GLOBAL SOLUTIONS FOR SMALL DATA TO THE HELE-SHAW PROBLEM

P. CONSTANTIN AND M. PUGH

Appeared in Nonlinearity, volume 6, pages 393-416, 1993.

Abstract. We analyze an equation governing the motion of an interface between two fluids in a pressure field. In two dimensions, the interface is described by a conformal mapping which is analytic in the exterior of the unit disc. This mapping obeys a nonlocal nonlinear equation. When there is no pumping at infinity, there is conservation of area and contraction of the length of the interface. We prove global in time existence for small analytic perturbations of the circle as well as nonlinear asymptotic stability of the steady circular solution. The same method yields well-posedness of the Cauchy problem in the presence of pumping.

1. Introduction

In the Hele-Shaw problem[1-4], two fluids are confined between close sheets of glass. A viscous incompressible fluid surrounds an inviscid incompressible fluid in which the pressure is constant. One assumes that this is a two dimensional problem, that the viscous fluid moves with a velocity proportional to the gradient of the pressure and that the interface moves with the viscous fluid. The boundary condition at the interface is assumed to be that the jump in pressure across the interface is a constant (the surface tension) multiple of the curvature at that point. The boundary condition at infinity can be either zero velocity (no pumping) or a sink of strength γ (fluid flux through circles asymptotes $2\pi\gamma$ at infinity).

The oversimplifications made above are discussed in the literature [5-8]. A local existence result (in the Lagrangian formulation and a different geometry) has been proven [9].

We approach this problem with the hodographic method [10-11]. We view the interface as the image of the unit circle under a conformal transformation. This conformal transformation maps the exterior of the unit circle in the w complex plane to the region of z plane containing the viscous fluid. The conformal transformation depends on time, as the interface moves in time:

$$z = f(w,t).$$

Let h be the derivative of f:

$$h(w,t) = \partial_w f(w,t).$$

For this to make sense, h can have neither zeroes nor poles in the exterior of the circle. Since there is a sink at infinity f must behave like w at infinity. In particular, $\frac{f(w)}{w}$ is bounded at infinity. This implies that a non-constant h must have singularities inside the unit circle.

The function $\alpha \in [0, 2\pi] \mapsto h(e^{i\alpha})$ obeys the evolution equation ([12]):

$$\frac{\partial h}{\partial t} = 2(I + \frac{1}{i}\partial_\alpha)hA(\frac{1}{|h|^2}(\frac{\gamma}{2\pi} - \tau\partial_\alpha H\kappa)). \tag{1}$$

We abuse notation and write h for $h(e^{i\alpha})$. Here τ is the surface tension, γ the rate of pumping at infinity, and $\kappa(h)$ is the curvature of the interface. H is the Hilbert transform on the unit circle:

$$Hf(\alpha) = \frac{1}{2\pi}P.V.\int_0^{2\pi} \cot(\frac{\alpha-\beta}{2})f(\beta)\,d\beta.$$

A is the operator

$$A = \frac{1}{2}(I - iH).$$

The $\tau = 0$ case is known to be integrable [10,13,14] but ill-posed. The case in which h has n zeroes and m poles $(n > m)$ in the unit circle has been analyzed in [15]. If h is initially a rational function, it is a rational function at least up to the time that the singularities of h hit the unit circle. The solution breaks down when a singularity hits the unit circle [16-18]. In [12], the authors analyze a localized version of the above equation for h and proved that in the $\tau = 0$ case singularities of h in the interior of the circle hit the circle in finite time. They also proved that for small $\tau > 0$ solutions exist up to the critical time for the corresponding zero τ solution, and are close to the zero surface tension solution at least up to this critical time. The non-zero surface tension problem is a singular perturbation of the zero surface tension problem [19-24].

Here are the basic questions regarding (1) and some answers provided in this paper.

- Do arbitrary initial shapes evolve if $\tau > 0$? (Yes)
- For how long? (At least a short time, depending on the radius of convergence of h)
- Do solutions exist for all time if $\tau > 0$ and there is no pumping ? (?)
- Do solutions exist for all time if the initial shape is near circular and no pumping is present? (Yes)
- Do nearly circular shapes relax to circles? (Yes)
- Is the property of area preservation and length contraction sufficient for well-posedness? (No)

We prove the following global existence and stability result:

Theorem 1.1. *When $\gamma = 0$, there exist constants $\epsilon > 0$, $C > 0$ such that, if the initial datum*

$$h_0(w) = \sum_0^\infty h_j(0) w^{-j}$$

satisfies

$$\sum_1^\infty |h_j(0)| \rho^j \le \epsilon$$

for some $\rho > 0$, then there exists a unique global solution to (1)

$$h(w,t) = \sum_0^\infty h_j(t) w^{-j}$$

defined for $t \ge 0$, $|w| > 1$. This solution satisfies

$$\sum_1^\infty |h_j(t)| \rho^j \le C \sum_1^\infty |h_j(0)| \rho^j.$$

Moreover, there exist a constant $A > 0$ such that for $\alpha \in [0, 2\pi]$,

$$|h(t, e^{i\alpha}) - r| \le ||h(0,\cdot) - r||_\infty e^{-At} am$$

for all $t \ge 0$, where r is such that πr^2 is the initial area of the bubble. Hence the solution tends asymptotically to the steady circular solution.

The paper is organized as follows: In Section 2 we discuss the equations of motion. We discuss the properties of area preservation and of length contraction and show by example (Appendix) that they alone are not sufficient for well-posedness. Section 3 is devoted to the proof of existence of solutions of an auxiliary problem, and Section 4 to that of stability of the circles in the auxiliary problem. In Section 5, the full problem is reduced to the auxiliary one.

2. Equations of Motion

There are two independent variables in what follows. w, representing space, is complex, and t, representing time, is real.

We analyze functions restricted to the unit circle, so $|w| = 1$, i.e. for $\alpha \in [0, 2\pi]$

$$w = e^{i\alpha}.$$

If h is a function of w which has a restriction to the unit circle, we refer to the restriction $\alpha \mapsto h(e^{i\alpha})$ by the same name, h. We hope that this causes no inconvenience to the reader. Functions can be represented by their Fourier series:

$$f(\alpha) = \sum_{-\infty}^{\infty} f_k e^{ik\alpha}.$$

The operator D is defined as follows:

$$Df = \frac{1}{i}\partial_\alpha f = w\partial_w f.$$

In the following, ∂ refers to ∂_α.

Recall the Hilbert transform on the unit circle:

$$Hf(\alpha) = \frac{1}{2\pi} P.V. \int_0^{2\pi} \cot(\frac{\alpha - \beta}{2}) f(\beta)\, d\beta = \sum_{k \neq 0} -isgn(k) f_k e^{ik\alpha},$$

and

$$(\partial Hf)_k = |k| f_k. \qquad \text{i.e.} \qquad \partial H = |\partial|.$$

We define the operator A by

$$A = \frac{1}{2}(I - iH), \qquad \text{that is} \qquad Af(\alpha) = \frac{1}{2} f_0 + \sum_{-\infty}^{-1} f_k e^{ik\alpha}.$$

The equation of motion is [12]:

$$\frac{\partial h}{\partial t} = 2(I + D) h A(\frac{1}{|h|^2}(\frac{\gamma}{2\pi} - \tau \partial H \kappa(h))). \tag{2}$$

Where τ is the surface tension, γ the rate of pumping at infinity, and

$$\kappa(h) = |h|^{-1}(1 + \mathrm{Re}\frac{Dh}{h}). \tag{3}$$

$\kappa(h)$ corresponds to the curvature of the interface when h is, as defined above, the derivative of f.

