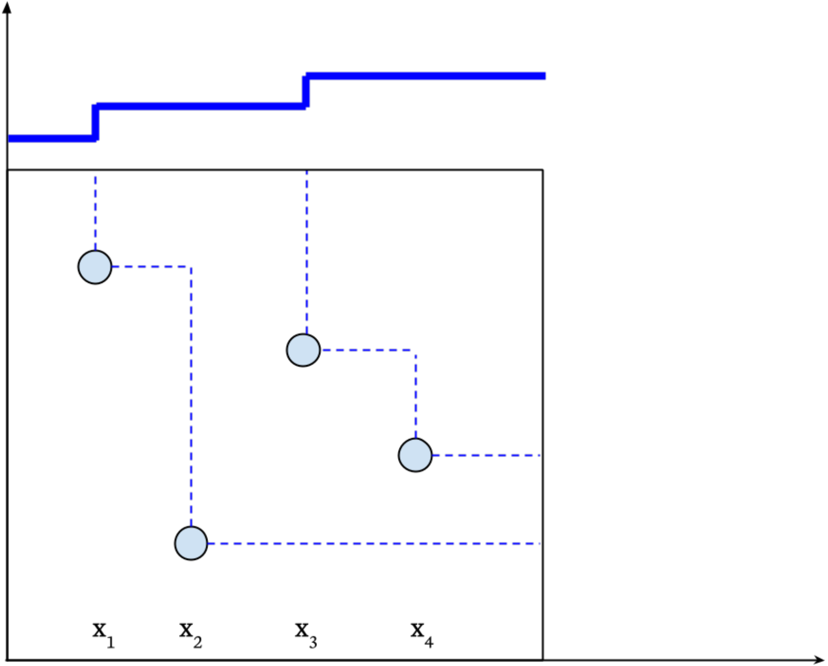
$\mathbf{x}_3$
$\mathbf{x}_4$



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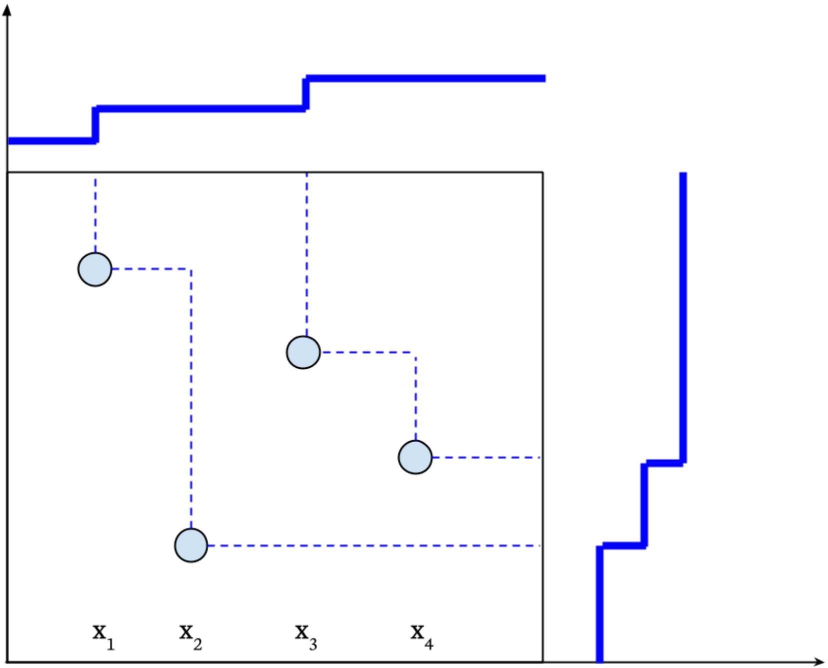

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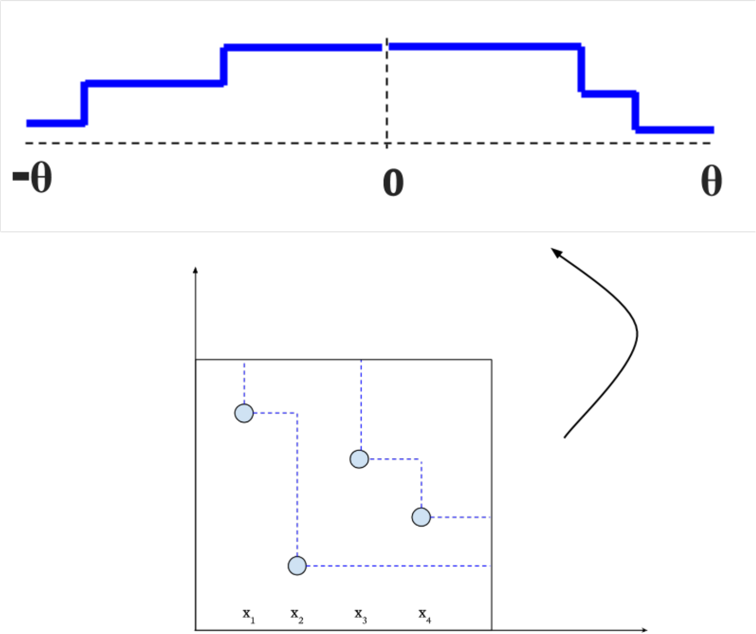

-θ
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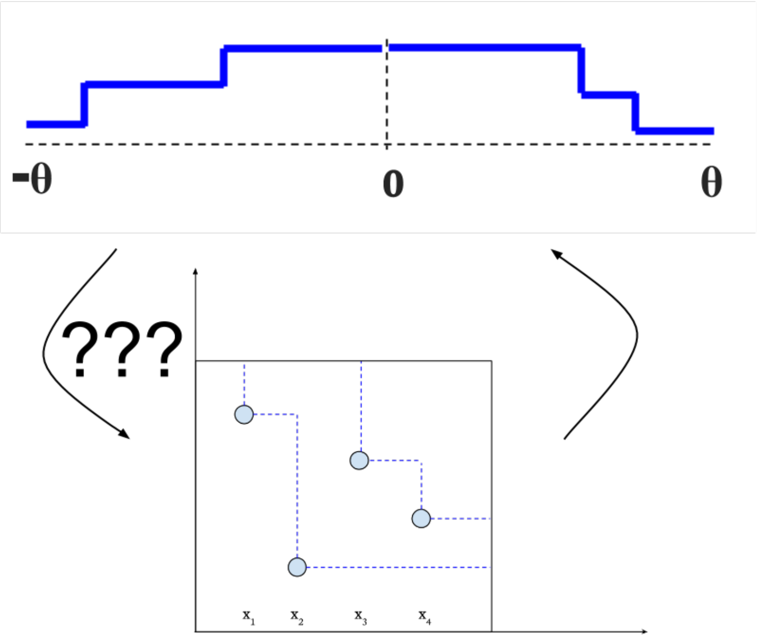

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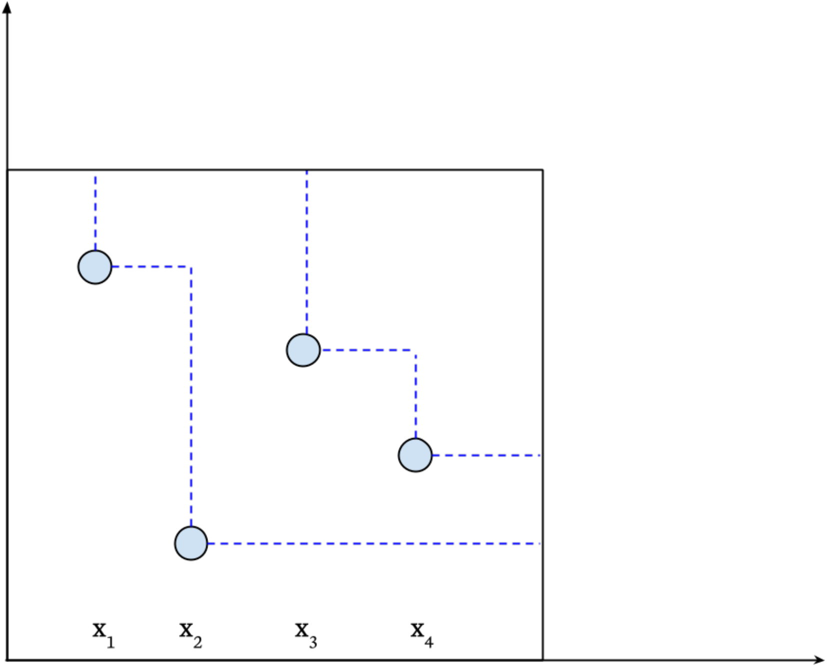

