Appeared in Arch Ration Mech Anal 154(2000)1:3-51.

LINEAR STABILITY OF STEADY STATES FOR THIN FILM AND CAHN–HILLIARD TYPE EQUATIONS

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ABSTRACT. We study the linear stability of smooth steady states of the evolution equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x - ah$$

under both periodic and Neumann boundary conditions. If $a \neq 0$ we assume $f \equiv 1$. In particular we consider positive periodic steady states of thin film equations, where $a = 0$ and f, g might have degeneracies such as $f(0) = 0$ as well as singularities like $g(0) = +\infty$.

If $a \leq 0$, we prove each periodic steady state is linearly unstable with respect to volume (area) preserving perturbations whose period is an integer multiple of the steady state's period. For area preserving perturbations having the *same* period as the steady state, we prove linear instability for all a if the ratio g/f is a convex function. Analogous results hold for Neumann boundary conditions.

The rest of the paper concerns the special case of $a = 0$ and power law coefficients $f(y) = y^n$ and $g(y) = \mathcal{B}y^m$. We characterize the linear stability of each positive periodic steady state under perturbations of the same period. For steady states that do not have a linearly unstable direction, we find all neutral directions. Surprisingly, our instability results imply a nonexistence result: for a large range of exponents m and n there cannot be two positive periodic steady states with the same period and volume.

Contents

1. Introduction

We study the evolution equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x - ah. \tag{1}$$

When $a = 0$ this is the one dimensional version of $h_t = -\nabla \cdot (f(h)\nabla\Delta h) - \nabla \cdot (g(h)\nabla h)$, which has been used to model the dynamics of a thin film of viscous liquid. The air/liquid interface is at height $z = h(x, y, t)$ and the liquid/solid interface is at $z = 0$. The one dimensional equation applies if the liquid film is uniform in the y direction.

The coefficient $f(h)$ of the fourth order term in the equation reflects surface tension effects; a typical choice is $f(h) = h^3 + \beta h^p$ where $0 < p < 3$ and $\beta \geq 0$ [9, 11, 16, 20, 29, 35]. The coefficient of the second order term can reflect additional forces such as gravity $g(h) \sim h^3$ [10], van der Waals interactions $g(h) \sim h^m, m < 0$ [8, 23, 35, 45], or thermocapillary effects

Date: July 20, 1999.

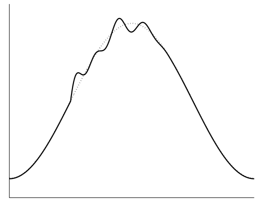

FIGURE 1. Zero-mean perturbation of a steady state.

$g(h) \sim h^2/(1 \pm ch)^2$ [36, 37, 43]. For a thin viscous film of liquid coating the inner wall of a straight pipe, the curvature of the pipe gives rise to $g(h) \sim h^3$ [21, 24]. Also, such an equation with $a = 0$ is used to model a gravity–driven Hele–Shaw cell, with $f(h) \sim h$ and $g(h) \sim h$ [14, 15]. The aggregation of aphids on a leaf can be modeled with $f(h) \sim h$ and $g(h) \sim h - c$, where h represents the population density [28].

Positive steady states correspond to the whole surface being wetted. If $f > 0$ and $g > 0$ then bounded positive steady states must be periodic or constant, by [26, Theorem B.1]. Periodic steady states certainly do exist for equation (1) with $a = 0$, by [33, 17],[26, §2.2], for example. Compactly supported (droplet) steady states exist only if g/f satisfies additional constraints [26, §2.2], and they can have relatively low regularity at the contact line. We treat only smooth steady states in this paper.

An example of equation (1) with $a > 0$ is the Sivashinsky equation that arises in the modeling of an alloy solidification problem [30, 38, 41], with $f \equiv 1$ and g linear. An equation with $f \equiv 1$ but $a = 0$ is the extensively studied Cahn–Hilliard equation [5, 44], for which g is a negative quadratic. We refer the reader to [2, 3, 18] for further references on the Cahn–Hilliard equation.

Equations like (1) are of mathematical interest as well: Bertozzi and Pugh [4] conjectured that finite-time blow-up ($||h(\cdot,t)||_\infty \to \infty$) is possible for certain such equations. In [26] we related the steady states and some of their properties to this blow-up conjecture.