We discuss the rate of change of the area. In order to do so, we express the area of the bubble in terms of h. From Stokes' theorem,

$$\text{Area of the bubble} = \frac{1}{2} \int_{\text{boundary}} -y dx + x dy.$$

In terms of the conformal transformation f, this becomes

$$S(h) = \text{Area} = \frac{1}{2} \mathrm{Re} \int_0^{2\pi} f(e^{i\alpha}) \overline{e^{i\alpha} \frac{\partial f}{\partial w}(e^{i\alpha})}\, d\alpha.$$

Since $h = \frac{\partial f}{\partial w}$, this equals

$$S(h) = \frac{1}{2} Re \int_0^{2\pi} h \left[(I + D)^{-1} h\right]^* \, d\alpha.$$

The equation of evolution (2) has the structure

$$\frac{\partial h}{\partial t} = 2(I + D)\,[hAE(h)] \tag{4}$$

for some real-valued functional E of $h, Dh, \ldots$. Whatever the functional form of the real expression E, if (4) holds then the area $S(h)$ obeys

$$\frac{d}{dt}S(h) = \int_0^{2\pi} |h|^2 E\,d\alpha.$$

In the case of the equation (2), E is given by

$$|h|^2 E = \frac{\gamma}{2\pi} - \tau\partial H\kappa.$$

Since $\partial H\kappa$ has mean zero, it follows that

$$\frac{d}{dt}S(h) = \gamma.$$

Hence the area grows linearly in time, at a rate equal to the rate of pumping.

Similarly, we express the length of the boundary of the bubble in terms of h:

$$2\pi L = \int_0^{2\pi} |h|\,d\alpha.$$

If h solves (4) then the length element $|h|$ obeys

$$\frac{\partial|h|}{\partial t} = \kappa|h|^2 E - \partial(|h|HE).$$

In particular,

$$\frac{d}{dt}L = \frac{1}{2\pi}\int_0^{2\pi} \kappa|h|^2 E\,d\alpha.$$

To produce a model (4) in which

$$\frac{d}{dt}S = 0,$$
$$\frac{d}{dt}L \le 0,$$

it suffices to require

$$\int_0^{2\pi} \kappa|h|^2 E\,d\alpha \le 0 \qquad \text{and} \qquad \int_0^{2\pi} |h|^2 E\,d\alpha = 0.$$

These requirements can be achieved when E is given by the rule:

$$|h|^2 E = -M\kappa$$

where M is any nonnegative self-adjoint linear operator which is zero on constant functions. M is determined by the pressure jump at the interface.

Therefore any nonnegative selfadjoint operator M which is zero on constant functions can be used to produce a "surface tension" model equation which exhibits both area conservation and length contraction. Naively one would expect this to lead to stability of circular solutions, and indeed linear neutral stability is true. However, the equation can be ill-posed: not only nonlinearly unstable, but catastrophically so (the higher the wave-length of the perturbation, the faster it grows). We give a simple example of such M in the appendix. In the case (2), $M = \tau\partial H$, which is nonnegative and self-adjoint. If $\gamma = 0$, then the area is conserved and the length does not increase in time.

The dependent variable in (4) is not the most natural one. We would prefer a variable which is more directly related to the geometry of the boundary. A first choice would be the curvature $\kappa(h)$ but the equation for it is not explicit. An even more natural choice is $|h|\kappa(h)$ because it corresponds to

κds in physical variables and by virtue of the Umlaufsatz, its integral is the winding number of the boundary:

$$\frac{1}{2\pi}\int_0^{2\pi} |h|\kappa(h)\,d\alpha = \text{the winding number of the interface} = 1.$$

Thus $|h|\kappa(h)$ should have mean equal to 1, for all time. Note that, because of (3),

$$1 - |h|\kappa(h) = \partial H u$$

where u is given by

$$u \;=\; \log(|h|).$$

It turns out that u is a natural choice for dependent variable.

Because f is a diffeomorphism, its derivative, h, never vanishes in the exterior of the disc. Since f equates asymptotically to w at infinity, h tends to a non-zero constant as w tends to infinity. This implies that h is analytic at infinity. Let $\tilde{h}(w) = h(\frac{1}{w})$. $\tilde{h}$ is analytic in the unit disc, as is $\frac{1}{\tilde{h}}$. Because the unit disc is simply connected, there is a holomorphic function on the unit disc such that $\tilde{h} = \exp(\tilde{\psi})$. It follows that $\psi(w) = \tilde{\psi}(\frac{1}{w})$ satisfies $h = \exp(\psi)$. This shows that we can define $\log(h)$ and it is analytic in the exterior of the disc, as well as at infinity. Because $\log h$ can be extended to an analytic function in the exterior of the unit circle, we can recover h from knowledge of u via

$$h = \rho e^{i\phi},$$

where

$$\rho \;=\; \exp(u)$$

and

$$\phi - \overline{\phi} = -H\log\rho, \qquad \text{and} \qquad \overline{\phi} = \frac{1}{2\pi}\int_0^{2\pi}\phi\,d\alpha.$$

The analyticity of $\log(h)$ at infinity also implies

$$\int_\gamma \log(h(z))\,dz = 0 \qquad \text{for any closed loop } \gamma.$$

Specifically,

$$\int_0^{2\pi} \log(h(e^{i\theta}))e^{i\theta}\,d\theta = 0.$$

Using the fact that $\log(h) = u + i(-Hu + \bar{\phi})$, we find that

$$\int_0^{2\pi} u(e^{i\theta})e^{i\theta}\,d\theta = 0.$$

That is, u has no ± 1 Fourier modes.

From the equation (4) and the above, we obtain the equation for u:

$$\frac{\partial u}{\partial t} = (I - \partial H)E(u) - (\partial H u)E(u) - (\partial u)(HE(u)), \tag{5}$$

$$\text{where} E(u) = e^{-2u}\left(\frac{\gamma}{2\pi} - \tau\partial H\kappa(u)\right) \text{ and } \kappa(u) = e^{-u}(1 - \partial H u).$$

The circle is a steady solution of the above equation when there is no pumping, i.e. if $\gamma = 0$. The circle corresponds to $u =$ constant. In this case, $\kappa(u) =$ constant, hence $E(u) = 0$. Thus $\frac{\partial u}{\partial t} = 0$ for the circle.

To show that the solution tends to a circle, it suffices to show that u tends to a constant function. For this reason, we analyze a pair of equations: one for the zero mode, and one for the non-zero modes. This is done by projecting the equation (5) onto the appropriate subspaces:

$$\frac{du_0}{dt} = e^{-3u_0(t)} P_0(E(v)) \tag{6}$$

$$\frac{\partial v}{\partial t} = e^{-3u_0(t)} \left[(1-\partial H)(E(v) - P_0E(v)) - (\partial Hv)E(v) - (\partial v)HE(v)\right]. \tag{7}$$

This is the notation that is used in the rest of the paper. u_0 represents the zero mode of u and v represents the remaining part, i.e. $v = u - u_0$. (Not to be confused with the common notation for initial data.)

The pair of equations (6) (7) are coupled. We decouple them by reparametrizing time in such a way as to remove the coupling $e^{-3u_0(t)}$ terms. To do this, take

$$\frac{\partial v}{\partial s}\frac{ds}{dt} = e^{-3u_0(t)} N(v),$$

where $N(v)$ represents the non-linear term. This would imply

$$\frac{\partial v}{\partial s} = N(v) \qquad \text{if} \qquad \frac{ds}{dt} = e^{-3u_0(t)}.$$

Reparametrizations of this sort are often not meaningful, as it is sometimes possible to reparametrize time in such a way that the new equation has global existence, even when it is known that the original equation blows up in finite time. A simple example of this can be found looking at Burger's equation viewed as a coupled system like ours and reparametrizing appropriately.

We prove that, in this particular case, the reparametrization is valid, and that global existence for the following reparametrized system implies global existence for the original system:

$$\frac{du_0}{ds} = P_0(E(v)), \tag{8}$$

$$\frac{\partial v}{\partial s} = \left[(1-\partial H)(E(v) - P_0E(v)) - (\partial Hv)E(v) - (\partial v)HE(v)\right]. \tag{9}$$

We keep two different notations for time in the following: t represents the original time and s represents the reparametrized time.

The method of proof is similar to that in [CK1]. We construct an iterative scheme and prove it converges to a solution.

3. Existence for the Reparametrized Equations

We have an equation (9) of the form

$$\frac{\partial v}{\partial s} = N(v),$$

where $N(v)$ is a nonlocal nonlinear operator. We use the following iterative scheme:

$$\frac{\partial v_n}{\partial s} = N(v_{n-1}) + N'(v_{n-1})(v_n - v_{n-1}) \text{ for } n > 0,$$

$$\frac{\partial v_0}{\partial s} = 0.$$

Each step has the same initial conditions: $v_n(z,0) = v_0(z)$. Here v_n is the nth iterate, not the nth Fourier coefficient. We use n consistently throughout for the nth iterate. Fourier coefficients appear with indices other than n. Also note that the 0th iterate is the initial data and is time independent.