$\mathbf{x}_1$
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$\mathbf{x}_3$
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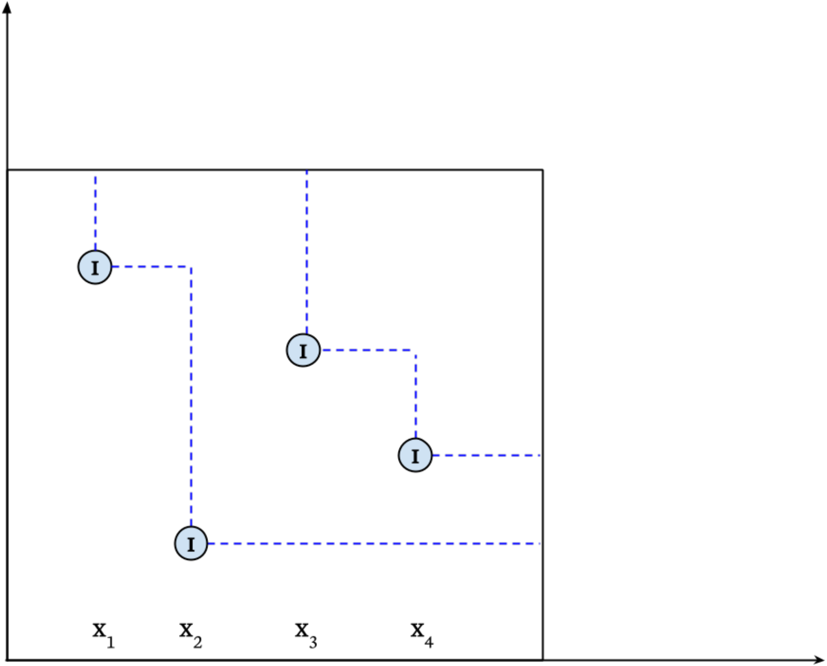
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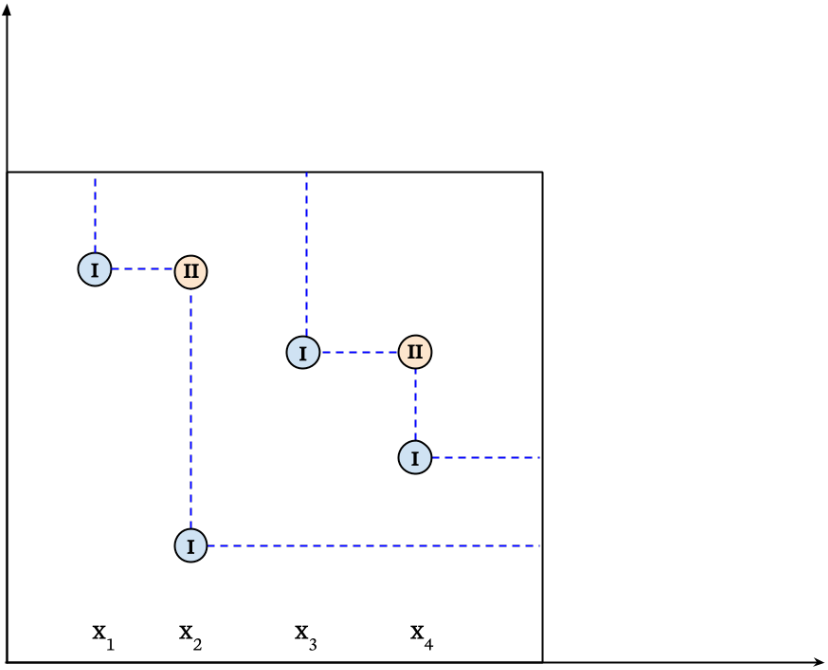
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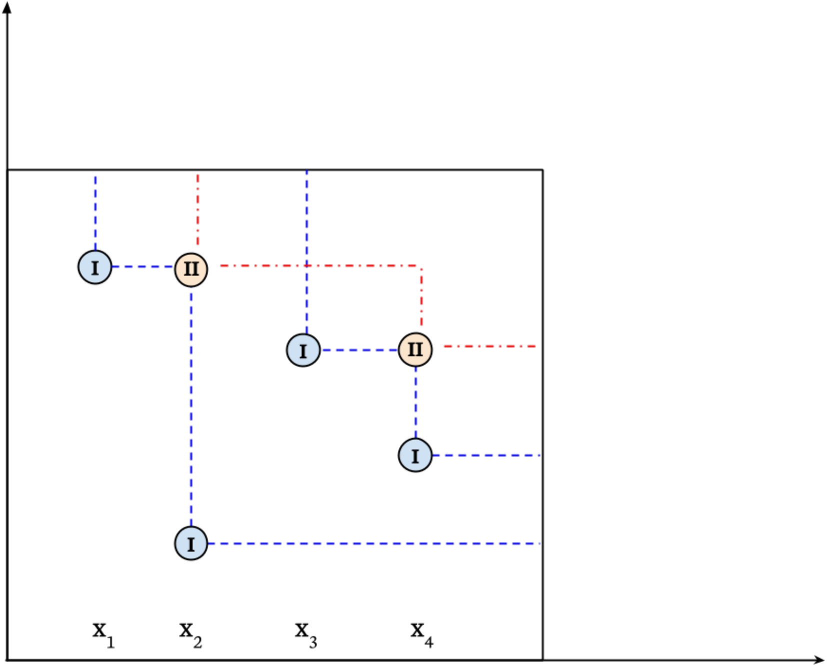
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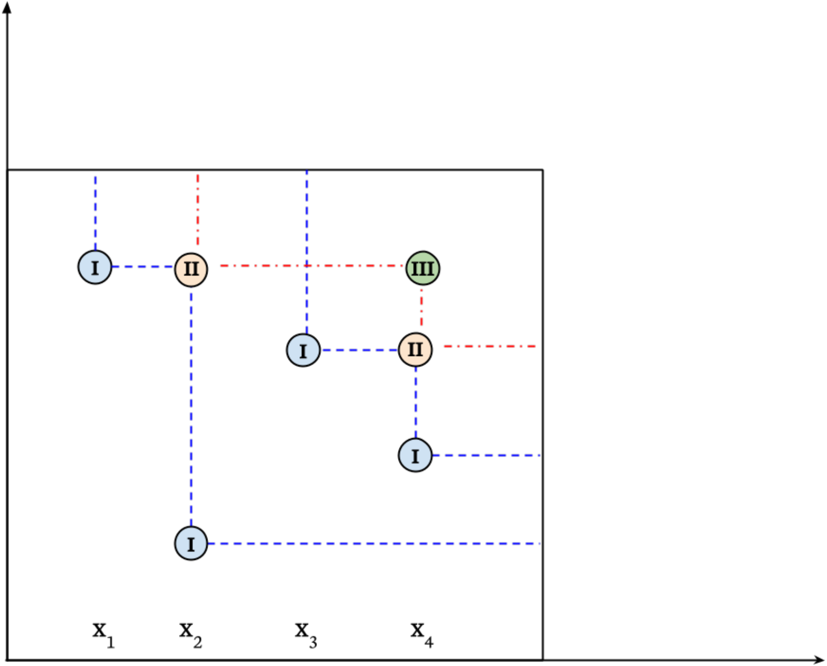
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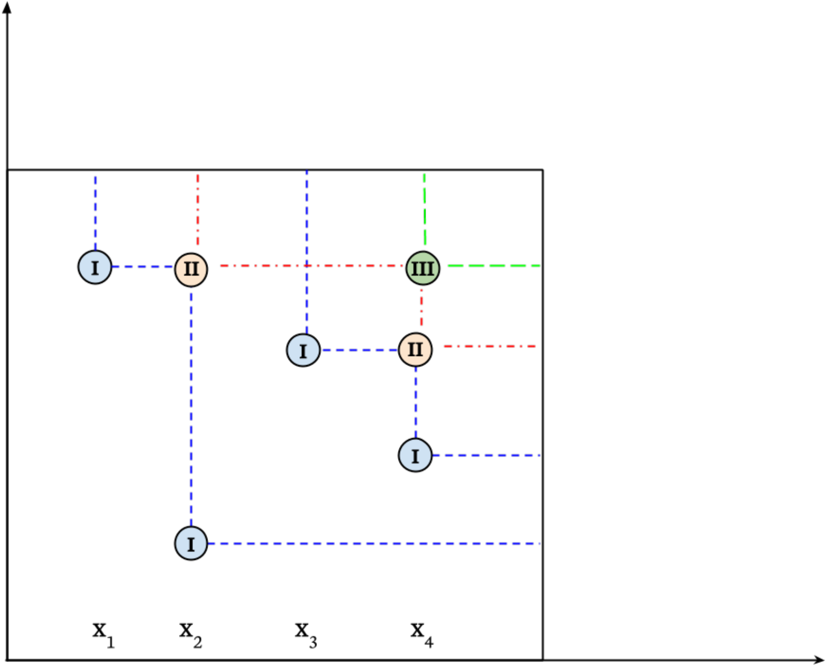
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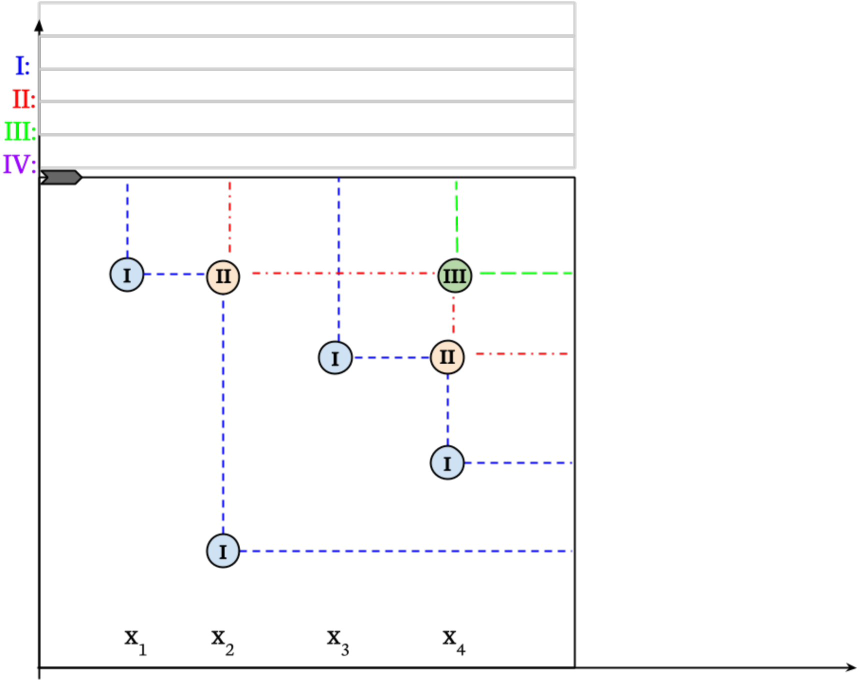

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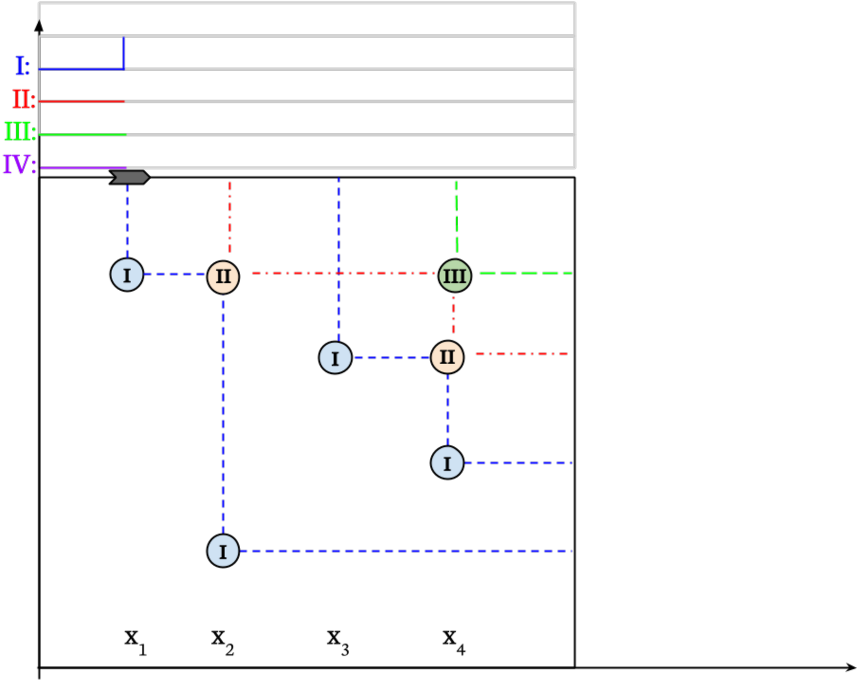
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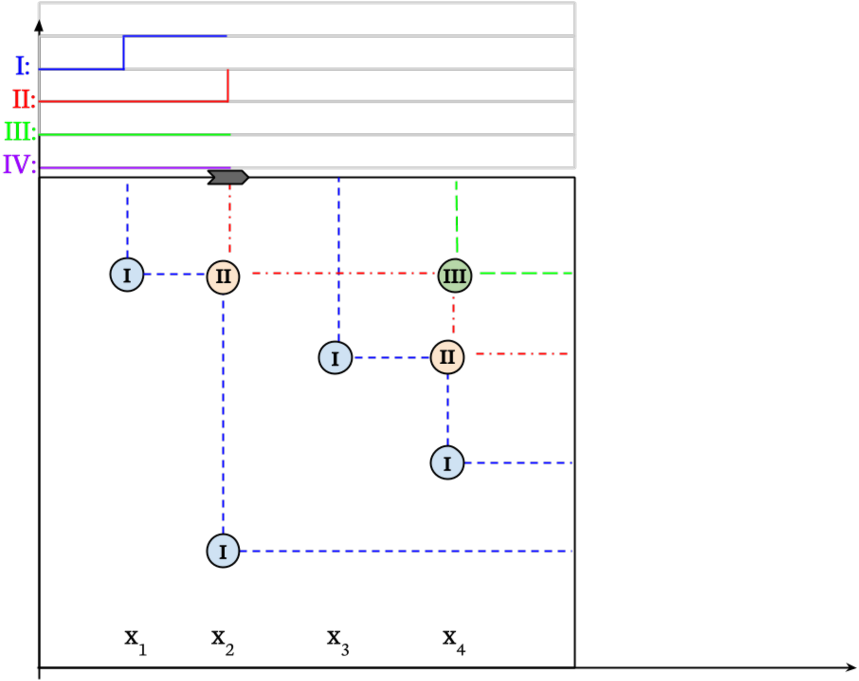
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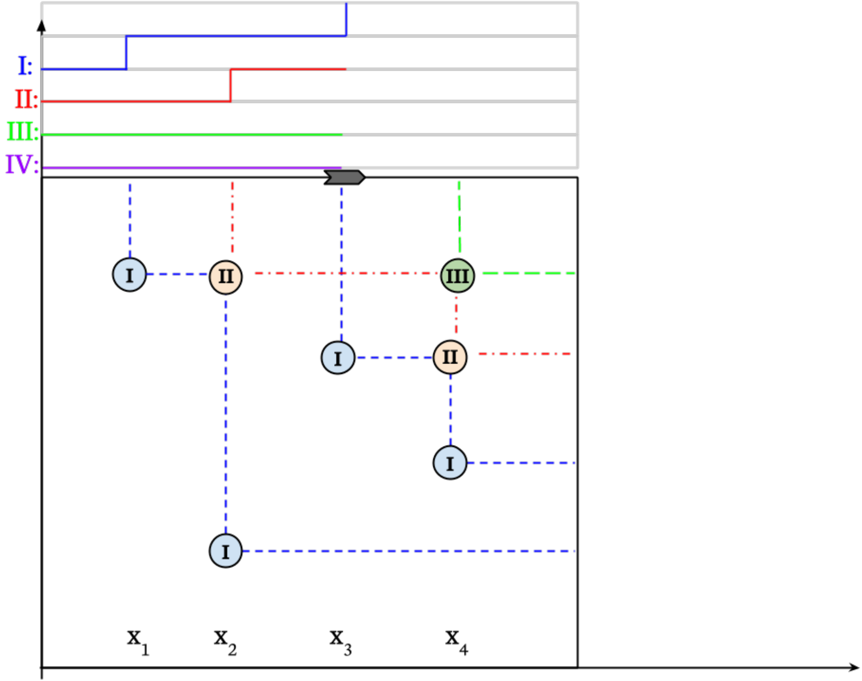