Our linear stability results are for zero-mean perturbations of steady states. The zero-mean requirement on the perturbation seems reasonable from a physical standpoint, as it corresponds to a disturbance of the fluid that alters the profile without adding additional fluid. When $a = 0$, zero-mean perturbations are natural from a mathematical standpoint because given periodic initial data $h(x, 0)$ and periodic boundary conditions, the evolution equation (1) preserves volume: $\int h(x,t)\,dx = \int h(x,0)\,dx$ for all time t. The same is true for Neumann boundary conditions. Thus zero-mean perturbations allow the possibility of relaxation back to the original steady state, whereas nonzero-mean perturbations do not. When $a \neq 0$ the equation does not conserve area in general, although solutions with zero mean will maintain zero mean. Since steady states automatically have zero mean when $a \neq 0$, it again seems natural to perturb with a function having mean zero. Further, if steady states are linearly unstable to zero–mean perturbations then they are certainly linearly unstable to more general perturbations.

In this paper we consider both periodic and Neumann boundary conditions for equation (1); we explain the relation between the two stability problems in Section 2.5. Note that nonmonotonic steady states of the Cahn–Hilliard equation are already known to be unstable under Neumann boundary conditions [6, 31].

The main results of the paper are roughly as follows:

• Theorem 1. If $a \leq 0$ then every periodic steady state is linearly unstable to periodic perturbations with longer period.

• Theorem 3. If $(g/f)'' \geq 0$ then every periodic steady state is linearly unstable to perturbations having the same period.

For example, Theorem 3 applies to a thin liquid film, $f(h) = h^3$, with positive or negative gravity and with net repulsive van der Waals interactions, $g(h) = \pm h^3 + \mathcal{B}h^{-1}$ where $\mathcal{B} > 0$.

• Theorems 4 and 5. Instability results analogous to Theorems 1 and 3 hold under Neumann boundary conditions.

The rest of the results are for $a = 0$, $f(y) = y^n$, and $g(y) = \mathcal{B}y^m$ with $\mathcal{B} > 0$ ('power law coefficients'). The perturbations have the same (or shorter) period as the steady state.

• Theorem 7. If $m < n$ or $m \geq n+1$ then any non-constant positive periodic steady state is linearly unstable.

• Theorem 9. When $m \in (n, n+1)$, the stability question reduces to stability of a particular nonlocal reaction-diffusion equation. We characterize this stability in terms of the time and area maps of a related nonlinear oscillator.

• Theorem 10. If a positive periodic steady state $h_{\rm ss}$ has no linearly unstable direction, then the linearly neutral directions are spanned either by the x-derivative of the steady state (which arises naturally out of the translation invariance of the problem), or by the x-derivative and a known function $\tilde{\kappa}$.

• Theorem 12. For $m < n$ and $m \geq n+1$ we prove that given a period P and an area A, there is at most one (up to translation) positive periodic steady state $h_{\rm ss}$ with that period and area. Remarkably, the theorem follows from the linear instability of these steady states.

To prove these results, we first linearize the evolution equation about a steady state and then transform it into a *self-adjoint* fourth order eigenvalue problem whose coefficients depend on the steady state. We use the steady state and its derivatives as trial functions in the Rayleigh quotient, obtaining the instability results Theorem 1–5. For those power law coefficients for which the instability result does not apply, we deduce enough properties of the steady states to enable us to characterize linear stability. The key role here is played by a function κ that arises from varying the minimum height of the steady state. Our arguments are similar to how stability of reaction-diffusion equations in one dimension can be characterized in terms of the time map, an approach used by Chafee [7, Theorem 6.2] and later authors, *e.g.* [39, §4.1],[42, Chapter 24D].

Notation. We write $\mathbb{T}_X$ for a circle of circumference $X > 0$. As usual, one identifies functions on $\mathbb{T}_X$ with functions on $\mathbb{R}$ that are X-periodic and calls them *even* or *odd* according to whether they are even or odd on $\mathbb{R}$.

2. Linear instability results

In this section, we treat instability for periodic boundary conditions first, then turn to Neumann conditions. At the end, we discuss the relation between the two and also briefly treat the stability question for constant steady states.