The equation for the differences $w_n = v_n - v_{n-1}$ is of the form

$$\frac{\partial w_n}{\partial s} - N'(v_{n-1})w_n = N(v_{n-1}) - N(v_{n-2}) - N'(v_{n-2})w_{n-1}. \tag{10}$$

The right hand side of the above equation suggests that the right choice of functional calculus would make it quadratic in w_{n-1}.

First we define function spaces, as in [12]:

$$||u||_\rho = \sum_{j=-\infty}^{\infty} |u_j|\rho^j,$$
$$||u||_{\rho,S} = \sum_{j=-\infty}^{\infty} \sup_{0\le s\le S} |u_j(s)|\rho^j.$$

These norms determine the following classes of functions:

$$B_\rho = \left\{ v \mid v = \sum_{j=-\infty}^{\infty} v_j z^j, ||v||_\rho < \infty \right\},$$

$$B_{\rho,S} = \left\{ v \mid v = \sum_{j=-\infty}^{\infty} v_j(s) z^j, 0 \le s \le S, ||v||_{\rho,S} < \infty \right\}.$$

The spaces B_ρ and $B_{\rho,S}$ satisfy the following:

$$||uv||_{\rho,S} \le ||u||_{\rho,S}||v||_{\rho,S},$$
$$||\partial H u||_{\rho',S} \le \frac{2}{1-\frac{\rho'}{\rho}}||u||_{\rho,S}.$$

Any analytic function f can be viewed as a map between these spaces. If f, f', f'' belong to some B_r, then f is an analytic change of variables in the ball of radius r in all the spaces $B_{\rho,S}$. If $||u_1||_{\rho,S}, ||u_2||_{\rho,S} \le r$, the following hold:

$$||f(u_1)||_{\rho,S} \le ||f||_\rho,$$
$$||f(u_2) - f(u_1)||_{\rho,S} \le ||f'||_\rho ||u_2 - u_1||_{\rho,S},$$
$$||f(u_2) - f(u_1) - f'(u_1)(u_2 - u_1)||_{\rho,S} \le \frac{1}{2}||f''||_\rho ||u_2 - u_1||^2_{\rho,S}.$$

Lemma 3.1. *If w satisfies*

$$\frac{\partial w}{\partial s} + (\partial H)^3 w = a(\partial H)^3 b \qquad\qquad w(0) = 0,$$

then the following bound on w is true:

$$||w||_{\rho,S} \le C||a||_{\rho,S}||b||_{\rho,S}.$$

Proof. The jth Fourier coefficient of the equation satisfies the following:

$$\dot{w}_j + |j|^3 w_j = \sum_{k+l=j} a_k |l|^3 b_l$$

$$\frac{d}{ds}(e^{|j|^3 s} w_j) = e^{|j|^3 s} \sum_{k+l=j} a_k |l|^3 b_l$$

$$e^{|j|^3 s} w_j(s) = \int_0^s e^{|j|^3 \sigma} \sum_{k+l=j} a_k |l|^3 b_l \, d\sigma$$

$$w_j(s) = \int_0^s e^{-|j|^3(s-\sigma)} \sum_{k+l=j} a_k |l|^3 b_l \, d\sigma$$

$$|w_j(s)| \le \int_0^s e^{-|j|^3(s-\sigma)} \sum_{k+l=j} |a_k||l|^3|b_l| \, d\sigma$$

$$|w_j(s)|\rho^{|j|} \le \int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|>2|j|} |a_k|\rho^{|k|}|l|^3|b_l|\rho^{|l|}\rho^{-(|l|+|k|-|j|)} \, d\sigma$$
$$+\rho^{|j|}\int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|\le 2|j|} |a_k||l|^3|b_l| \, d\sigma = I_1 + I_2.$$

For the first integral, use the following: if $l+k=j$ and $|l|>2|j|$, then $|k|>|j|$ and hence $\rho^{-(|l|+|k|-|j|)} < \rho^{-|l|}$, so

$$|w_j(s)|\rho^{|j|} \le \int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|>2|j|} |a_k|\rho^{|k|}|l|^3|b_l|\rho^{|l|}\rho^{-|l|} \, d\sigma + I_2.$$

Let $K = \sup_{l\in Z}(|l|^3\rho^{-|l|})$. Then

$$|w_j(s)|\rho^{|j|} \le K\int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|>2|j|} |a_k|\rho^{|k|}|b_l|\rho^{|l|} \, d\sigma + I_2$$

$$\sup_{0\le s\le S}|w_j(s)|\rho^{|j|} \le K\sum_{|l|>2|j|}\sup_{0\le s\le S}|a_k(s)|\rho^{|k|}\sup_{0\le s\le S}|b_l(s)|\rho^{|l|}\int_0^s e^{-|j|^3(s-\sigma)}\,d\sigma$$
$$+ I_2.$$

The integral in time above is bounded, since $j \neq 0$, and we get

$$\sup_{0\le s\le S}|w_j(s)|\rho^{|j|} \le K\sum_{k+l=j}\sup_{0\le s\le S}|a_k(s)|\rho^{|k|}\sup_{0\le s\le S}|b_l(s)|\rho^{|l|} + \text{ 2nd integral.}$$

Now to bound I_2:

$$I_2 = \rho^{|j|}\int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|\le 2|j|} |a_k||l|^3|b_l| \, d\sigma$$
$$\le \int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|\le 2|j|} 8|j|^3|a_k|\rho^{|k|}|b_l|\rho^{|l|}\rho^{-(|k|+|l|-|j|)} \, d\sigma.$$

We know $\rho^{-(|k|+|l|-|j|)} \le 1$. Hence

$$I_2 \le \int_0^s e^{-|j|^3(s-\sigma)} \sum_{|l|\le 2|j|} 8|j|^3|a_k|\rho^{|k|}|b_l|\rho^{|l|} \, d\sigma$$
$$\le 8\sum_{|l|\le 2|j|}\sup_{0\le s\le S}|a_k(s)|\rho^{|k|}\sup_{0\le s\le S}|b_l(s)|\rho^{|l|}\int_0^s |j|^3 e^{-|j|^3(s-\sigma)}\,d\sigma.$$

The last integral is less than 1. Note that the boundedness of this integral depended on the nonlinear term having no more derivatives than the linear part. Again, we sum over all terms

$$\le 8\sum_{k+l=j}\sup_{0\le s\le S}|a_k(s)|\rho^{|k|}\sup_{0\le s\le S}|b_l(s)|\rho^{|l|}.$$

Putting these two bounds together, and taking $C = K + 8$, we have

$$\sup_{0\le s\le S} |w_j(s)|\rho^{|j|} \le C \sum_{k+l=j} \sup_{0\le s\le S} |a_k(s)|\rho^{|k|} \sup_{0\le s\le S} |b_l(s)|\rho^{|l|}.$$

Summing in j,

$$\sum_{j=-\infty}^{\infty} \sup_{0\le s\le S} |w_j(s)|\rho^{|j|} \le C \sum_{k=-\infty}^{\infty} \sup_{0\le s\le S} |a_k|\rho^{|k|} \sum_{l=-\infty}^{\infty} \sup_{0\le s\le S} |b_l(s)|\rho^{|l|}.$$

The above inequality is related to our norm. Our norm involves ρ^js, whereas we have $\rho^{|j|}$ above. However, since our functions are real valued, the two norms are equivalent. This implies

$$||v||_{\rho,S} \le C||a||_{\rho,S}||b||_{\rho,S} \qquad \text{as desired.}$$

□

The above lemma also holds for equations in which the nonlinearity is of the form

$$a(\partial H)^2 b,\ a\partial H b,\ \partial H(a(\partial H)^2 b), \text{ or } (\partial H)^2(a\partial H b),$$

as well as those of the form $(\partial a)(Hb)$.