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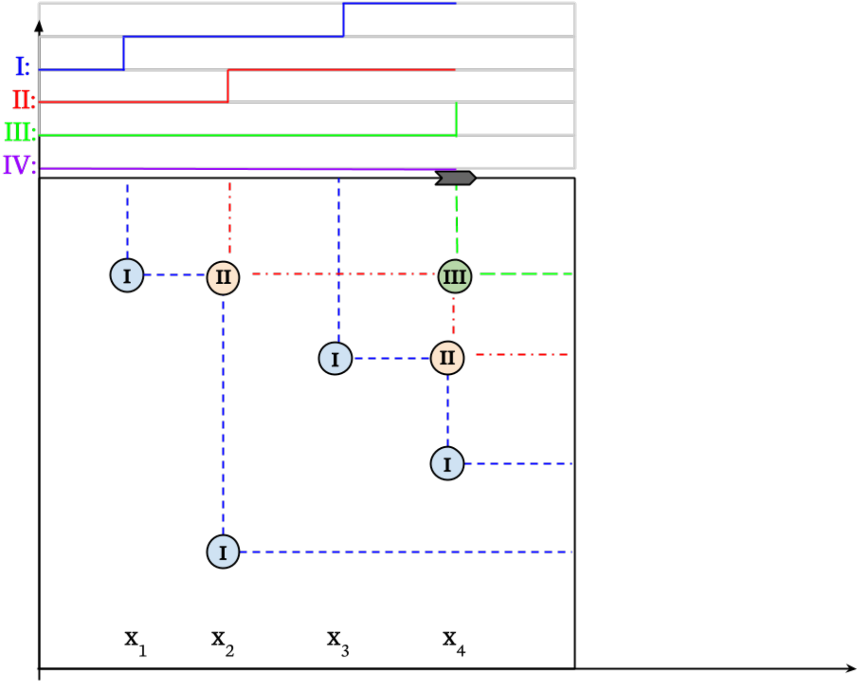
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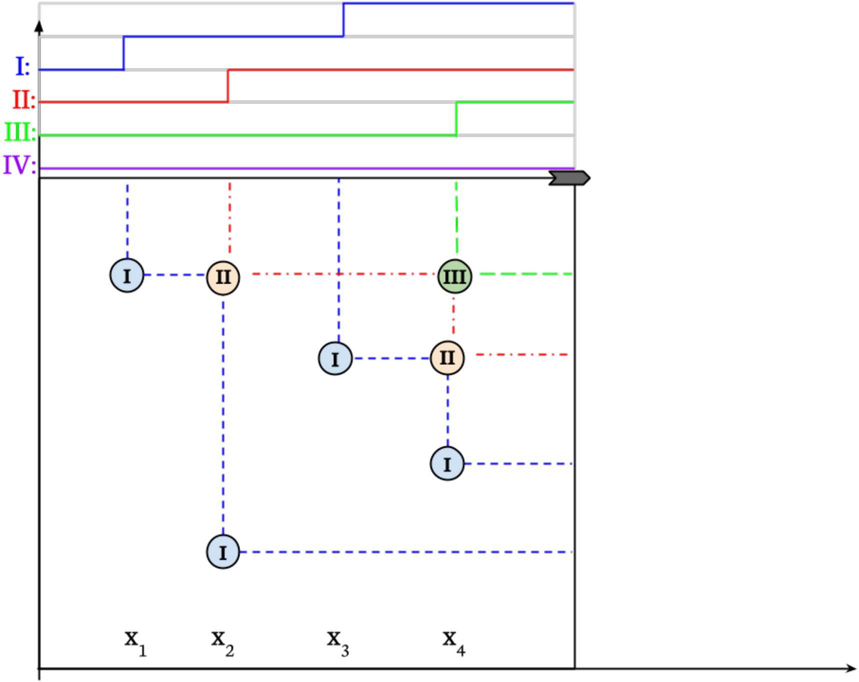
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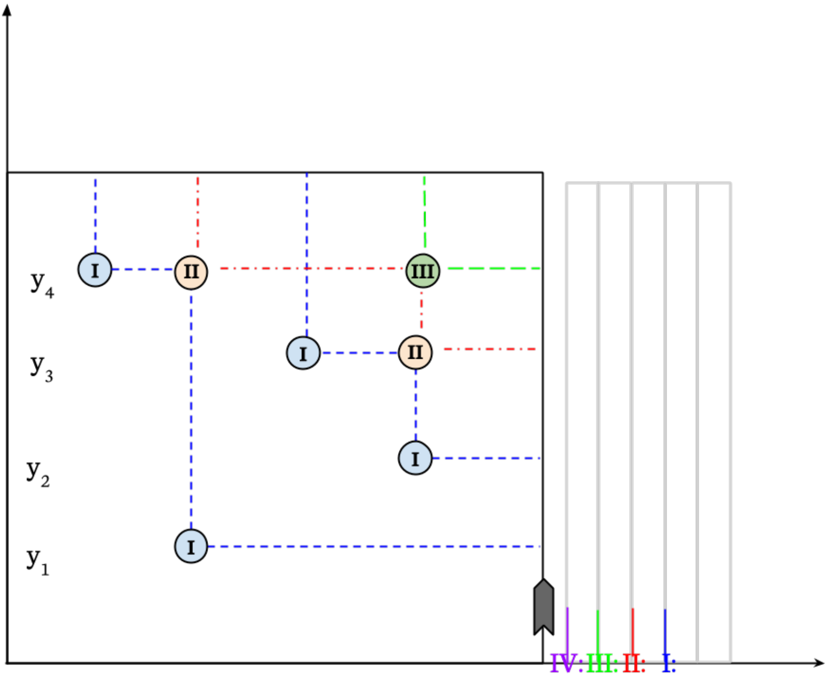

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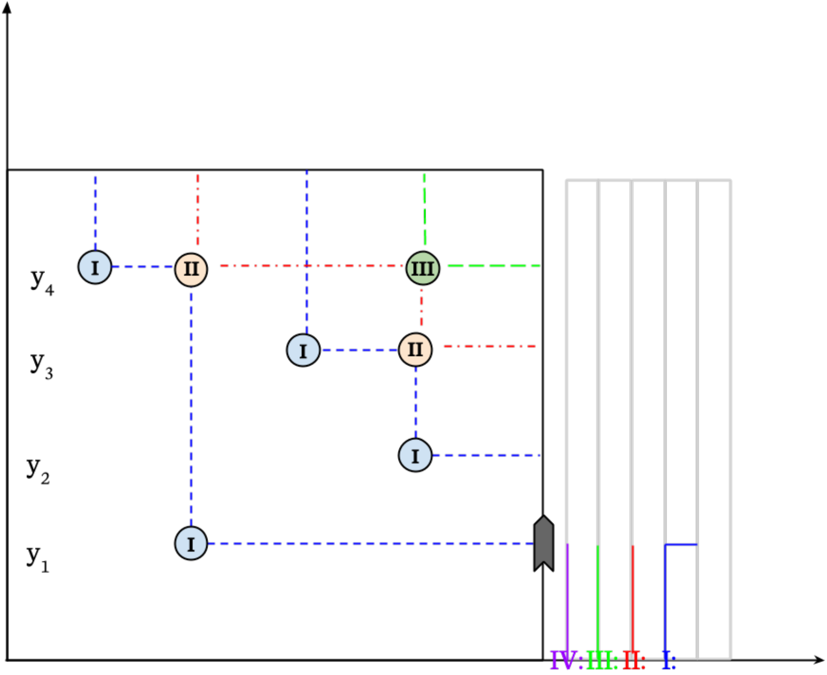
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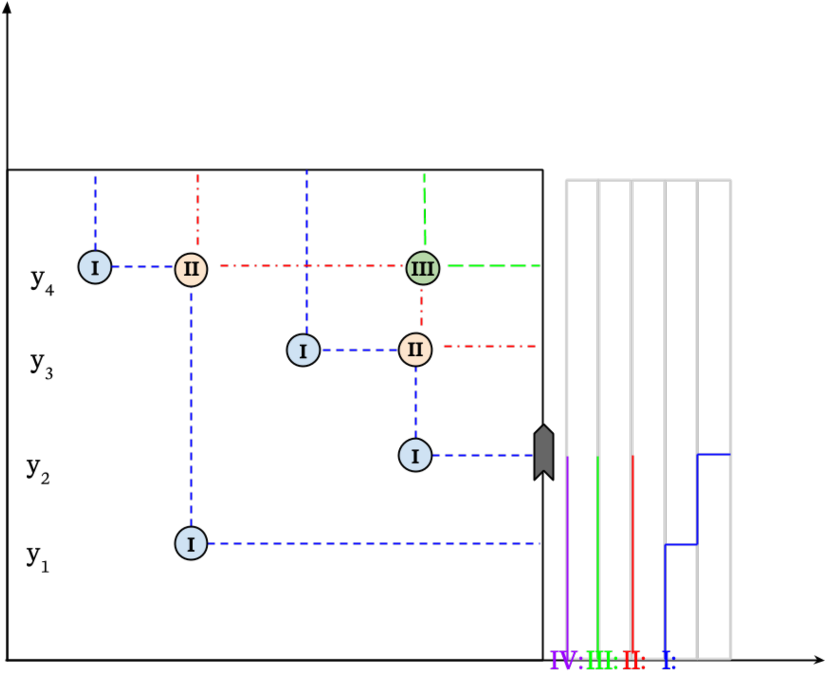
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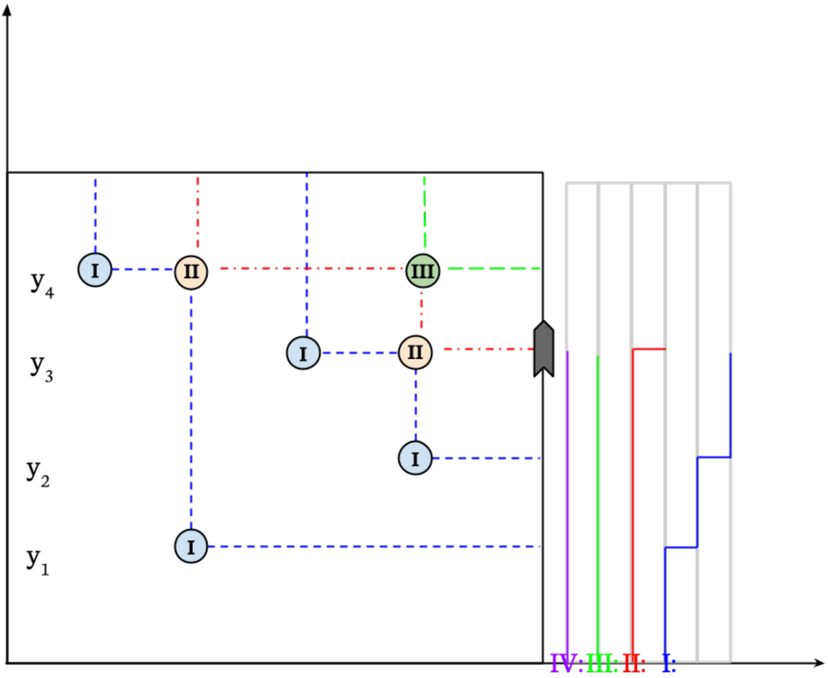
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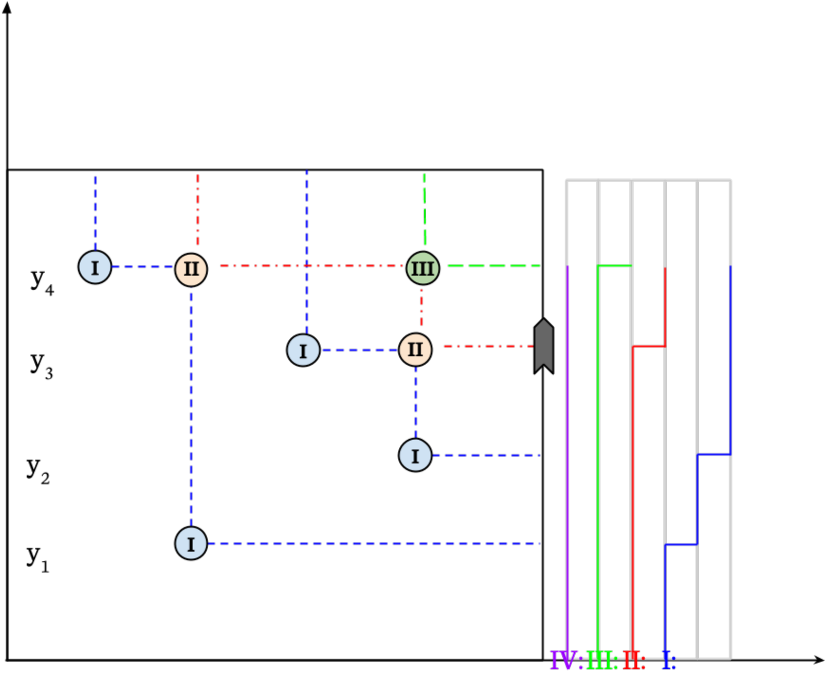