Throughout this section we assume $f, g \in C^1(\mathbb{R})$, but our results still apply if the coefficient functions $f(y)$ and $g(y)$ are only defined and C^1 for a restricted range of y-values. (For example, for a thin film equation with coefficients $f(y) = y^{3/2}$ and $g(y) = y^{-1}$, the coefficients are defined and C^1 for $y > 0$.) Indeed we really only need f and g to be C^1-smooth on a neighborhood of the range of the steady state h_{ss} under consideration; we can then modify f and g to make them C^1 away from this range. Such a modification does not affect the linearized problem since whenever f and g appear they are evaluated at $h_{\text{ss}}(x)$. With this understood, we assume $f, g \in C^1(\mathbb{R})$ from now on.

We also assume $f > 0$, and define the ratio

$$r = \frac{g}{f}.$$

Fix $a \in \mathbb{R}, X > 0$. We assume that

$$\text{if } a \neq 0 \text{ then } f \equiv 1.$$

We use this assumption in the linearization (see Appendices A and B).

2.1. **Linearizing the equation around a smooth periodic steady state.**

We first linearize the evolution equation (1) around a periodic steady state and then define linear stability for the steady state. We give an expanded treatment of this linearization and spectral theory in Appendix A.

Readers willing to accept the Rayleigh quotient definition of stability given in (5) below may skip to Theorems 1–3. They are warned that it is w' (not w) in the Rayleigh quotient that corresponds to the zero-mean perturbation in the original problem.

Assume $h_{\rm ss} \in C^4(\mathbb{R})$ is a steady state of the evolution equation (1):

$$(f(h_{\rm ss})h_{\rm ss}''' + g(h_{\rm ss})h_{\rm ss}')' + ah_{\rm ss} = 0. \tag{2}$$

[Here and throughout the paper, if a function has only one independent variable we use $'$ to denote derivatives with respect to that variable, *e.g.* $h_{\rm ss}' = (h_{\rm ss})_x$.] Assume also $h_{\rm ss}$ is X-periodic. Provided $h_{\rm ss}$ is non-constant, its *smallest* period is either X or an integer fraction of X.

When $a = 0$ the steady state satisfies a simpler equation than (2). Indeed, (2) implies $f(h_{\rm ss})h_{\rm ss}''' + g(h_{\rm ss})h_{\rm ss}' = C$ for some constant C, and one shows the constant C (the flux) is zero by dividing the equation by $f(h_{\rm ss})$ and integrating over the interval $(0, X)$. Then since $C = 0$,

$$h_{\rm ss}''' + r(h_{\rm ss})h_{\rm ss}' = 0 \qquad \text{when } a = 0. \tag{3}$$

This can be integrated up again to get a nonlinear oscillator formulation for the steady states. For later use, we note that (3) also holds when $a = 0$ and the steady state satisfies Neumann conditions $h_{\rm ss}' = h_{\rm ss}''' = 0$ at $x = 0$, because then $C = 0$ by evaluating the equation at $x = 0$.

Now we linearize the evolution equation about $h_{\rm ss}$. Let $h = h_{\rm ss} + \varepsilon\phi$, where the perturbation $\phi(x,t)$ is X-periodic in x and has mean value zero, $\int_0^X \phi(x,t)\,dx = 0$, at each time t. We only consider such area preserving perturbations, for reasons explained in the Introduction.

In Appendix A we expand the evolution equation (1) in orders of ε; at lowest order, $O(1)$, one recovers the steady state condition (2) as expected. At order $O(\varepsilon)$, the problem is linearized: $\phi_t = \mathcal{L}\phi$ where the linear operator $\mathcal{L}$ is

$$\mathcal{L}\phi := -\left[f(h_{\rm ss})\left(\phi_{xx} + r(h_{\rm ss})\phi\right)_x\right]_x - a\phi. \tag{4}$$

We call the steady state $h_{\rm ss}$ *linearly unstable* if $\mathcal{L}$ has an eigenvalue σ_1 with positive real part, since if ϕ_1 is the corresponding eigenfunction then $\phi(x,t) = e^{\sigma_1 t}\phi_1(x)$ is an exponentially growing solution of $\phi_t = \mathcal{L}\phi$. But in the absence of a general 'linearization theorem' for the evolution (1), we are not guaranteed that for initial data near $h_{\rm ss}$ the solution of the nonlinear evolution equation will behave like the solution of the linearized evolution. Note that a linearization theorem *is* known in the special case $f \equiv 1$, using semilinear operator theory; see for example [31, §6]. In any case, even if a linearization theorem is known, it is still unclear what happens in null directions of $\mathcal{L}$. One null direction arises from an infinitesimal translation of the steady state in the x direction, corresponding to $\phi = h_{\rm ss}'$. This is a 0-eigenfunction of $\mathcal{L}$ since (2) and (3) imply $\mathcal{L}h_{\rm ss}' = 0$, using also that $f \equiv 1$ when $a \neq 0$.