The lemma is also true for equations of the form

$$\frac{\partial w}{\partial s} - \partial H(1 - (\partial H)^2)w = a(\partial H)^3 b.$$

Lemma 3.2. *If w satisfies*

$$\frac{\partial w}{\partial s} + (\partial H)^3 w = \partial H(a\partial H(b\partial H c)),$$

then the following bound on w is true:

$$||w||_{\rho,S} \le C||a||_{\rho,S}||b||_{\rho,S}||c||_{\rho,S}.$$

Proof. The right hand side is of the form

$$|j| \sum_{k+l=j} a_k|l| \sum_{p+q=l} b_p|q|c_q.$$

Repeat the above argument, breaking the sum into four sums:

$$\sum_{|l|<2|j|\ |q|<3|j|} \qquad \sum_{|l|\ge 2|j|\ |q|<3|j|} \qquad \sum_{|l|<2|j|\ |q|\ge 3|j|} \qquad \sum_{|l|\ge 2|j|\ |q|\ge 3|j|}.$$

□

We use these lemmas to prove convergence of the iteration scheme.

Lemma 3.3. *Let ρ, S be arbitrary. Then the solution w_n of*

$$\frac{\partial w_n}{\partial s} - N'(v_{n-1})w_n = N(v_{n-1}) - N(v_{n-2}) - N'(v_{n-2})w_{n-1},$$

$$w_n(z,0) = 0 \qquad \textit{for } n \ge 2,$$

satisfies the inequality

$$||w_n||_{\rho,S} \le \frac{CB||w_{n-1}||^2_{\rho,S}}{1 - CA},$$

where given $\epsilon > 0$ there exists a $\delta > 0$ such that $||v_{n-1}||_{\rho,S} < \delta$ implies Max$(A, B) < \epsilon$.

We sketch the proof in what follows.

Writing the equation (10) out in full we get a large number of terms. One of the terms from $N'(v_{n-1})w_n$ is $-\partial H(1-(\partial H)^2)w_n$. We keep this term on the left hand side of the equation, and move the remaining terms to the right hand side. We then use our lemma to bound each term on the right hand side.

There are two types of terms on the right hand side: those from $N'(v_{n-1})w_n$ and those from $N(v_{n-1}) - N(v_{n-2}) - N'(v_{n-2})w_{n-1}$. Each is treated as follows:

A sample term from $N'(v_{n-1})w_n$ is

$$\partial H((e^{-2v_{n-1}} - 1)\partial H(w_n e^{-v_{n-1}})).$$

This is bounded by

$$Ce^r ||v_{n-1}||_{\rho,S} ||w_n||_{\rho,S} ||e^{v_{n-1}}||_{\rho,S}.$$

Here we are assuming $||v_i||_{\rho,S} \leq r$ and use one of the Taylor expansion inequalities. Note that the above term is of the form

$$||w_n||_{\rho,S} \cdot (\text{ something that is small if } ||v_{n-1}||_{\rho,S} \text{ is small }).$$

All the terms from $N'(v_{n-1})w_n$ are bounded in this way.

We group the terms from $N(v_{n-1}) - N(v_{n-2}) - N'(v_{n-2})w_{n-1}$ in such a way that as to get the desired quadratic behavior in w_{n-1}. For example, one group of 3 terms reduces to

$$\partial H\left(e^{-2v_{n-1}} \partial H((e^{-v_{n-1}} - e^{-v_{n-2}} + w_{n-1}e^{-v_{n-1}})\partial H v_{n-1})\right).$$

The lemma bounds this by

$$C||e^{-2v_{n-1}}||_{\rho,S} ||e^{-v_{n-1}} - e^{-v_{n-2}} + w_{n-1}e^{-v_{n-2}}||_{\rho,S} ||v_{n-1}||_{\rho,S}$$

$$C \leq e^{||v_{n-1}||_{\rho,S}} \frac{1}{2} e^r ||w_{n-1}||^2_{\rho,S} ||v_{n-1}||_{\rho,S}.$$

Again, we used one of the Taylor expansion inequalities and the fact that $B_{\rho,S}$ is a Banach algebra. We assume throughout that $||v_i||_{\rho,S} \leq r$. Proceeding in this way, we arrive at an inequality of the form

$$||w_n||_{\rho,S} \leq CA||w_n||_{\rho,S} + CB||w_{n-1}||^2_{\rho,S}.$$

Since A and B can be taken arbitrarily small,

$$||w_n||_{\rho,S} \leq \frac{CB||w_{n-1}||^2_{\rho,S}}{1 - CA}$$

when A is such that $AC < 1$.

Terms like $P_0(E(v_{n-1}))$ were not mentioned in the above analysis. Observe that for our equations,

$$w_{n0}(0) = 0 \qquad \text{and} \qquad \frac{dw_{n0}}{ds} = 0.$$

Here w_{n0} represents the 0th Fourier coefficient of the nth difference. Hence $w_{n0}(s) = 0$ for all s. For our norms, this implies that

$$||w_n||_{\rho,S} = ||w_n - w_{n0}||_{\rho,S} \leq ||a_n - a_{n0}||_{\rho,S} \leq ||a_n||_{\rho,S}.$$

Here $a_n - a_{n0}$ represents any term from the right hand side. (They all have no zero mode or have had their zero mode subtracted off.) This is why the projected terms have no contribution.

Thus we have bounds for all the difference equations except the first one:

$$\frac{\partial w_1}{\partial s} = N(v_0) + N'(v_0)w_1.$$

Proceeding in the same manner as above, we find a bound

$$\text{terms from } N'(v_0)w_1 \ \leq A||w_1||_{\rho,S},$$

where A is the same constant as above, and

$$\text{terms from } N(v_0) \ \leq \tilde{B}||v_0||_{\rho,S}.$$

This proves the following lemma:

Lemma 3.4. *Let ρ, S be arbitrary. Then the solution w_1 of*

$$\frac{\partial w_1}{\partial s} = N(v_0) + N'(v_0)w_1,$$
$$w_1(z,0) = 0,$$

satisfies the inequality

$$||w_1||_{\rho,S} \leq \frac{C\tilde{B}||v_0||_{\rho,S}}{1-CA}.$$

A is the same as in the previous lemma and $\tilde{B}$ can be taken arbitrarily small.

We therefore prove by induction that the series $\sum ||w_n||_{\rho,S}$ converges. This implies the following

Theorem 3.5. *There exist two numbers ϵ and C such that, if initial datum v_0 satisfies*

$$||v_0||_\rho \leq \epsilon$$

for some $\rho > 0$, there exists, for all s, a solution to the reparametrized equation $v(w,s)$, belonging to $B_{\rho,S}$ for all $S > 0$ and satisfying

$$||v||_{\rho,S} \leq C||v_0||_\rho.$$

4. Stability

We now show that the solution decays to a circle. This requires showing that all the non-constant modes decay to zero.

First we define a new norm.

$$||v||_\alpha^2 = \sum_{j=-\infty}^{\infty} |j|^{2a}|v_j|^2.$$

This is a Sobolev norm, it is the same as

$$||v||_\alpha^2 = \frac{1}{2\pi}\int_0^{2\pi} |\partial^\alpha v|^2\, d\theta$$

when α is an integer. Since all of our functions have zero mean, this is a norm, rather than a seminorm.

We use these the following inequalities:

Hölder's Inequality:

$$\int |fg|\, d\mu \leq (\int |f|^p\, d\mu)^{\frac{1}{p}}(\int |g|^q\, d\mu)^{\frac{1}{q}} \qquad \text{where } \frac{1}{p}+\frac{1}{q}=1.$$

Young's Inequality:

$$|a||b| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$$

where p and q are conjugate exponents as above.

Boundedness of the Hilbert transform as an operator from $L^p([0,2\pi])$ to $L^p([0,2\pi])$ for $1 < p < \infty$:

$$\int_0^{2\pi} |Hf(x)|^p\, dx \leq C_p \int_0^{2\pi} |f(x)|^p\, dx.$$

Gagliardo-Nirenberg Inequality:

$$\int |\partial^\alpha f(x)|^{\frac{2s}{\alpha}}\, dx \leq C_{\alpha,s}||f||_\infty^{\frac{2s}{\alpha}-2}\int |\partial^s f(x)|^2\, dx \qquad \text{where } 1 \leq \alpha \leq s.$$

(9) is an equation of the form

$$\frac{dv}{ds} = N(v).$$

Taking the inner product with $(\partial H)^5 v$, we bound the term $\langle N(v), (\partial H)^5 v\rangle$ as follows:

$$\frac{1}{2}\frac{d}{ds}||v||_{5/2}^2 = \langle N(v), (\partial H)^5 v\rangle \leq -A||v||_4^2,$$

where A depends on $||v||_\alpha$ in such a way that, if $||v||_\alpha$ is sufficiently small, A is positive.