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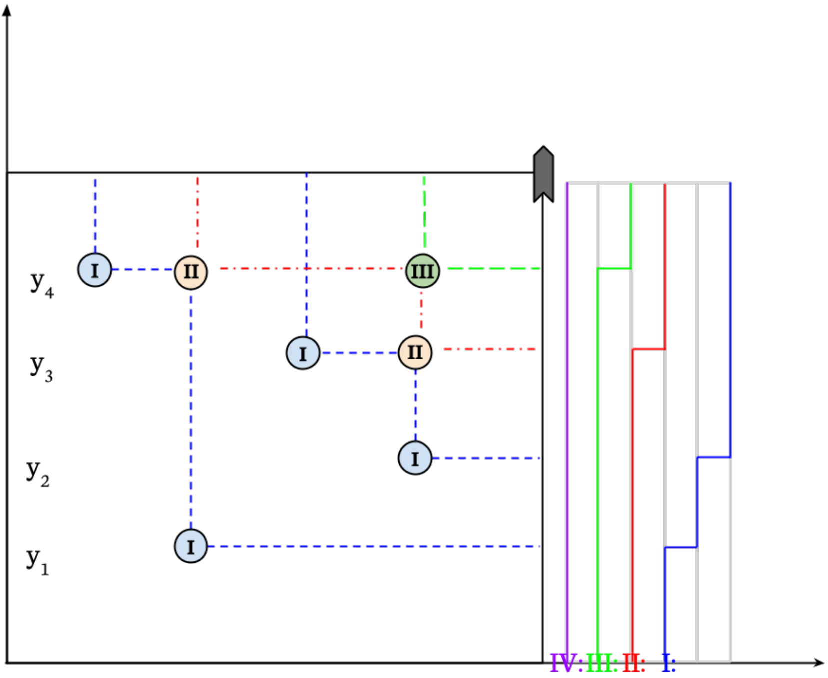

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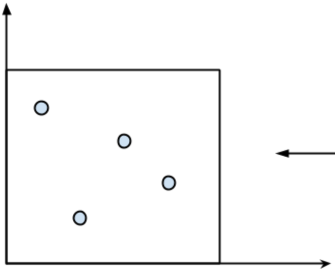
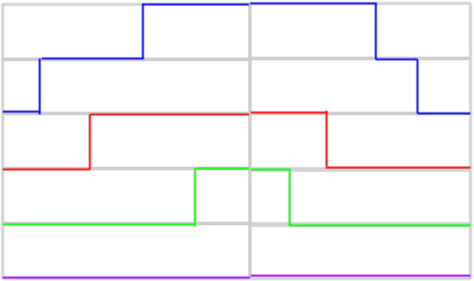
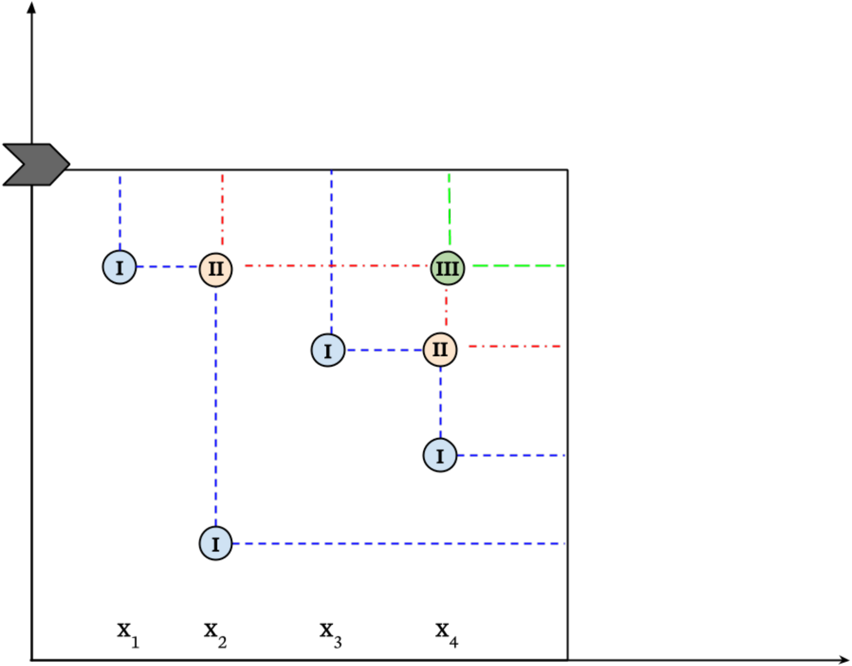
Robinson Schensted (Knuth) Bijection

Configurations of points ⟷ Non-intersecting line ensembles



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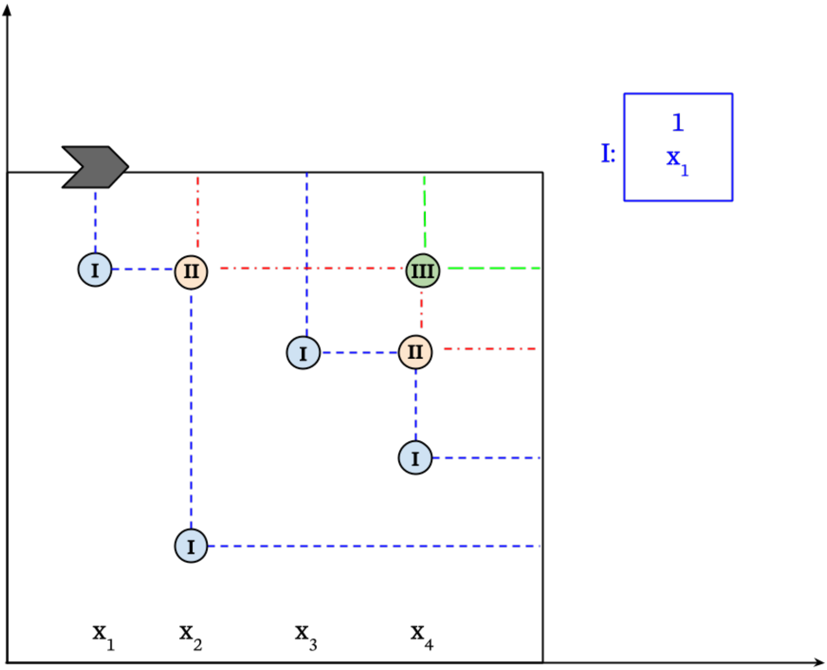
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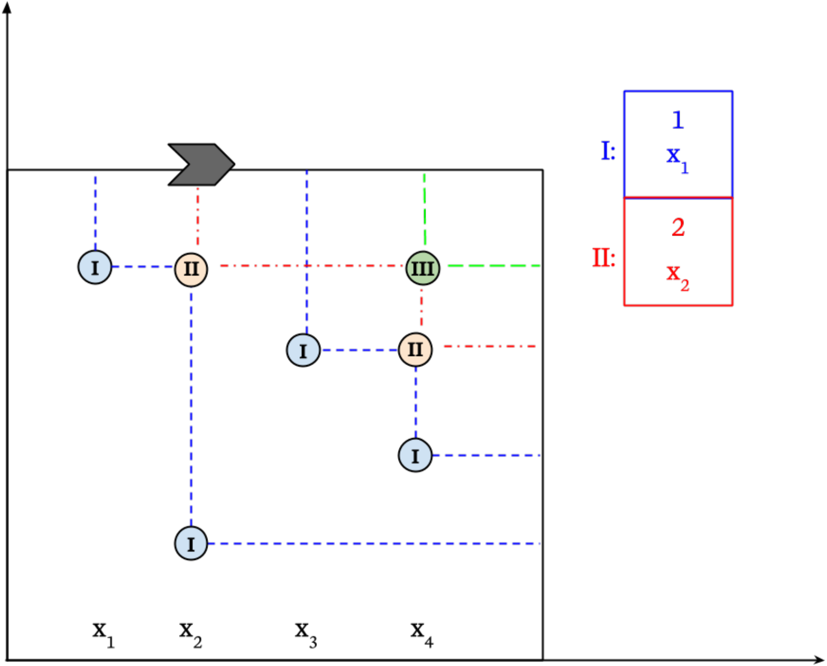
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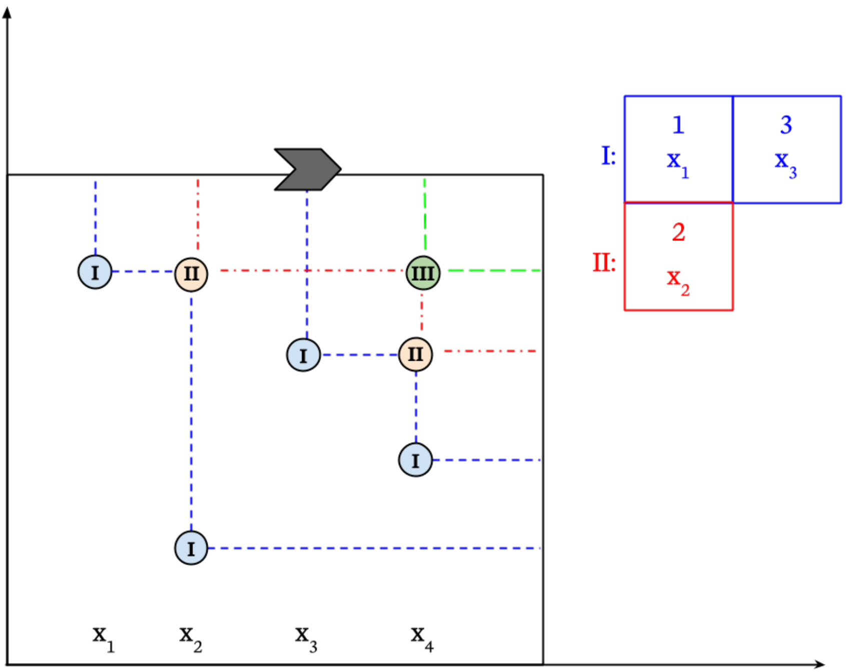

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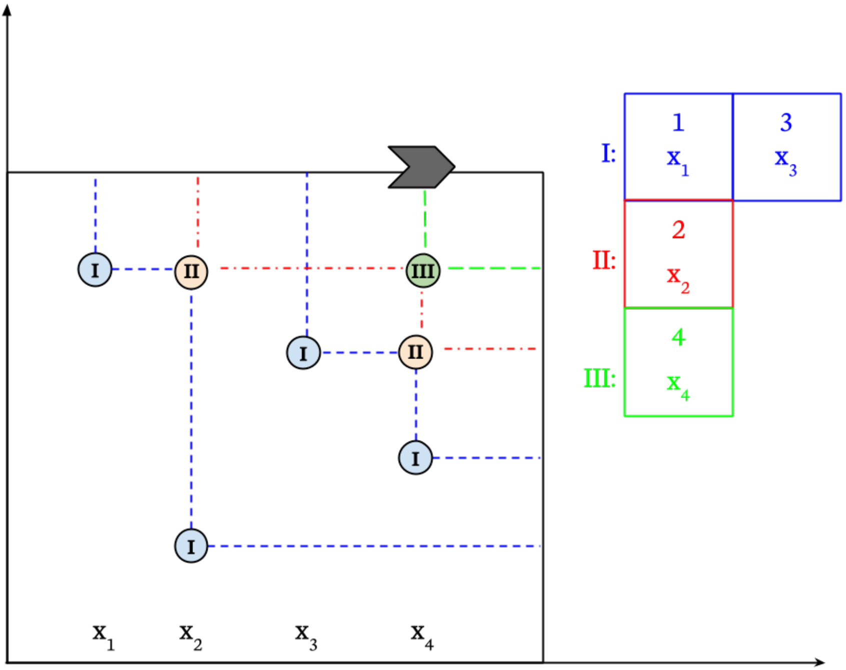
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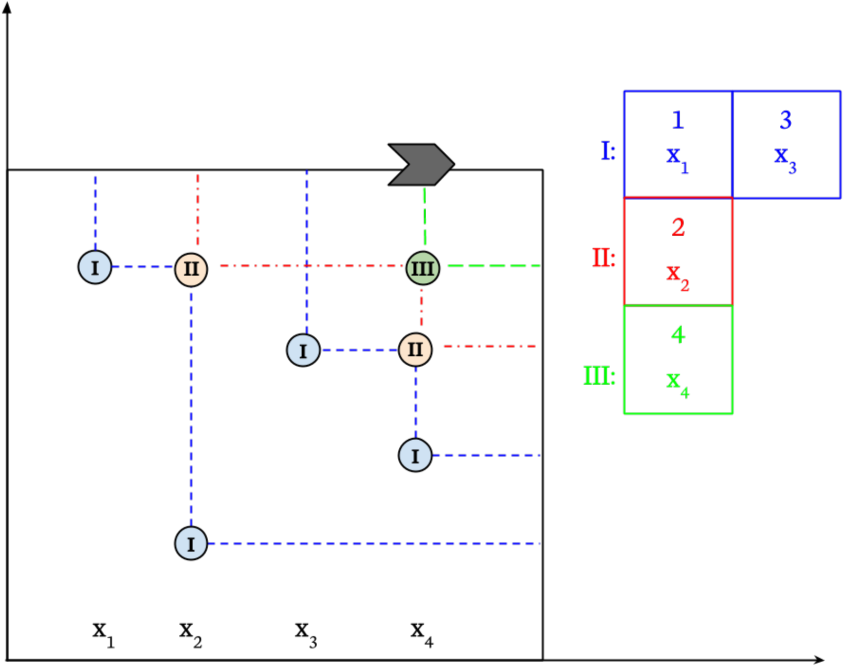
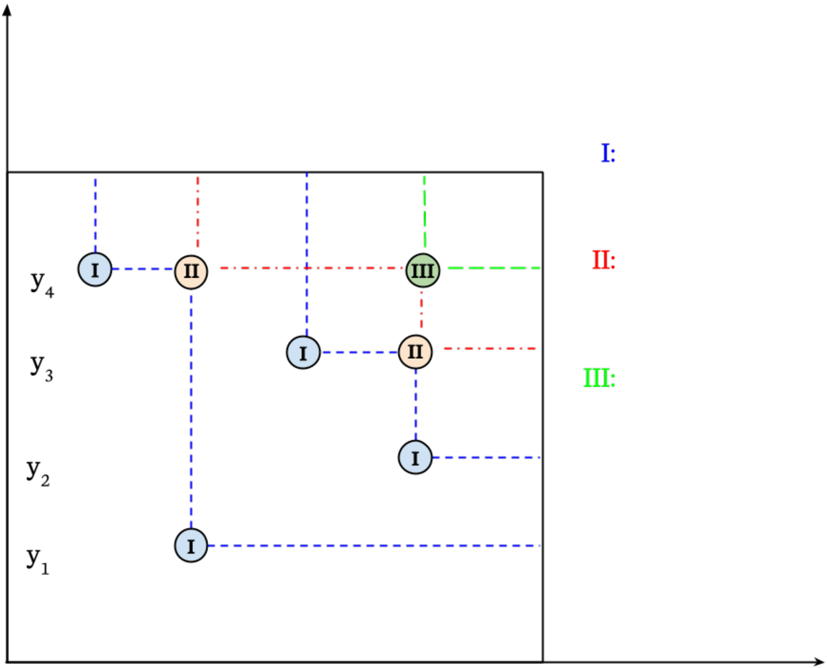
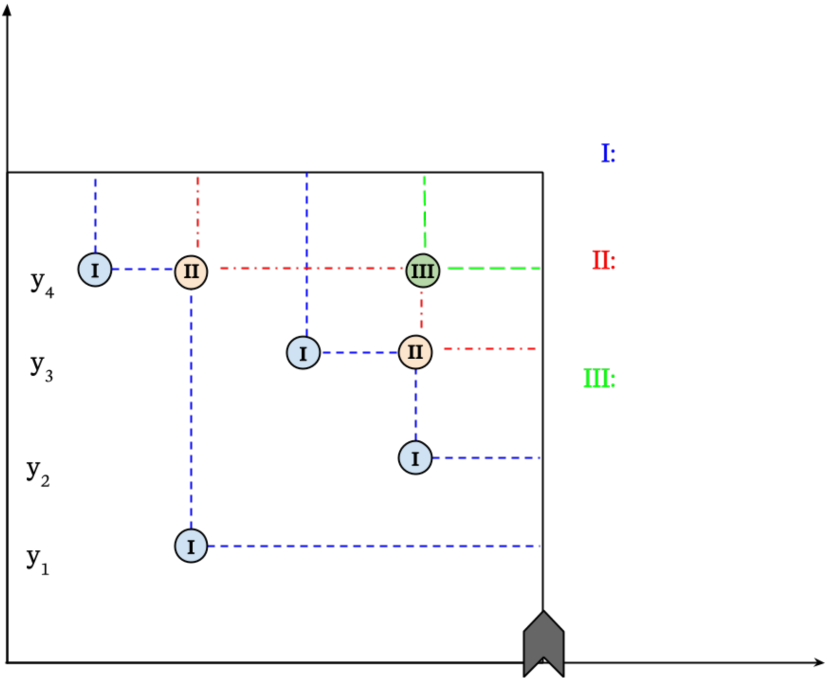