To reformulate the linearized equation $\phi_t = \mathcal{L}\phi$ as a self-adjoint problem we introduce the antiderivative $\psi(x,t) = \int^x \phi(\xi,t)\,d\xi$, which is X-periodic because ϕ has mean value zero.

Appendix A shows ψ satisfies $\psi_t = -f(h_{ss})\mathcal{I}\psi$ where

$$\mathcal{I}\psi := \psi_{xxxx} + (r(h_{ss})\psi_x)_x + af(h_{ss})^{-1}\psi.$$

The associated eigenvalue problem is $\mathcal{I}w = \lambda f(h_{ss})^{-1}w$, and a linearly unstable direction for the original problem corresponds to a *negative* eigenvalue λ. The lowest eigenvalue, λ_1, has the Rayleigh quotient

$$\lambda_1(h_{ss}) = \min_w \frac{\int_0^X [(w'')^2 - r(h_{ss})(w')^2 + af(h_{ss})^{-1}w^2]\,dx}{\int_0^X f(h_{ss})^{-1}w^2\,dx}, \tag{5}$$

where the minimum is taken over $w \in H^2(\mathbb{T}_X) \setminus \{0\}$ normalized by $\int_0^X wf(h_{ss})^{-1}\,dx = 0$. We discuss all this in detail in Appendix A.

Definition. The steady state h_{ss} is *linearly unstable at period* X if $\lambda_1(h_{ss}) < 0$, and *linearly stable* otherwise.

The phrase 'h_{ss} is linearly unstable at period X' does not mean the most unstable direction has wavelength X; it means there exists an unstable direction with least period either X or an integer fraction of X. For example, with $X = 2\pi, f \equiv 1, g \equiv r \equiv 9$ and $a = 0$, the minimizers for λ_1 are $w(x) = \sin 2x$ and $\cos 2x$.

The 0-eigenfunction $v = h'_{ss}$ that arose from translational symmetry of the original problem integrates up to a 0-eigenfunction $w = h_{ss} - c$ for $\mathcal{I}$. Hence $\lambda_1(h_{ss}) \le 0$ for all periodic steady states, and so linear stability is equivalent to $\lambda_1 = 0$ while instability requires $\lambda_1 < 0$. If $\lambda_1 = 0$ one then asks whether there are null directions *other* than $h_{ss} - c$. We address this in Theorem 10, for power law coefficients.

2.2. **Linear instability of periodic steady states.** For the evolution equation (1) with $a \le 0$, we find periodic steady states are linearly unstable when the perturbations have longer period than the steady state:

Theorem 1. *Let $f, g \in C^1(\mathbb{R})$ with $f > 0$, and take $a \le 0, X > 0$. If $a < 0$ then assume $f \equiv 1$. Suppose $h_{ss} \in C^4(\mathbb{R})$ is a non-constant periodic steady state of (1) with least period X/j for some integer $j \ge 2$.*

Then $\lambda_1(h_{ss}) < 0$, so that h_{ss} is linearly unstable with respect to area preserving perturbations at period X.

In Section 4.1 we prove the theorem by taking a truncation of $w = h_{ss} - c$ as a trial function in the Rayleigh quotient (5). At heart, this is the method of Carr, Gurtin, and Slemrod [6, Theorem 8.2] for $a = 0$ and Neumann boundary conditions, adapted to the periodic case.

Our proof breaks down for $a > 0$, which seems reasonable since the term $-ah$ in (1) is then stabilizing when one linearizes around the trivial steady state $h_{ss} \equiv 0$. A linearly unstable steady state might still exist for $a > 0$; the following observation determines whether it can be non-constant.