Assume this bound is true. Since we know $||v||_{5/2} \leq ||v||_4$,

$$\frac{d}{ds}||v||_{5/2}^2 \leq -A||v||_4^2 \leq -A||v||_{5/2}^2 < 0.$$

Taking $\alpha \leq \frac{5}{2}$, the decay in $||v||_{5/2}$ ensures that $||v||_\alpha$ remains sufficiently small to keep $||v||_{5/2}$ decreasing. Hence

$$||v||_{5/2}^2(s) \leq ||v||_{5/2}^2(0)e^{-As},$$

as desired.

This shows that $||v||_{5/2}$ decays to zero as time goes to infinity.

Thus it suffices to prove the above bound.

The equation (9) is

$$\begin{aligned}\frac{dv}{ds} = \partial H(1-(\partial H)^2)v - (1-\partial H)\left[E(v) + \partial H(1+\partial H)v\right] - P_0(E(v))& \\ - (\partial H v)(E(v)) - (\partial v)(HE(v)),&\end{aligned}$$

where

$$E(v) = -e^{-2v}(\partial H(e^{-v}(1-\partial H v))).$$

The first term of the right hand side of the equation is linear and has negative eigenvalues. (The ± 1 modes are zero.) The second term is at least quadratic in v, although this is not immediate on inspection. The fourth and fifth terms are also at least quadratic in v. Since $v = 0$ is a solution, if v is small in some appropriate norm, the right hand side should be dominated by the linear term, implying decay of v in time.

The first thing we do is write the second term so that, rather than being a sum of at most linear terms whose sum is at least quadratic, it is a sum of at least quadratic terms:

$$\begin{aligned}E(v) + \partial H(1+\partial H)v = \partial H(v\partial H v + (e^{-v} - 1 + v) - (e^{-v} - 1 + v)\partial H v)& \\ + (e^{-2v} - 1)\partial H(e^{-v}(1-\partial H v)).&\end{aligned}$$

Take the inner product, $\langle N(v), (\partial H)^5 v\rangle$, and use linearity to write it as a sum of inner products.

We bound specific terms.

First, the linear term:

$$\langle \partial H(1-(\partial H)^2)v, (\partial H)^5 v\rangle = \sum_{-\infty}^{\infty} |k|^6(1-|k|^2)|v_k|^2 \leq \sum_{-\infty}^{\infty} -|k|^8|v_k|^2 = -||v||_4^2.$$

Now, the projected term:

$$\langle P_0(E(v)), (\partial H)^5 v\rangle = (E(v))_0((\partial H)^5 v)_0 = 0.$$

This vanishes since $\partial H v$ has no zero mode.

The quadratic terms are all that remain. We now bound a specific term:

$$\frac{1}{2}\frac{d}{dt}||v||_{5/2}^2 = \text{Terms } - \langle(\partial H v)(E(v)), (\partial H)^5 v\rangle.$$

First use the fact that ∂H is self-adjoint:

$$\begin{aligned}-\langle(\partial Hv)(E(v)),(\partial H)^5 v\rangle &= -\langle\partial H((\partial Hv)(E(v))),(\partial H)^4 v\rangle\\ &\le |\langle\partial H((\partial Hv)(E(v))),(\partial H)^4 v\rangle| \le ||v||_4||\partial H((\partial Hv)(E(v))||_{L^2}.\end{aligned}$$

It suffices to bound $||\partial H((\partial Hv)(E(v)))||_{L^2}$ by $C||v||_4||v||_\alpha$, since, we can take the coefficient of $||v||_4^2$ to be small.

In the following we use C for the constant, although it may be a different (larger) constant after each step.

$$\begin{aligned}||\partial H((\partial Hv)(E(v)))||_{L^2} &= \sqrt{\int_0^{2\pi}|\partial H((\partial Hv)(E(v)))|^2}\\ &= \sqrt{\int_0^{2\pi}|H\partial((\partial Hv)(E(v)))|^2} \le C\sqrt{\int_0^{2\pi}|\partial((\partial Hv)(E(v)))|^2}.\end{aligned}$$

Here we used the fact that H and ∂ commute and that the Hilbert transform is a bounded operator.

$$\begin{aligned}&\le C\sqrt{\int_0^{2\pi}|(\partial\partial Hv)(E(v))|^2+\int_0^{2\pi}|(\partial Hv)(\partial E(v))|^2}\\ &\le C\left(\sqrt{\int_0^{2\pi}|(\partial\partial Hv)(E(v))|^2}+\sqrt{\int_0^{2\pi}|(\partial Hv)(\partial E(v))|^2}\right).\end{aligned}$$

We bound each of the above terms separately. In the first term, notice that it is the product of two terms, each of which has two derivatives. We use Young's inequality with $p=2$ and $q=2$. This choice of p and q does not put a higher L^p norm on one or another, since they are essentially the same, each having two derivatives.

$$\begin{aligned}\le C\sqrt{\int_0^{2\pi}|(\partial\partial Hv)(E(v))|^2} &\le C(\sqrt{\int_0^{2\pi}|\partial\partial Hv|^4+\int_0^{2\pi}|E(v)|^4})\\ &\le C\left(\sqrt{\int_0^{2\pi}|H(\partial)^2v|^4}+\sqrt{\int_0^{2\pi}|E(v)|^4}\right)\\ &\le C\left(\sqrt{\int_0^{2\pi}|(\partial)^2v|^4}+\sqrt{\int_0^{2\pi}|e^{-2v}\partial H(e^{-v}(1-\partial Hv))|^4}\right)\\ &\le C\left(\sqrt{\int_0^{2\pi}|(\partial)^2v|^4}+||e^{-2v}||_\infty^2\sqrt{\int_0^{2\pi}|H\partial(e^{-v}(1-\partial Hv))}\right)=X.\end{aligned}$$

Now use the Gagliardo-Nirenberg inequality with $\alpha = 2$ and $s = 4$ to bound $\int_0^{2\pi} |(\partial)^2 v|^4$

$$\begin{aligned}
X &\leq C(||v||_\infty ||v||_4 + ||e^{-2v}||_\infty^2 \sqrt{\int_0^{2\pi} |\partial(e^{-v}(1-\partial H v))|^4})\\
&\leq C(||v||_\infty ||v||_4 + ||e^{-2v}||_\infty^2 (\sqrt{\int_0^{2\pi} |\partial v|^4 |e^{-v}|^4} + \sqrt{\int_0^{2\pi} |\partial v|^4 |\partial H v|^4 |e^{-v}|^4}\\
&\qquad + \sqrt{\int_0^{2\pi} |e^{-v}|^4 |(\partial)^2 H v|^4}))\\
&\leq C(||v||_\infty ||v||_4 + ||e^{-2v}||_\infty^2 ||e^{-v}||_\infty^2 (\sqrt{\int_0^{2\pi} |\partial v|^4}\\
&\qquad + \sqrt{\int_0^{2\pi} |\partial v|^4 |\partial H v|^4} + \sqrt{\int_0^{2\pi} |(\partial)^2 v|^4})).
\end{aligned}$$

The Gagliardo-Nirenberg inequality bounds the first and third term above. We use Young's inequality with $p = 2$ and $q = 2$ (because the number of derivatives is equal in each term) so that we can then apply the Gagliardo-Nirenberg inequality.

$$\begin{aligned}
X &\leq C(||v||_\infty ||v||_4 + ||e^{-2v}||_\infty^2 ||e^{-v}||_\infty^2 (||v||_\infty ||v||_2 + \sqrt{\int_0^{2\pi} |\partial v|^8} + ||v||_\infty ||v||_4))\\
&\leq C(||v||_\infty ||v||_4 + ||e^{-2v}||_\infty^2 ||e^{-v}||_\infty^2 (||v||_\infty ||v||_2 + ||v||_\infty^3 ||v||_4 + ||v||_\infty ||v||_4)).
\end{aligned}$$