I
II
III
II
I
I
I
x1
x2
x3
x4
I:
1
x1
3
x3
II:
2
x2
III:
4
x4


I:
II:
III:
y_4
y_3
y_2
y_1
I
II
III



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y_4
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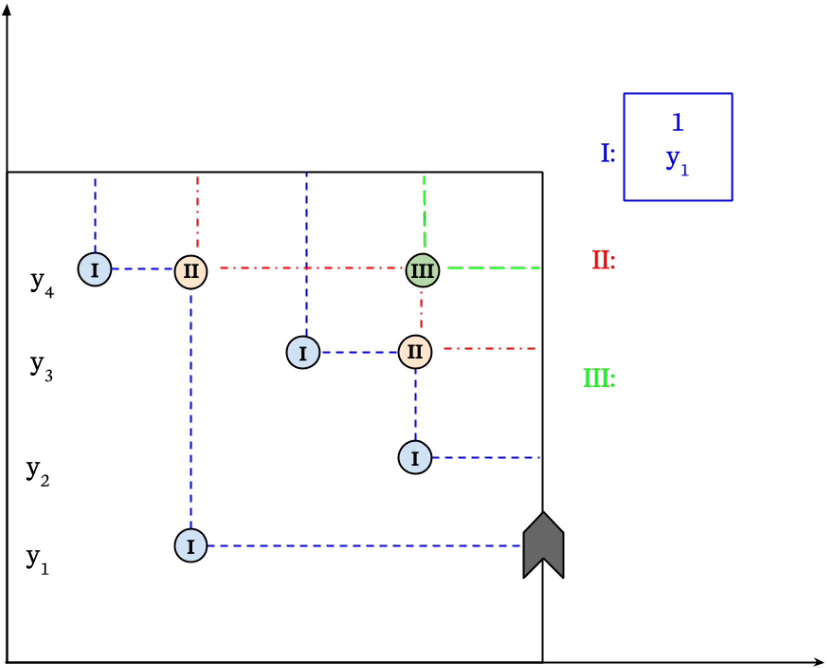
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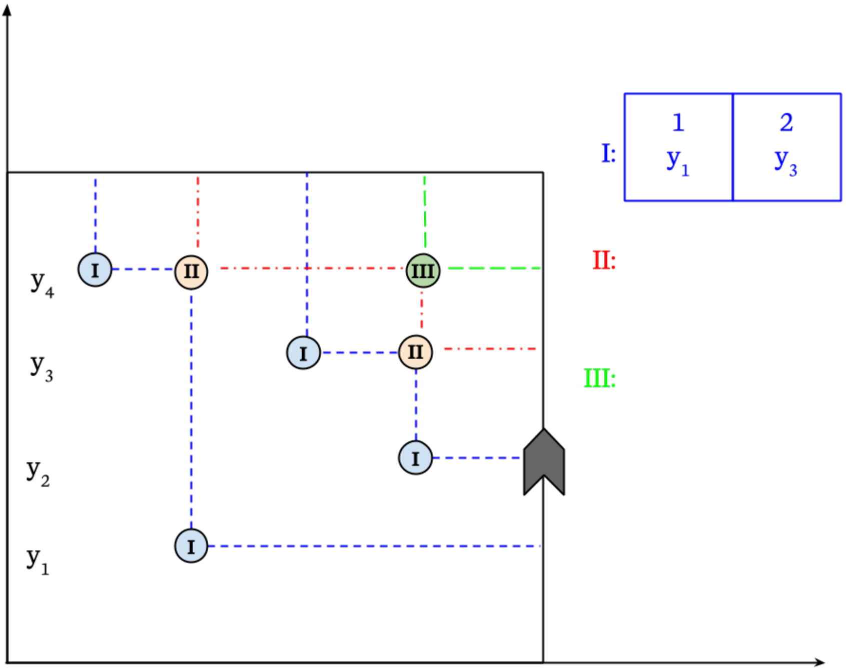
I:
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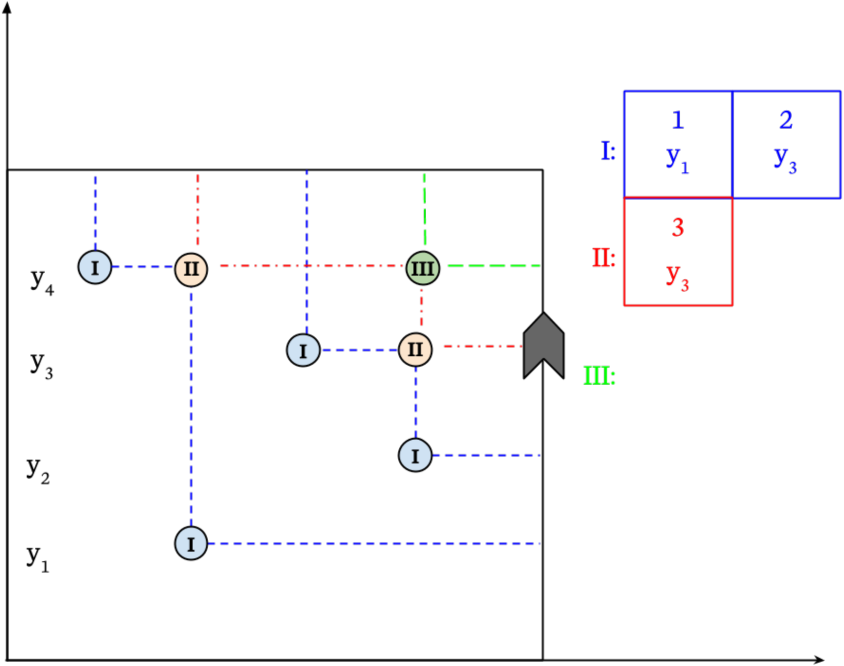

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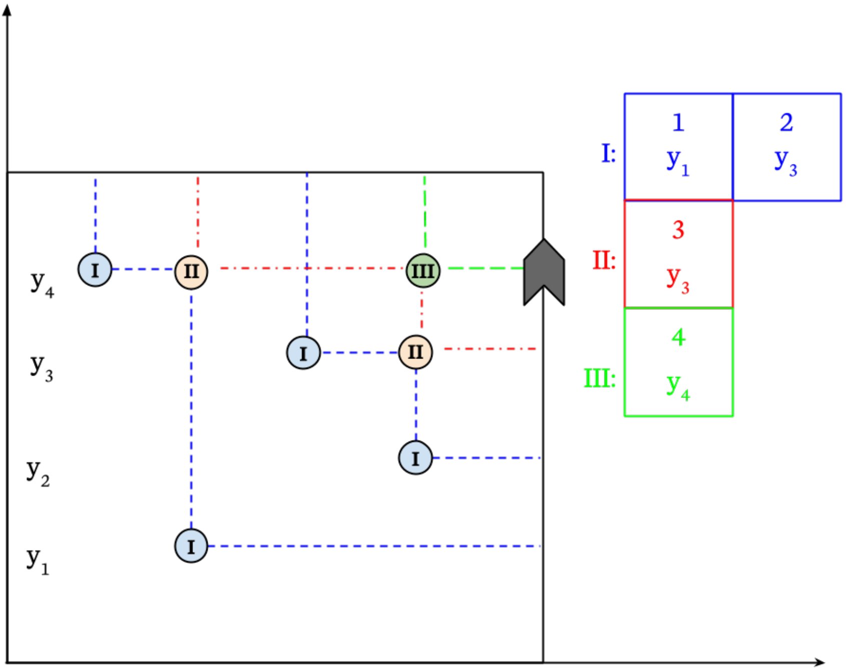

y_4
y_3
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I
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y_3
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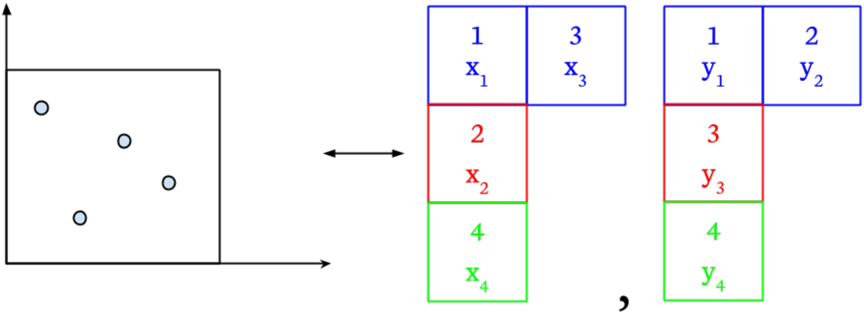
y_4

Robinson Schensted (Knuth) Bijection

Configurations of points ⟷ Pairs of “decorated” Young Tableaux of the same shape



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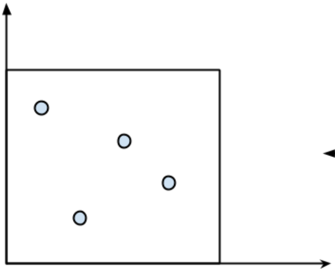
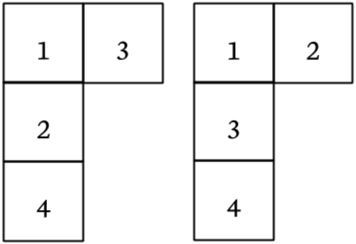


⟷



(x_1, x_2, x_3, x_4) (y_1, y_2, y_3, y_4)

Problem:

What are the finite dimensional distributions?