Lemma 2. *Let $f, g \in C^1(\mathbb{R})$ with $f > 0$, and take $a, X > 0$. Suppose $h_{ss} \in C^4(\mathbb{R})$ is non-constant and X-periodic and satisfies the steady state equation (2). Then*

$$\max_x \frac{1}{4} \frac{g(h_{ss}(x))^2}{f(h_{ss}(x))} \geq a.$$

The lemma, proved in Section 4.2, tells us that if a is very large, say $a \gg g(0)^2/4f(0)$, then any non-constant steady state must achieve rather extreme values in order to exist at all. The term $-ah$ might then fail to stabilize the evolution near the steady state. Sarocka, Bernoff and Rossi [38, §3] find computationally steady states with extreme amplitudes, for the Sivashinsky equation with large a.

In Section 4.3 we take $w = h'_{ss}$ as a trial function and prove steady states are linearly unstable when the ratio g/f is convex:

Theorem 3. *Let $f, g \in C^2(\mathbb{R})$ with $f > 0$, and take $a \in \mathbb{R}, X > 0$. If $a \neq 0$ then assume $f \equiv 1$. Let $h_{ss} \in C^4(\mathbb{R})$ be an X-periodic non-constant steady state of (1).*

If $r = g/f$ is convex ($r'' \geq 0$) and non-constant on the range of h_{ss} then $\lambda_1(h_{ss}) < 0$, so that h_{ss} is linearly unstable with respect to area preserving perturbations at period X.

In view of Theorem 1, this theorem provides new information when the least period of the steady state is X, rather than a fraction of X, so that the perturbations have period no longer than that of the steady state.

2.3. **Linear instability for Neumann boundary conditions.** The instability results above extend to Neumann boundary conditions with proofs that are slightly simpler. The main technical point is that we change the space of trial functions for the fourth order selfadjoint problem from $H^2(\mathbb{T}_X) \cap \{\int w f(h_{ss})^{-1}\, dx = 0\}$ to $H^2(0, X) \cap H^1_0(0, X)$.

Consider the evolution (1) under the Neumann boundary conditions

$$h_x = h_{xxx} = 0 \quad \text{at } x = 0, X.$$

These conditions are equivalent to $h_x = 0$ and the 'zero flux' condition $f(h)h_{xxx} + g(h)h_x = 0$.

A steady state $h_{ss} \in C^4[0, X]$ satisfies the steady state equations (2) and (3) and the Neumann conditions $h'_{ss} = h'''_{ss} = 0$ at $x = 0, X$. We continue to assume that if $a \neq 0$ then $f \equiv 1$. The linearization and spectral theory proceed much like in the periodic case; the few significant changes are outlined in Appendix B.

The linearized eigenvalue problem is

$$\mathcal{L}u := -\left[f(h_{ss})\,(u'' + r(h_{ss})u)'\right]' = \sigma u, \qquad \int_0^X u\, dx = 0,$$

with boundary conditions $u' = u''' = 0$ at $x = 0, X$. The 'integrated' symmetric eigenproblem is

$$\mathcal{I}w := w'''' + (r(h_{\text{ss}})w')' + af(h_{\text{ss}})^{-1}w = \nu f(h_{\text{ss}})^{-1}w, \tag{6}$$

with boundary conditions $w = w'' = 0$ at $x = 0, X$. These boundary conditions and the fact that $h'_{\text{ss}} = 0$ at $x = 0, X$ imply $w'''' = 0$ at the endpoints. The eigenvalues of the two problems are related by $\sigma = -\nu$, and the eigenfunctions by $u = w'$.

The symmetric problem (6) has discrete spectrum, with lowest eigenvalue given by

$$\nu_1(h_{\text{ss}}) = \min_w \frac{\int_0^X [(w'')^2 - r(h_{\text{ss}})(w')^2 + af(h_{\text{ss}})^{-1}w^2]\, dx}{\int_0^X f(h_{\text{ss}})^{-1}w^2\, dx} \tag{7}$$

where the minimum is over $w \in H^2(0, X) \cap H_0^1(0, X) \setminus \{0\}$. The weak eigenfunctions are C^4-smooth on the closed interval $[0, X]$ and solve (6) classically. They satisfy $w = 0$ at the endpoints by virtue of lying in $H_0^1(0, X)$. They also satisfy the 'natural boundary condition' $w'' = 0$ at $x = 0, X$; this follows from the Euler–Lagrange analysis.

As before, we call h_{ss} *linearly unstable at length* X if $\nu_1(h_{\text{ss}}) < 0$.