Using the fact that $||v||_2 \leq ||v||_4$ and a Sobolev theorem that tells us $||v||_\infty \leq ||v||_\alpha$ if $\alpha > \frac{\dim}{2}$, we bound the above as follows:

$$X \leq C(||v||_\alpha ||v||_4 + ||e^{-2v}||_\alpha^2 ||e^{-v}||_\alpha^2 (||v||_\alpha ||v||_4 + ||v||_\alpha^3 ||v||_4 + ||v||_\alpha ||v||_4)).$$

In this way, we have shown

$$||(\partial \partial H v)(E(v))||_{L^2} \leq C ||v||_\alpha ||v||_4.$$

The constant C above depends on $||v||_\alpha$ in a way that one can easily see from the bounds, but we are not too interested in its behavior. We now bound the second term from the inner product:

$$||(\partial H v)(\partial E(v))||_{L^2} = \sqrt{\int_0^{2\pi} |\partial H v|^2 |\partial E(v)|^2} = Y.$$

The first thing we do is apply Young's inequality. We know that the $\partial E(v)$ term yields a term with three derivatives on v. The Gagliardo-Nirenberg inequality would suggest a power of $\frac{8}{3}$ on this term. This would come from a use of $p = 4$ and $q = \frac{4}{3}$ in Young's inequality. This looks promising, as it yields a power of 8 on the $\partial H v$ term, which is consistent with our bounds from above. This type of logic determines p and q in the following.

$$\begin{aligned}
Y &\leq C\Big(\sqrt{\int_0^{2\pi} |\partial H v|^8} + \sqrt{\int_0^{2\pi} |\partial E(v)|^{\frac{8}{3}}}\Big)\Big) \\
&\leq C(||v||_\infty^3 ||v||_4 + \sqrt{\int_0^{2\pi} |\partial v|^{\frac{8}{3}} |E(v)|^{\frac{8}{3}}} + \sqrt{\int_0^{2\pi} |e^{-2v}\partial^2 H(e^{-v} - e^{-v}\partial H v)|^{\frac{8}{3}}}) \\
&\leq C(||v||_\infty^3 ||v||_4 + \sqrt{\int_0^{2\pi} |\partial v|^8} + \sqrt{\int_0^{2\pi} |E(v)|^4} \\
&\qquad + \sqrt{\int_0^{2\pi} |e^{-2v}|^{\frac{8}{3}} |\partial^2 H(e^{-v} - e^{-v}\partial H v)|^{\frac{8}{3}}}).
\end{aligned}$$

We used Young's inequality again here, with $p = 3$ and $q = \frac{3}{2}$. The L^4 norm of $E(v)$ was bounded above.

$$Y \leq C(||v||_\infty^3 ||v||_4 + ||v||_\alpha ||v||_4 + ||e^{-2v}||_\infty^{\frac{4}{3}} \sqrt{\int_0^{2\pi} |\partial^2(e^{-v} - e^{-v}\partial H v)|^{\frac{8}{3}}}.$$

As before, we expand this out and bound each term.

$$\begin{aligned}
Y \leq C(&||v||_\infty^3 ||v||_4 + ||v||_\alpha ||v||_4 + (||e^{-2v}||_\infty ||e^{-v}||_\infty)^{\frac{4}{3}} \Big(\sqrt{\int_0^{2\pi} |\partial^2 v|^{\frac{8}{3}}} \\
&+ \sqrt{\int_0^{2\pi} |\partial v|^{\frac{16}{3}}} + \sqrt{\int_0^{2\pi} |(\partial^2 v)(\partial H v)|^{\frac{8}{3}}} + \sqrt{\int_0^{2\pi} |((\partial v)^2)(\partial H v)|^{\frac{8}{3}}} \\
&+ \sqrt{\int_0^{2\pi} |(\partial v)(\partial^2 H v)|^{\frac{8}{3}}} + \sqrt{\int_0^{2\pi} |\partial^3 H v|^{\frac{8}{3}}}\Big)\Big).
\end{aligned}$$

Proceed as before, applying Young's inequality and then the Gagliardo-Nirenberg inequality. We apply Young's inequality as follows: on the third term, $p = \frac{3}{2}$, on the fourth term, $p = \frac{3}{2}$, and on the fifth term, $p = 3$. We find a bound of the form:

$$||(\partial H v)(\partial E(v))||_{L^2} \leq C||v||_\alpha^{\frac{1}{3}} ||v||_4.$$

Hence:

$$-\langle(\partial H v)(E(v)), (\partial H)^5 v\rangle \leq C(||v||_\alpha + ||v||_\alpha^{\frac{1}{3}})||v||_4^2.$$

Repeating these arguments for each term of $N(v)$ with quadratic behavior, we arrive at the following:

$$\frac{1}{2}\frac{d}{ds}||v||_{5/2}^2 \leq -||v||_4^2 + C(||v||_\alpha^{\frac{1}{3}} + ||v||_\alpha + \dots)||v||_4^2$$

C multiplies more terms than shown above, but they are all of the sort that become small when $||v||_\alpha$ is small. We take $||v||_\alpha$ small enough to make

$$1 - C(||v||_\alpha^{\frac{1}{3}} + ||v||_\alpha) > 0.$$

This proves

$$\frac{1}{2}\frac{d}{ds}||v||_{5/2}^2 \leq -A||v||_4^2.$$

Taking $\frac{1}{2} < \alpha \leq \frac{5}{2}$ ensures that the decay of $||v||_{5/2}$ keeps $||v||_\alpha$ small enough to keep $||v||_{5/2}$ decaying. We have therefore proven the following:

Theorem 4.1. *There exists $\delta > 0$ and $A > 0$ such that a solution v to equation (9) with initial data $v(0)$ such that*

$$||v_0||_{5/2} < \delta$$

implies

$$||v||_{5/2}(s) \leq ||v(0)||_{5/2} e^{-As}.$$

Therefore the solution to the reparametrized equation decays. We now argue that if we had a global solution to the original equation, it would also decay.

The original equation is of the form

$$\frac{\partial v}{\partial t} = e^{-3u_0(t)} N(v).$$

Given a solution to this equation, we again take the inner product with $(\partial H)^5 v$ to find:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||v||_{5/2}^2 =& \langle e^{-3u_0(t)} N(v), (\partial H)^5 v\rangle \\ =& e^{-3u_0(t)} \langle N(v), (\partial H)^5 v\rangle \leq -e^{-3u_0(t)} A||v||_4^2.\end{aligned}$$

Recall the length of the interface,

$$L(t) = \frac{1}{2\pi}\int_0^{2\pi} e^u \, d\alpha.$$

Jensen's inequality implies that

$$u_0(t) \leq \log(L(t)) \leq \log(L(0)) = u_0(0).$$

Hence

$$\frac{1}{2}\frac{d}{dt}||v||_{5/2}^2 \leq -e^{-3u_0(t)} A||v||_4^2 \leq -e^{-3u_0(0)} A||v||_4^2 \leq -e^{-3u_0(0)} A||v||_{5/2}^2.$$

Therefore decay for the original equation follows from decay for the reparametrized equation. Now we prove that global existence for the reparametrized equation implies global existence for the original equation.

5. Existence for the Original Equations

We have already proven global existence for small initial data for the equation

$$\frac{\partial \tilde{v}}{\partial s} = N(\tilde{v}) \qquad \text{where} \qquad \tilde{v}(\alpha, 0) = v_0(\alpha).$$

We use the solution to this equation to define a solution for

$$\frac{du_0}{dt} = e^{-3u_0(t)} P_0(E(v))$$

$$\frac{\partial v}{\partial t} = e^{-3u_0(t)} N(v) \qquad \text{where} \qquad v(\alpha, 0) = v_0(\alpha).$$

Note that we have the same initial conditions, from the time reparametrization. We suppress the dependence on α in what follows.

Define the function $v(t)$ as follows

$$\tilde{v}(s) = v(t(s)).$$

$v(t)$ is to satisfy the original equation, so we must have

$$N(\tilde{v}(s)) = \frac{\partial \tilde{v}}{\partial s} = \frac{\partial v}{\partial t}(t(s))\frac{dt}{ds}(s) = e^{-3u_0(t(s))} N(v(t(s)))\frac{dt}{ds}(s).$$

Hence the reparametrization $t(s)$ satisfies

$$1 = e^{-3u_0(t(s))}\frac{dt}{ds}(s) \qquad \text{for all } s.$$

and $\frac{dt}{ds}$ never vanishes.