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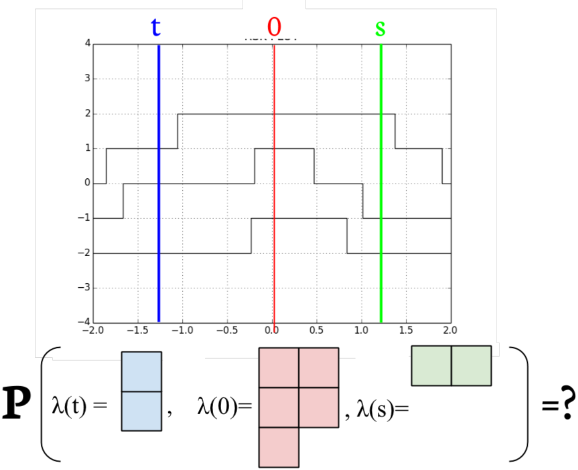
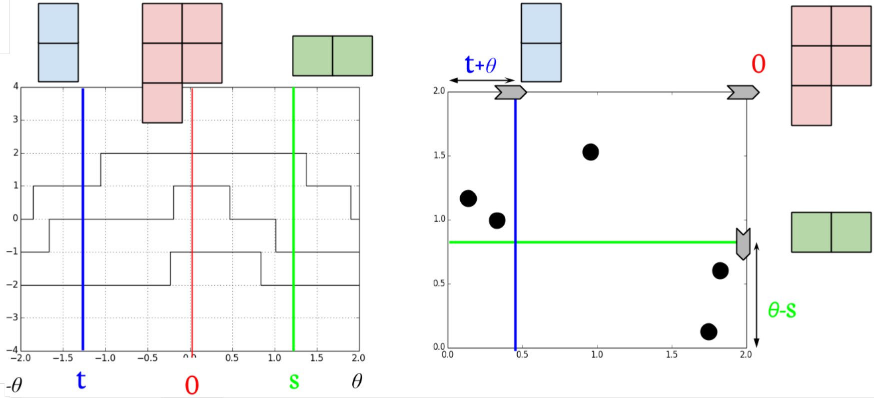
e.g.



Remark:

A priori, there is some complicated dependence due the fact that the regions overlap in non-trivial ways.



Recall the “decorated tableaux” point of view.
Let $L, \vec{x}, R, \vec{y}$ denote the two Tableaux and their decorations.

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Proof idea:

Use the fact that Poisson points are uniformly distributed and that Robinson-Schensted correspondence is a bijection.

Theorem: (N.)

$$\mathbf{P}\left(\lambda(t)=\lambda, \lambda(0)=\nu, \lambda(s)=\mu\right)$$
$$= e^{-\theta^2} s_\lambda(\rho_{t+\theta}) s_{\nu/\lambda}(\rho_{-t}) s_{\nu/\mu}(\rho_s) s_\mu(\rho_{\theta-s})$$

where s are Schur functions, ρ_t is the "exponential" specialization.

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This is a Schur Process! The diagram for this Schur process is:

$$\begin{array}{ccccccc} & & & \lambda(0) & & & \\ & & \rho_{0-t}\nearrow & & \searrow\rho_{s-0} & & \\ & & \lambda(t) & & \lambda(s) & & \\ & \rho_{t-(-\theta)}\nearrow & & & & \searrow\rho_{\theta-s} & \\ \emptyset & & & & & & \emptyset \end{array}$$

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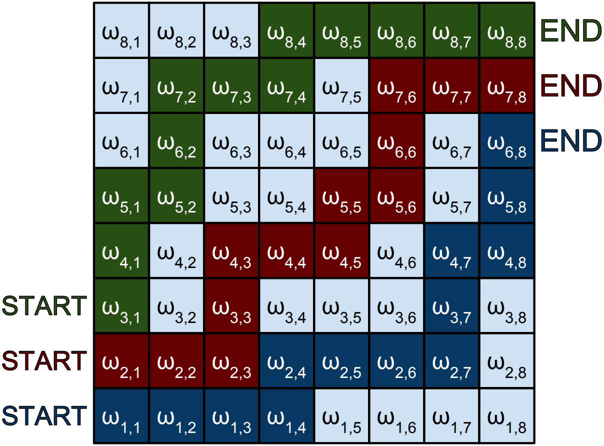
$$s_{\lambda/\mu}(\rho_t) = \det\left[W\Big((\lambda_i - i) - (\mu_j - j)\Big)\right]_{1 \le i,j \le n}$$

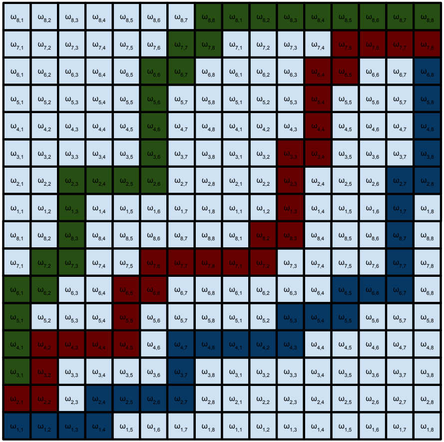
where $W(x) = t^x/x!$

Limits of Multi-Layer Random Polymers



"Soft-Max" (Generalized) Longest Increasing Subsequence:

$$Z_d^\beta = \mathbf{E}_{X_1,\ldots,X_d}\left[\exp\left(\beta \sum_{i=1}^{d}\sum_{t=1}^{2N} \omega_{X_i(t)}\right)\right]$$



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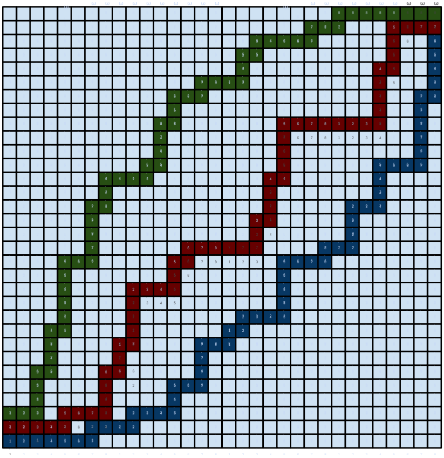

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Continuum Directed Random Polymer

Introduced by Alberts-Khanin-Quastel '14

Space	Time	Paths	Disorder

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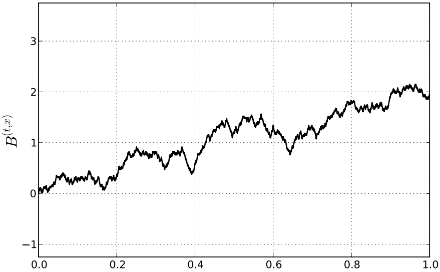

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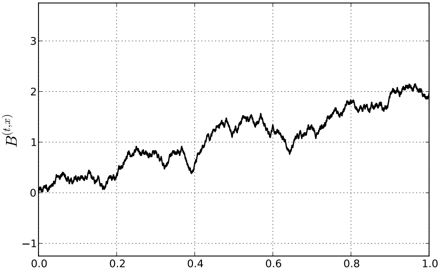

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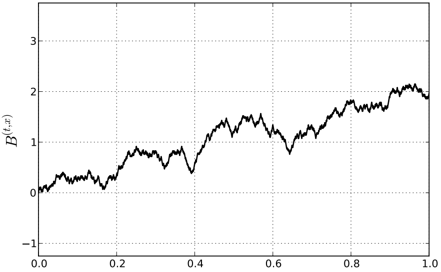

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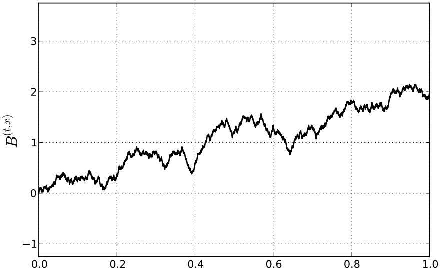

$$\mathcal{Z}^{\beta}(t,x) = \rho(t,x)\mathbf{E}_{B^{(t,x)}(\cdot)}\left[:\!\exp\!:\left(\beta \int_0^t \xi\big(s, B^{(t,x)}(s)\big)\,ds\right)\right]$$

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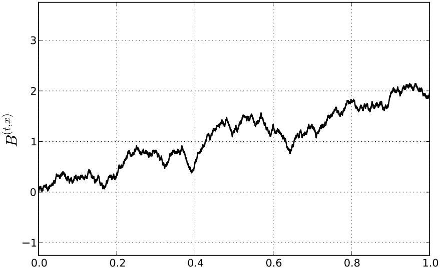

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Here $\rho = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ and :exp: is the **Wick exponential**. This is formally a **chaos series**.

Continuum Directed Random Polymer

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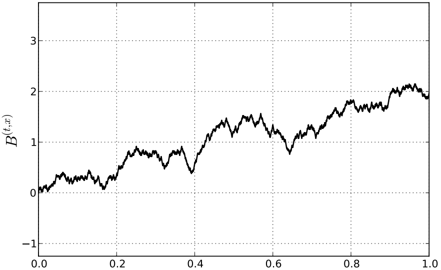

$$\mathcal{Z}_d^{\beta}(t,x) = \rho(t,x) \sum_{k=0}^{\infty} \beta^k \iint_{\substack{\vec{t} \in \Delta_k(0,t) \\ \vec{z} \in \mathbb{R}^k}} \psi_k^{(t,x)}\big((t_1,x_1),\ldots,(t_k,x_k)\big) \xi^{\otimes k}(d\vec{t}, d\vec{x})$$

$\psi_k^{(t,x)}$ is k-point correlation function for $B^{(t,x)}$. $\Delta_k(0,t)$ is ordered k-tuples.

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$$\partial_t \mathcal{Z}^\beta = \frac{1}{2}\partial_{xx}\mathcal{Z}^\beta + \beta \mathcal{Z}^\beta \xi$$

- $\mathcal{H} = \log(\mathcal{Z}^\beta)$ is the Hopf-Cole solution to the KPZ equation:

$$\partial_t \mathcal{H} = \frac{1}{2}\partial_{xx}\mathcal{H} - \frac{1}{2}\left(\partial_x \mathcal{H}\right)^2 + \beta\xi$$

Remarks:

- $\mathcal{Z}^\beta$ is a function of only the white noise field.
- $\mathcal{Z}^\beta$ solves (as a mild solution) the **stochastic heat equation (SHE)** (with delta initial data)

$$\partial_t \mathcal{Z}^\beta = \frac{1}{2}\partial_{xx}\mathcal{Z}^\beta + \beta \mathcal{Z}^\beta \xi$$

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- $\mathcal{Z}^\beta$ is the limit of discrete polymer partition function when $\beta_N = \beta N^{-\frac{1}{4}}$. (other paper by Alberts-Khanin-Quastel '14)

d - **layer CDRP**

"multi-layer extension of stochastic heat equation" (O'Connell-Warren'15)

Space	Time	Paths	Disorder

d - **layer CDRP**

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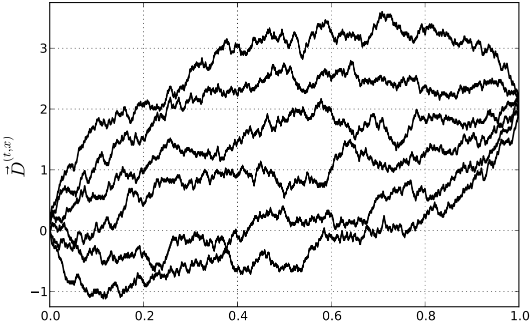

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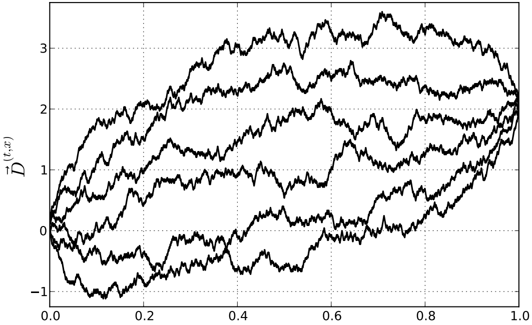

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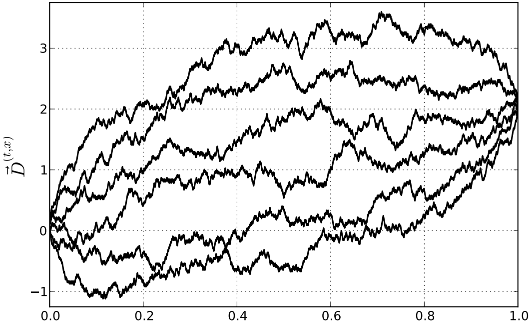

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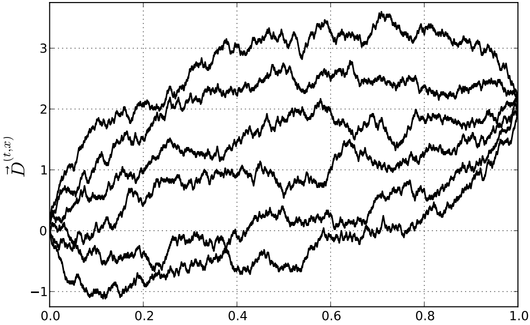

$$\mathcal{Z}^{\beta}_{m}(t,x) := \mathbf{E}\Big[:\!\exp\!:\Big(\beta\sum_{j=1}^{m}\int_0^t \xi\big(s, D_j^{(t,x)}(s)\big)\,ds\Big)\Big]$$

d - layer Discrete Polymer

Space	Time	Paths	Disorder

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$x \in \mathbb{Z}$	$t \in \mathbb{N}$		