Theorem 4. *Let $f, g \in C^1(\mathbb{R})$ with $f > 0$, and take $a \leq 0, X > 0$. If $a < 0$ then assume $f \equiv 1$. Assume $h_{\text{ss}} \in C^4[0, X]$ is a non-constant Neumann steady state of (1) and suppose $h_{\text{ss}}(0) = h_{\text{ss}}(b)$ and $h'_{\text{ss}}(b) = h'''_{\text{ss}}(b) = 0$ for some $b \in (0, X]$.*

Then $\nu_1(h_{\text{ss}}) < 0$, so that h_{ss} is linearly unstable with respect to area preserving perturbations at length X.

We prove this in Section 4.4. In particular it shows for $a \leq 0$ that if a Neumann steady state is X-periodic with minima at 0 and X, then it is linearly unstable. Lemma 2 still constrains the existence of steady states when $a > 0$, as it holds in the Neumann case also.

Novick–Cohen [31, Theorem 6.1] proved Theorem 4 with $a = 0, f \equiv 1$. In [30, Theorem 2.1] she further stated the theorem for the Sivashinsky equation, where $a > 0, f \equiv 1$ and g is a linear function, but her proof actually holds for $a < 0$, not $a > 0$ as stated. Nonetheless, steady states of the Sivashinsky equation *are* unstable by Theorem 5 below, no matter what the value of a: Theorem 5 applies since the linear function $g = g/f$ is convex.

Steady states are linearly unstable when the ratio g/f is convex, even if the steady state is monotone:

Theorem 5. *Let $f, g \in C^2(\mathbb{R})$ with $f > 0$, and take $a \in \mathbb{R}, X > 0$. If $a \neq 0$ then assume $f \equiv 1$. Suppose $h_{\text{ss}} \in C^4[0, X]$ is a non-constant Neumann steady state of (1).*

If $r = g/f$ is convex ($r'' \geq 0$) and non-constant on the range of the steady state h_{ss} then $\nu_1(h_{\text{ss}}) < 0$, so that h_{ss} is linearly unstable with respect to area preserving perturbations at length X.

We prove this in Section 4.5 by taking $w = h'_{\text{ss}}$ as a trial function.

The theorem does *not* apply to the Cahn–Hilliard equation $h_t = -h_{xxxx} - ((1-3h^2)h_x)_x$ because $r(y) = 1 - 3y^2$ is concave rather than convex. Indeed, Grinfeld and Novick–Cohen [18, Theorem 6.2] proved linear stability for certain monotonic steady states of this equation.

2.4. **Linear stability of the constant steady states, when $\mathbf{a = 0}$.** The simplest steady state of the general evolution equation (1) with $a = 0$ is the constant function $h_{\text{ss}} \equiv \overline{h}$; when $a \neq 0$ the only constant steady state is $h_{\text{ss}} \equiv 0$. The linear stability analysis of these constant steady states is direct. Take the perturbed steady state: $h = \overline{h} + \varepsilon\phi$ where $\phi(x,t)$ has mean value zero in x, for each t. Linearizing (1) around $\overline{h}$ gives

$$\phi_t = -f(\overline{h})\phi_{xxxx} - g(\overline{h})\phi_{xx} - a\phi.$$

Periodic boundary conditions. Assuming periodic boundary conditions on $(0, X)$ we expand ϕ in Fourier modes, $\phi(x,t) = \sum_{j \neq 0} a_j(t) \exp[ij2\pi x/X]$, obtaining

$$a_j'(t) = \left[f(\overline{h})\left(r(\overline{h}) - (2\pi/X)^2 j^2\right)(2\pi/X)^2 j^2 - a\right] a_j(t).$$

Assume $a = 0$ for simplicity. Then the constant steady state is linearly asymptotically stable with respect to area preserving perturbations of period X if $r(\overline{h})X^2 < 4\pi^2$; it is linearly neutrally stable if $r(\overline{h})X^2 = 4\pi^2$, and is linearly unstable if $r(\overline{h})X^2 > 4\pi^2$. In the unstable case, the most unstable direction corresponds to the integer j_m that maximizes $(2\pi/X)^2 j^2 \left(r(\overline{h}) - (2\pi/X)^2 j^2\right)$.

If $g < 0$ (and hence $r < 0$) then the constant steady states for $a = 0$ are linearly stable, but if $g > 0$ then the constant steady states are linearly unstable for all large X, justifying the description of the equation as 'long-wave unstable' when $g > 0, a = 0$.