We also know

$$\begin{aligned}
\frac{du_0}{dt} &= e^{-3u_0(t)}P_0(E(v(t))) \\
\frac{d}{dt}e^{3u_0(t)} &= 3P_0(E(v(t))) \\
e^{3u_0(t)} &= e^{3u_0(0)} + 3\int_0^t P_0(E(v(\tau)))\, d\tau.
\end{aligned}$$

We now change coodinates, viewing each t and τ as $t(s)$ and $t(\sigma)$ for appropriate s and σ. Set $t(0) = 0$.

$$\begin{aligned}
e^{3u_0(t(s))} &= e^{3u_0(0)} + 3\int_0^{t(s)} P_0(E(v(t(\sigma))))\frac{dt}{ds}(\sigma)\, d\sigma \\
e^{3u_0(t(s))} &= e^{3u_0(0)} + 3\int_0^{s} P_0(E(\tilde{v}(\sigma)))\frac{dt}{ds}(\sigma)\, d\sigma.
\end{aligned}$$

For the reparametrization to be consistent with the equations for $\frac{du_0}{dt}$ and $\frac{\partial v}{\partial t}$, it must satisfy

$$\frac{dt}{ds}(s) = e^{3u_0(0)} + 3\int_0^{s} P_0(E(\tilde{v}(\sigma)))\frac{dt}{ds}(\sigma)\, d\sigma.$$

Differentiating with respect to s, we find

$$\begin{aligned}
\frac{d}{ds}\frac{dt}{ds}(s) &= 3P_0(E(\tilde{v}(s)))\frac{dt}{ds}(s) \\
\frac{dt}{ds}(s) &= \frac{dt}{ds}(0)e^{3\int_0^s P_0(E(\tilde{v}(\sigma)))\, d\sigma}.
\end{aligned}$$

To prove global existence for the original equations, we show that as s goes to infinity, $t(s)$ goes to infinity. It suffices to show

$$\frac{dt}{ds}(s) = \frac{dt}{ds}(0)e^{3\int_0^s P_0(E(\tilde{v}(\sigma)))\, d\sigma} \geq C > 0 \quad \text{for all } s > 0.$$

This requires that

$$\int_0^s P_0(E(\tilde{v}(\sigma)))\, d\sigma > C \quad \text{for all } s > 0.$$

It suffices to find a C such that

$$\begin{aligned}
\left|\int_0^s P_0(E(\tilde{v}(\sigma)))\, d\sigma\right| &< C \quad \text{for all } s > 0. \\
\left|\int_0^s P_0(E(\tilde{v}(\sigma)))\, d\sigma\right| &\leq \int_0^\infty |P_0(E(\tilde{v}(\sigma)))|\, d\sigma \\
&= \int_0^\infty \frac{1}{2\pi}\left|\int_0^{2\pi} E(\tilde{v}(\alpha,\sigma))\, d\alpha\right| d\sigma \\
&\leq C\int_0^\infty \sqrt{\int_0^{2\pi} |E(\tilde{v}(\alpha,\sigma))|^2\, d\alpha}\, d\sigma.
\end{aligned}$$

Here the L^2 norm of $E(\tilde{v})$ is viewed as a function of σ. It is bounded by the H^2 norm of $\tilde{v}$.

$$\begin{aligned}\int_0^{2\pi} |E(\tilde{v}(\alpha,\sigma))|^2\,d\alpha &= \int_0^{2\pi} |e^{-2\tilde{v}}\partial H(e^{-\tilde{v}} - e^{-\tilde{v}}\partial H\tilde{v})|^2\,d\alpha \\ &\le ||e^{-2\tilde{v}}(\cdot,\sigma)||_\infty^2 \int_0^{2\pi} |H\partial(e^{-\tilde{v}} - e^{-\tilde{v}}\partial H\tilde{v})|^2\,d\alpha \\ &\le C||e^{-2\tilde{v}}(\cdot,\sigma)||_\infty^2 \int_0^{2\pi} |\partial(e^{-\tilde{v}} - e^{-\tilde{v}}\partial H\tilde{v})|^2\,d\alpha \\ &\le C||e^{-2\tilde{v}}(\cdot,\sigma)||_\infty^2 ||e^{-\tilde{v}}(\cdot,\sigma)||_\infty^2 (\int_0^{2\pi} |\partial\tilde{v}|^2 + |(\partial\tilde{v})(\partial H\tilde{v})|^2 + |\partial^2 H\tilde{v}|^2) \\ &\le C||e^{-2\tilde{v}}(\cdot,\sigma)||_\infty^2 ||e^{-\tilde{v}}(\cdot,\sigma)||_\infty^2 (||\tilde{v}||_2^2 + ||\tilde{v}||_\infty^2 ||\tilde{v}||_2^2 + ||\tilde{v}||_2^2).\end{aligned}$$

From our previous work we know

$$||e^{-2\tilde{v}}(\cdot,\sigma)||_\infty^2 \le C||e^{-2\tilde{v}}(\cdot,\sigma)||_{5/2}^2 \le C||e^{-2\tilde{v}}||_{5/2}^2(0)e^{-A\sigma} \le C||e^{-2\tilde{v}}||_{5/2}^2(0).$$

We bound all the terms except for the $||\tilde{v}(\cdot,\sigma)||_2$ term in this way:

$$\int_0^{2\pi} |E(\tilde{v}(\alpha,\sigma))|^2\,d\alpha \le C||\tilde{v}(\cdot,\sigma)||_2^2.$$

In this way the problem is reduced to

$$\int_0^\infty \sqrt{\int_0^{2\pi} |E(\tilde{v}(\alpha,\sigma))|^2\,d\alpha}\,d\sigma \le C\int_0^\infty ||\tilde{v}(\cdot,\sigma)||_2\,d\sigma.$$

Since

$$||\tilde{v}||_2(\sigma) \le ||\tilde{v}||_{5/2}(\sigma) \le Ce^{-\frac{A}{2}\sigma},$$

$$\int_0^\infty \sqrt{\int_0^{2\pi} |E(\tilde{v}(\alpha,\sigma))|^2\,d\alpha}\,d\sigma \le C\int_0^\infty e^{-A\sigma}\,d\sigma \le C.$$

This is the desired bound

$$\frac{dt}{ds}(s) = \int_0^s P_0(E(\tilde{v}(\sigma)))\,d\sigma \ge C \quad \text{for all } s > 0.$$

Therefore

$$Cs \le t(s) \quad \text{for all } s > 0,$$

and global existence for $\tilde{v}(s)$ implies global existence for $v(t)$.

A simple argument using length contraction shows there is another constant K such that

$$t(s) \le Ks,$$

so the reparametrization is controlled as follows:

$$Cs \le t(s) \le Ks.$$

We use the lower bound $Cs \le t(s)$ to prove the following:

Theorem 5.1. *There exist two numbers ϵ and C such that, if initial datum v_0 satisfies*

$$||v_0|| \le \epsilon$$

for some $\rho > 0$, then there is a solution $v(\alpha,t)$ to the original equations which belongs to $B_{\rho,T}$ for all $T > 0$ and satisfies

$$||v||_{\rho,T} \le C||v_0||_\rho.$$

6. Conclusions

We found a framework for the proof of existence of solutions of Hele-Shaw problems based on a real dependent variable. We identified the structure of the equations, the role played by the surface tension and showed by example that there exist equations of Hele-Shaw type with artificial surface tension which display length contraction, area preservation and neutral linear stability of circular puddles but for which, nevertheless, these puddles are nonlinearly catastrophically unstable. For the original Hele-Shaw problem we proved that small analytic perturbations of circular puddles lead to global analytic solutions which become circular in infinite time. The rate of decay in time is exponential. Local (in time) existence of analytic solutions for arbitrary initial data, in the presence of pumping, can be obtained easily by the same technique. Although there is a simple Lyapunov function for this problem - the length of the interface - there is no argument based solely on it. In particular, we were unable to prove that large perturbations of the circle decay to a single circle. If large perturbations actually do not do this, another possible behavior is pinching off and decaying to several circles - which, in our language, means passing through singularities.

7. Acknowledgements

It is our pleasure to thank Leo Kadanoff for numerous interesting discussions. P.C. acknowledges partial support by NSF.