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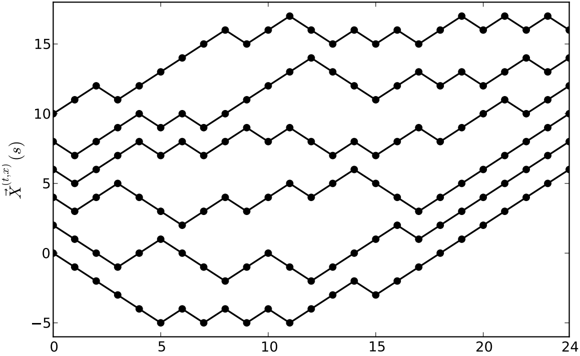
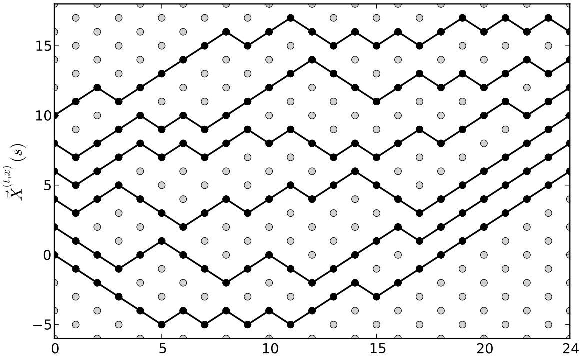


d - layer Discrete Polymer

Space	Time	Paths			Disorder
		N.I. Random Walks	Start	End	iid Random Variables
$x \in \mathbb{Z}$	$t \in \mathbb{N}$	$\vec{X}^{(t,x)}(\cdot)$	$\vec{X}^{(t,x)}(0) = (0, 2, \ldots, 2d-2)$	$\vec{X}^{(t,x)}(t) = (x, \ldots, x+2d-2)$	$\omega(\cdot,\cdot)$



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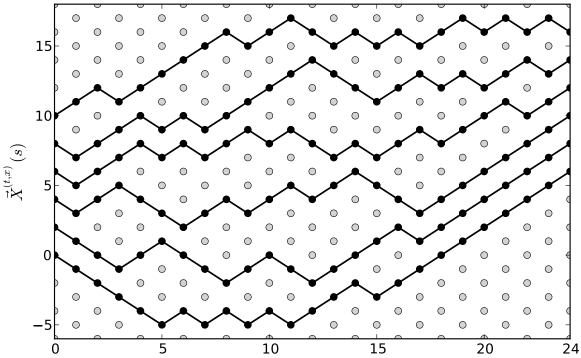

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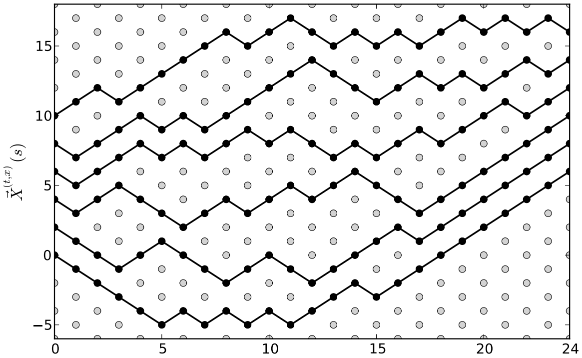

$$Z_m^{\beta,disc}(t,x) = \mathbf{E}\left[\exp\left(\beta \sum_{j=1}^{m}\sum_{s=1}^{t} \omega\left(s, X_j^{(t,x)}(s)\right)\right)\right]$$

Remarks

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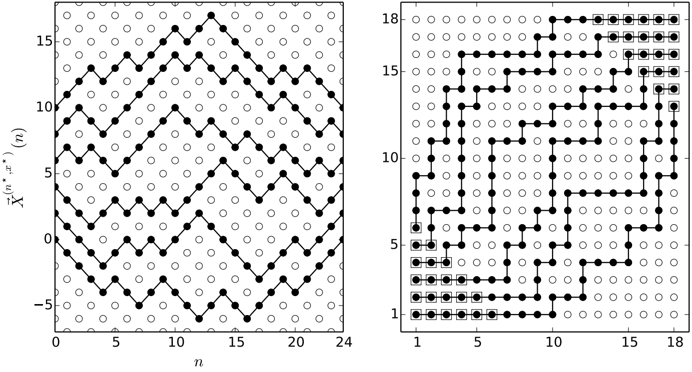
- Rotate by 45 degrees to interpret as up-right lattice paths



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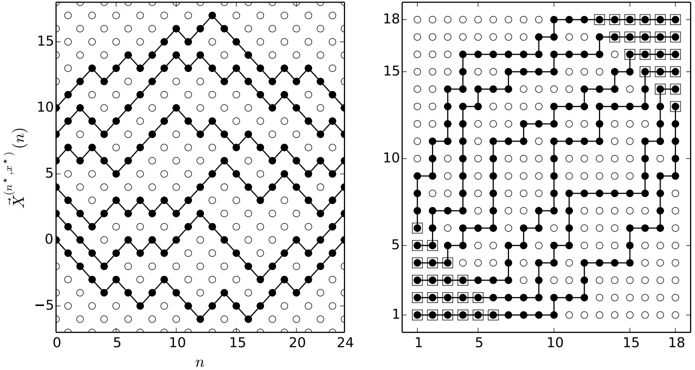

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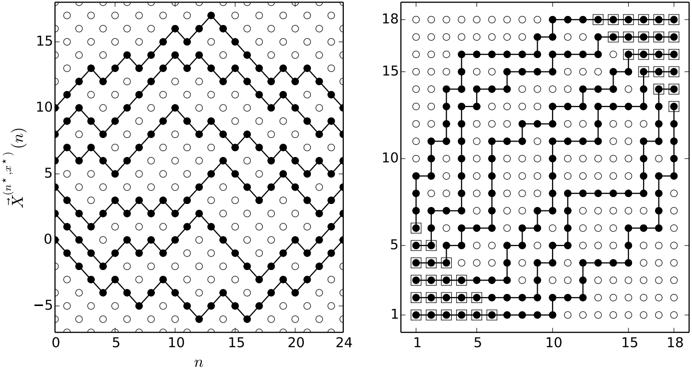

- $Z_d^{\beta,disc}$ is then the d-th row in geometric RSK
- This is a "tropicalization" of Last Passage Percolation

Convergence of Discrete to Continuum

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Name	Space	Time	Paths	Disorder	Start	End
Multi-Layer CDRP $\mathcal{Z}_d^{\beta}(t,x)$	$x \in \mathbb{R}$	$t \in \mathbb{R}^+$	$\vec{D}(\cdot)$ N.I. Brownian Bridges	$\xi(\cdot,\cdot)$ White Noise	$\vec{D}(0) = (0,0,\ldots,0)$	$\vec{D}(t) = (x,x,\ldots,x)$
Multi-Layer Discrete $Z_d^{\beta,disc}(t,x)$	$x \in \mathbb{Z}$	$t \in \mathbb{N}$	$\vec{X}(\cdot)$ N.I. Random Walks	$\omega(\cdot,\cdot)$ i.i.d. random variables	$\vec{X}(0) = (0,2,\ldots,2d-2)$	$\vec{X}(t) = (x,\ldots,x+2d-2)$

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Suppose the variables ω are centered, unit variance and have finite exponential moments:

$$\Lambda(\beta) := \log\left(\mathcal{E}(e^{\beta\omega(0,0)})\right)$$

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$$Z_d^{\beta_N,disc}\left(\lfloor Nt \rfloor, \lfloor \sqrt{N}x \rfloor\right)\exp\left(-dNt\Lambda(\beta_N)\right) \Rightarrow \frac{\mathcal{Z}_d^{\sqrt{2}\beta}(t,x)}{\rho(t,x)^d}$$

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- Conjectured to have universality for *fixed* $\beta > 0$ but this is hard. (The method of using chaos series expansions does not seem to apply)

d - layer Semi-Discrete Polymer

Space Time	Paths	Disorder

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$x \in \mathbb{N}$	$t \in \mathbb{R}$		

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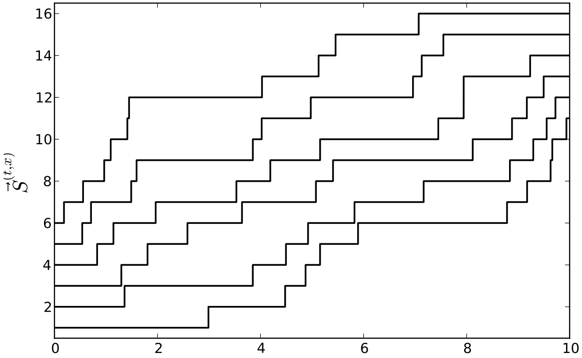

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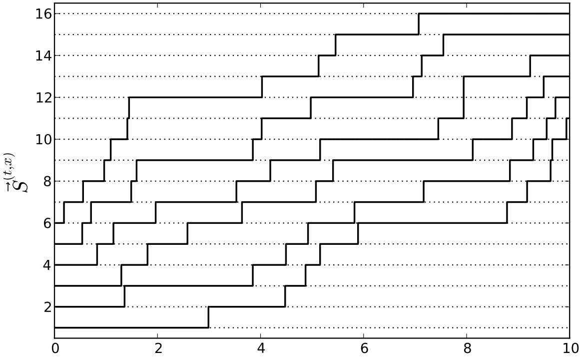

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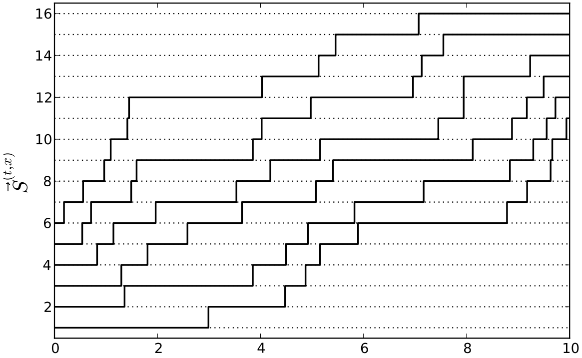

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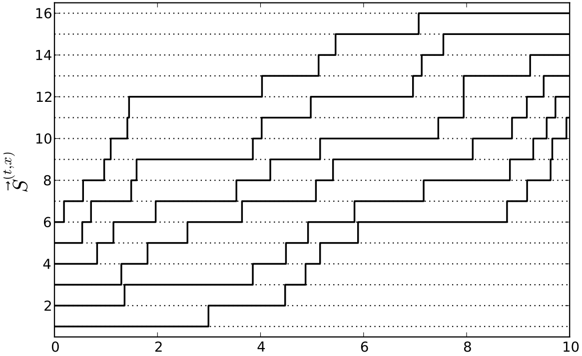

$$H^B(\vec{S}^{(t,x)}) = \sum_{j=1}^{m} \int_0^t dB_{S_j^{(t,x)}(s)}(s), \quad Z_d^{\beta,sd}(t,x) = \mathbf{E}\Big[\exp\Big(\beta H^B(\vec{S}^{(t,x)}) \Big) \Big]$$

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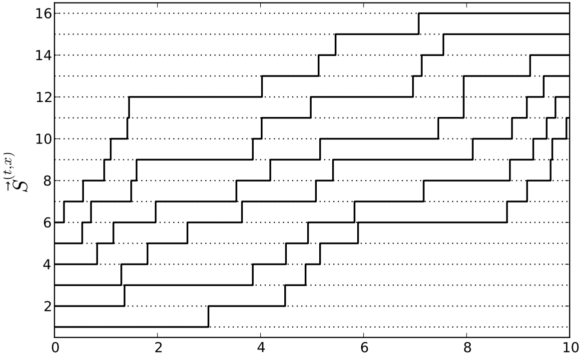

$Z_d^{\beta,sd}$ has nice structure (O'Connell '12). "Positive Temperature" generalization of:

$$\lim_{\beta\to\infty} \beta^{-1} \log\left(Z_k^{\beta,sd}(t,N)\right) \stackrel{d}{=} k\text{-th eigenvalue of } N \times N \text{ GUE (Variance } t)$$

Convergence Result for Semi-Discrete Polymer Partition Function

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Multi-Layer CDRP $\mathcal{Z}_d^{\beta}(t,x)$	$x \in \mathbb{R}$	$t \in \mathbb{R}^+$	$\vec{D}(\cdot)$ N.I. Brownian Bridges	$\xi(\cdot,\cdot)$ White Noise	$\vec{D}(0) = (0,0,\ldots,0)$	$\vec{D}(t) = (x,x,\ldots,x)$
Multi-Layer Semi-Discrete $Z_d^{\beta,sd}(t,x)$	$x \in \mathbb{N}$	$t \in \mathbb{R}^+$	$\vec{S}(\cdot)$ N.I. Poisson paths	$B_{\cdot}(\cdot)$ i.i.d. Brownian motions	$\vec{S}(0) = (1,2,\ldots,d)$	$\vec{S}(t) = (x+1,\ldots,x+d)$

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Theorem (N.)

For $\beta > 0$, set $\beta_N = N^{-\frac{1}{4}}\beta$.

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- Verifies conjecture that $\{\mathcal{Z}_d^{\beta}\}_{d=1}^{\infty}$ yields a KPZ line ensemble (Corwin-Hammond '15).

Corollary (Conjecture from KPZ line ensemble modulo constants)

There are explicit constants $c_{m,t}$ so that if we set set:

$$\mathcal{H}^t_m(x) = \log\left(\frac{c_{m,t}\mathcal{Z}^1_m(t,x)}{c_{m-1,t}\mathcal{Z}^1_{m-1}(t,x)}\right)$$

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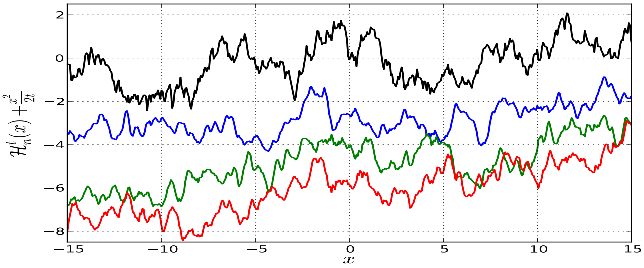
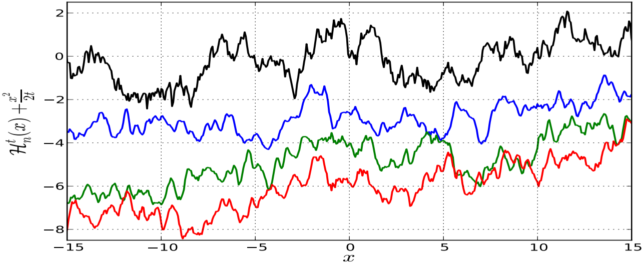
The KPZ line ensemble is a multi-layer generalization of the KPZ equation:





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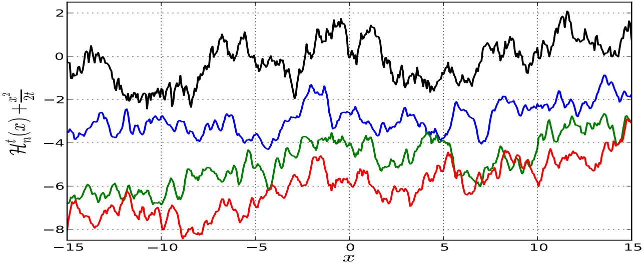

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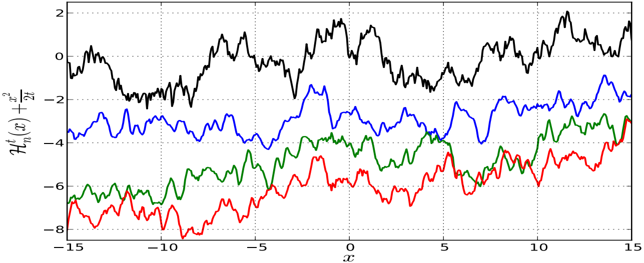

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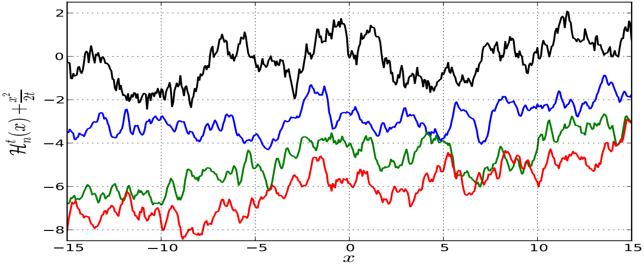

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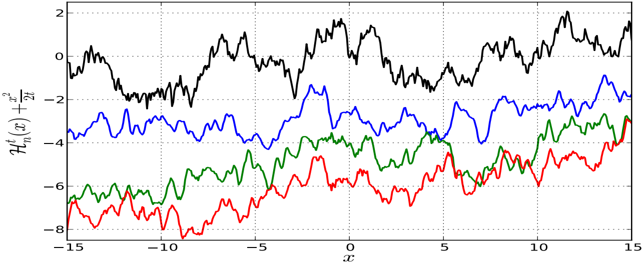

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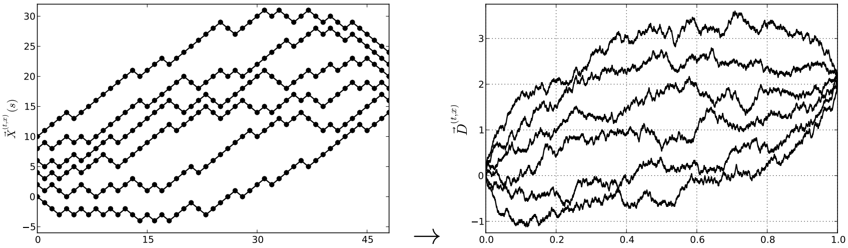
- $\mathcal{H}_m^t(x) + \frac{x^2}{2t}$ conjectured to converge to Airy line ensemble as $t \to \infty$

Proof ideas



$\vec{X}^{(t,x)}(s)$
30
25
20
15
10
5
0
−5
0
15
30
45
$\rightarrow$
$\vec{D}^{(t,x)}$
3
2
1
0
−1
0.0
0.2
0.4
0.6
0.8
1.0

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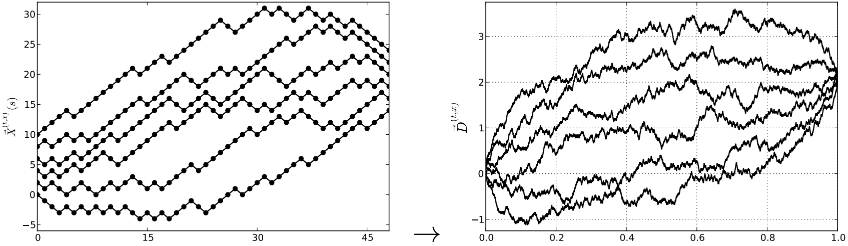

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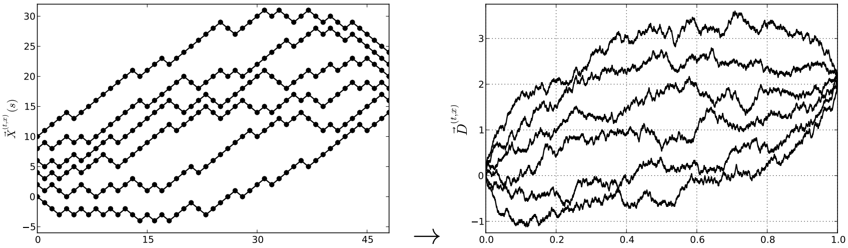

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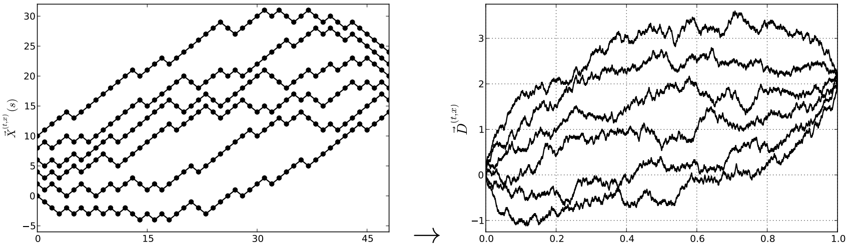

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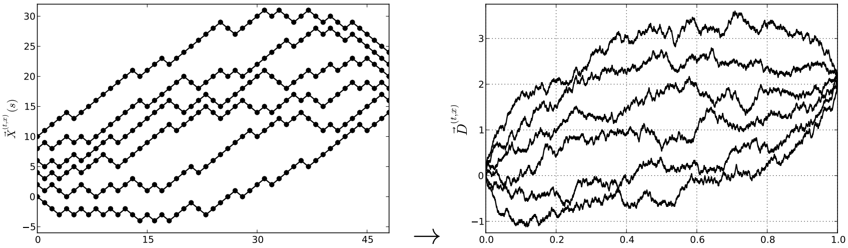

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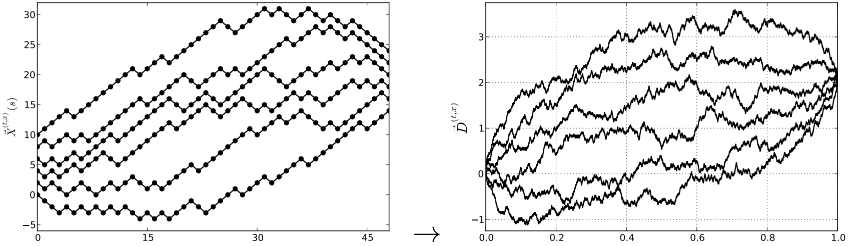

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Stabilization Time Distribution for a Type of Exclusion Process

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Example:

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2: ●●○●●○●○

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Start from a random initial condition: How long until all particles on the left and all holes on the right?

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where $\chi_3 \overset{d}{=} \sqrt{Z_1^2 + Z_2^2 + Z_3^2}$, the norm of a 3D standard Gaussian.

Pf Ideas: Convert the starting string to a down right path by setting $\bigcirc$ to a right step and $\bullet$ to a downstep.

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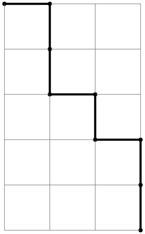

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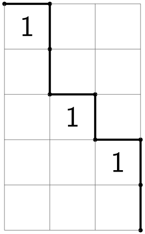

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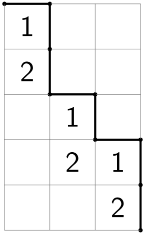

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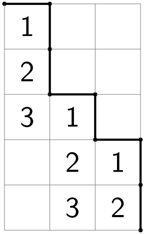

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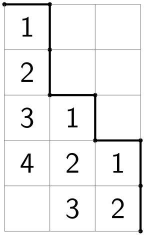

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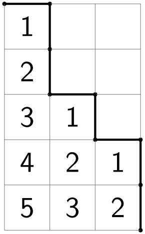

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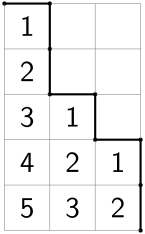

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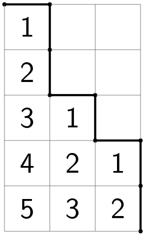

This leads to:

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which leads to

$$T_n \stackrel{d}{=} \frac{1}{2}n + \max_{0 \le k \le n} S_k - \frac{1}{2}S_n$$

where S_k is a Bernoulli-p random walk.

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Works since ○● → ●○ corresponds to

$$\begin{array}{|c|c|}\hline 0 & 1\\\hline 1 & 0\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 0\\\hline 0 & 1\\\hline\end{array}.$$

which reduces Last Passage Time by exactly 1.

Asymptotics

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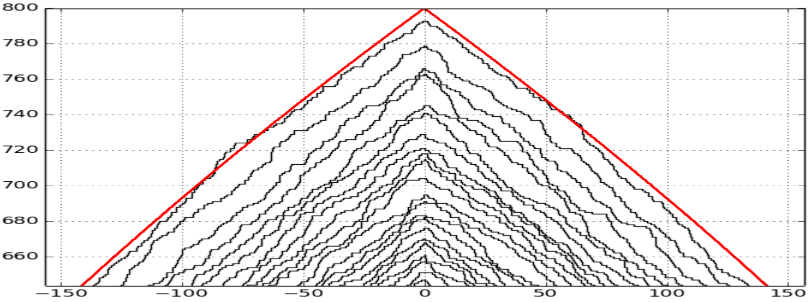

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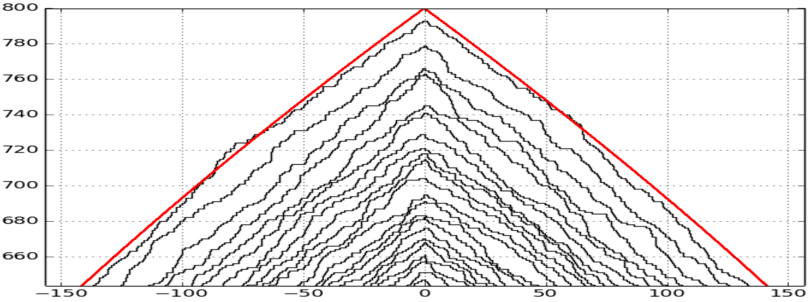

Asymptotics

(Comes by applying results in Borodin-Olshanski 2006)

$$\frac{\lambda_1(2\theta^{2/3}\tau) - 2\left(\theta - \theta^{2/3}\,|\tau|\right)}{\theta^{1/3}} \Rightarrow \mathcal{A}_2(\tau) - \tau^2,$$

where $\mathcal{A}_2(\cdot)$ is the Airy 2 process.