8. Appendix

We now discuss how the choice of M in (4) affects the problem. The equation is:

$$\frac{\partial u}{\partial t} = (I - \partial H)E(u) - (\partial H u)(E(u)) - (\partial u)(HE(u)) \tag{11}$$

where

$$E(u) = e^{-2u}(-M\kappa(u)).$$

The linearization about the steady solution $u = 0$ is of interest:

$$\frac{\partial v}{\partial t} = -(I - \partial H)M\kappa'(0)v.$$

Assume that M acts on κ by convolution, hence by multiplication on the Fourier modes. In this case, M commutes with differentiation and the Hilbert transform. If we take $\kappa(h)$ to be the curvature of the interface,

$$\kappa(h) = e^{-u}(1 - \partial H u),$$

the linearization is of the form

$$\frac{\partial v}{\partial t} = (I - (\partial H)^2)Mv \qquad \text{and} \qquad \frac{dv_k}{dt} = (1 - |k|^2)M_k v_k.$$

Since $M_k \geq 0$ for length contraction, this implies that no matter what the choice of M, the circular solution is linearly neutrally stable.

In the following, we show that length contraction and area conservation are not sufficient for asymptotic decay to the cicular solution by showing the quadratic approximation of (11) to be ill-posed for a certain choice of M that provides length contraction and area conservation.

If we assume $\kappa(u) = e^{-u}(1 - \partial H u)$, the quadratic approximation of (11) is

$$\begin{aligned}\frac{\partial u}{\partial t} = (1 - (\partial H)^2)Mu - 2(1 - \partial H)(uM(1 + \partial H)u) - (1 - \partial H)M(u\partial H u + \frac{u^2}{2}) \\ - (\partial H u)(M(1 + \partial H)u) - (\partial u)(HM(1 + \partial H)u).\end{aligned}$$

I.e.

$$\frac{du_j}{dt} = (1-|j|^2)M_j u_j - M_j(1-|j|)\sum_{l=-\infty}^{\infty}(|l|+\frac{1}{2})u_l u_{j-l}$$
$$- \sum_{l=-\infty}^{\infty} M_l(1+|l|)[2(1-|j|)+|j-l|+(j-l)sgn(l)]u_l u_{j-l}.$$

It is clear that if M_j decay fast at infinity the equation is ill posed. As an extreme example let M to be the operator determined by the folowing multipliers:

$$M_{\pm 2} = 1 \qquad\qquad M_j = 0 \quad \text{otherwise.}$$

For this choice of M, the quadratic approximation is

$$\begin{aligned}
\dot{u}_2 &= -3u_2 + \sum(\frac{1}{2}+|l|)u_{2-l}u_l + 6u_4u_{-2} \\
\dot{u}_{-2} &= -3u_{-2} + \sum(\frac{1}{2}+|l|)u_{l-2}u_l + 6u_{-4}u_2 \\
\dot{u}_j &= 6u_{j-2}u_2 + 6(|j|-1)u_{j+2}u_{-2} \qquad \text{for } j > 2 \\
\dot{u}_j &= 6(|j|-1)u_{j-2}u_2 + 6u_{j+2}u_{-2} \qquad \text{for } j < -2.
\end{aligned}$$

The equation for $\dot{u}_j$ makes it clear that this system is ill-posed. The lack of a stable linear part allows arbitrarily small perturbations to travel arbitrarily fast.

References

[1] Hele-Shaw, J.H.S. 1898 Nature 58, 34.
[2] Saffman, P.G. & Taylor, G.I. 1958 Proc. R. Soc. London A 245, 312.
[3] Saffman, P.G. 1986 J. Fluid Mech. 173, 73.
[4] Bensimon, D., Kadanoff, L., Liang, S., Shraiman, B. & Tang, C. 1986 Rev. Mod. Phys. 58, 977.
[5] Tabeling, P., Zocchi, G. & Libchaber, A. 1987 J. Fluid Mech. 177, 67.
[6] Park, C.W. & Homsy, G.M. 1984 J. Fluid Mech. 139, 291.
[7] Sarkar, S. & Jasnow, D. in press Phys. Rev. A.
[8] Tanveer, S. 1990 Proc. R. Soc. London A 428, 511.
[9] Duchon, R. & Robert, R. 1984 Ann. Inst. Henri Poincaré 1, 5.
[10] Shraiman, B. & Bensimon, D. 1984 Phys. Rev. A 30, 2840.
[11] Tanveer, S. 1986 Phys. Fluids 29, 3537.
[12] Constantin, P. & Kadanoff, L. 1991 Physica D. 47, 450-460.
[13] Bensimon, D. & Pelce, P. 1986 Phys. Rev. A 33, 4477.
[14] Mineev, A., private communication.
[15] Howison, S.D. 1986 J. Fluid Mech. 167, 439.
[16] Peterson, M.A. 1989 Phys. Rev. Lett. 62, 284.
[17] Lacey, A.A., Howison, S.D., Ockendon, J.R. & Wilmott, P. 1990 Quart. J. Mech. Appl. Math. 43, 387-405.
[18] Lacey, A.A., Howison, S.D. & Ockendon, J.R. 1988 Quart. J. Mech. Appl. Math. II 41, 183-193.
[19] Pelce, P. Dynamics of Curved Fronts (Academic Press, New York, 1988).
[20] Hong, D.C. & Langer, J.S. 1986 Phys. Rev. Lett. 56, 2032.
[21] Combescot, R., Dombre, T., Hakim, V., Pomeau, Y. & Pumir, A. 1986 Phys. Rev. Lett. 56, 2036.
[22] Shraiman, B. 1986 Phys. Rev. Lett. 56, 2028.
[23] Zeldovitch, Ya.B. 1980 Combust. Sci. Technol. 241.
[24] Kruskal, M. & Segur, H., unpublished.

References

[1] Hele-Shaw, J.H.S. 1898 Nature 58, 34.
[2] Saffman, P.G. & Taylor, G.I. 1958 Proc. R. Soc. London A 245, 312.
[3] Saffman, P.G. 1986 J. Fluid Mech. 173, 73.
[4] Bensimon, D., Kadanoff, L., Liang, S., Shraiman, B. & Tang, C. 1986 Rev. Mod. Phys. 58, 977.

[5] Tabeling, P., Zocchi, G. & Libchaber, A. 1987 J. Fluid Mech. 177, 67.
[6] Park, C.W. & Homsy, G.M. 1984 J. Fluid Mech. 139, 291.
[7] Sarkar, S. & Jasnow, D. in press Phys. Rev. A.
[8] Tanveer, S. 1990 Proc. R. Soc. London A 428, 511.
[9] Duchon, R. & Robert, R. 1984 Ann. Inst. Henri Poincaré 1, 5.
[10] Shraiman, B. & Bensimon, D. 1984 Phys. Rev. A 30, 2840.
[11] Tanveer, S. 1986 Phys. Fluids 29, 3537.
[12] Constantin, P. & Kadanoff, L. 1991 Physica D. 47, 450-460.
[13] Bensimon, D. & Pelce, P. 1986 Phys. Rev. A 33, 4477.
[14] Mineev, A., private communication.
[15] Howison, S.D. 1986 J. Fluid Mech. 167, 439.
[16] Peterson, M.A. 1989 Phys. Rev. Lett. 62, 284.
[17] Lacey, A.A., Howison, S.D., Ockendon, J.R. & Wilmott, P. 1990 Quart. J. Mech. Appl. Math. 43, 387-405.
[18] Lacey, A.A., Howison, S.D. & Ockendon, J.R. 1988 Quart. J. Mech. Appl. Math. II 41, 183-193.
[19] Pelce, P. Dynamics of Curved Fronts (Academic Press, New York, 1988).
[20] Hong, D.C. & Langer, J.S. 1986 Phys. Rev. Lett. 56, 2032.
[21] Combescot, R., Dombre, T., Hakim, V., Pomeau, Y. & Pumir, A. 1986 Phys. Rev. Lett. 56, 2036.
[22] Shraiman, B. 1986 Phys. Rev. Lett. 56, 2028.
[23] Zeldovitch, Ya.B. 1980 Combust. Sci. Technol. 241.
[24] Kruskal, M. & Segur, H., unpublished.

Department of Mathematics, The University of Chicago, Chicago, Illinois 60637, U.S.A.