Goldstein, Pesci and Shelley [14, §IIIB] have *non*linear instability results for the constant steady state in the special case $f \equiv g$.

Neumann boundary conditions. Under Neumann boundary conditions on $(0, X/2)$, one expands ϕ in Fourier cosine modes, obtaining the same stability results as in the periodic case.

2.5. **Relation between the periodic and Neumann stability problems.** Suppose h_{ss} is an even X-periodic steady state of the evolution equation (1) with extrema at $x = 0, X/2, X$, so that $h'_{\text{ss}} = h'''_{\text{ss}} = 0$ at these points also. When $a = 0$, linear instability of this steady state with respect to periodic boundary conditions on $(0, X)$ *implies* instability with respect to Neumann conditions but not conversely in general, as we shortly explain.

The Rayleigh quotients of the two problems are the same. For the Neumann eigenvalue ν_1, the minimization is over $H^2(0,X) \cap H^1_0(0,X)$. For the periodic eigenvalue λ_1, trial functions are in $H^2(\mathbb{T}_X)$ on the circle and have zero mean with respect to the weight $1/f(h_{\text{ss}})$. Assume $a = 0$, so that the numerator of the Rayleigh quotient (5) for λ_1 is unchanged by addition of a constant to w; thus the sign of the minimal Rayleigh quotient is unaffected if we drop the weighted zero mean requirement on w. Further, we can assume $w(0) = w(X) = 0$. Thus to determine the sign of λ_1 with $a = 0$, we may minimize over $H^2(\mathbb{T}_X) \cap H_0^1(0, X)$. This space is a proper subset of $H^2(0, X) \cap H_0^1(0, X)$. Hence if $\lambda_1(h_{ss}) < 0$ then $\nu_1(h_{ss}) < 0$, so that linear instability with respect to periodic boundary conditions on $(0, X)$ *implies* instability with respect to Neumann conditions. The converse is clearly false: if $X = 2\pi$, $a = 0$ and $f, g, r \equiv 1$, then $\lambda_1 \geq 0$ by the sharp Poincaré inequality, but $\nu_1 < 0$ by using $w(x) = \sin(\pi x/X)$ in the Rayleigh quotient (7).

For arbitrary a, Neumann instability on the half-period does imply periodic instability on the whole period: functions in $H^2(0, X/2) \cap H_0^1(0, X/2)$ extend by odd reflection to functions in $H^2(\mathbb{T}_X)$ with weighted mean zero, and so one deduces

$$\text{`}\lambda_1(h_{ss}) \text{ over } \mathbb{T}_X\text{'} \leq \text{`}\nu_1(h_{ss}) \text{ over } (0, X/2)\text{'}.$$

Furthermore when $a = 0$, if $\lambda_1(h_{ss})$ over $\mathbb{T}_X$ is negative then so is $\nu_1(h_{ss})$ over $(0, X/2)$. For assume $\lambda_1(h_{ss})$ is negative, so that the lowest eigenvalue $\tau_1(h_{ss})$ of the associated second order problem (see Section 5) is also negative. The proof of Lemma 21(a) shows the corresponding zero-mean eigenfunction u_1 is even, and so $w := \int_0^x u_1(\xi)\, d\xi$ is odd and X-periodic, so that $w(X/2) = 0$ and $w \in H_0^1(0, X/2)$. The Rayleigh quotient of w for $\nu_1(h_{ss})$, over $(0, X/2)$, is negative since $\tau_1(h_{ss}) < 0$, and so $\nu_1(h_{ss}) < 0$. Thus for $a = 0$, Neumann instability on a half-period is equivalent to periodic instability over the whole period.

3. Linear stability results for power law thin film equations

Our linear instability results have been for fairly general coefficients f and g. We will prove linear *stability* results for a restricted class of coefficients: power-law coefficients. Then we establish some qualitative properties of these steady states that follow from our linear stability and instability results.

3.1. **Power law coefficients: rescaling the problem.** The results in this section concern the case of power-law coefficients:

$$f(y) = y^n, \qquad g(y) = \mathcal{B} y^m, \qquad \text{for } y > 0,$$

for some exponents $n, m \in \mathbb{R}$ and some positive constant $\mathcal{B} > 0$. In this section we take $a = 0$ and consider only periodic boundary conditions. The evolution equation (1) becomes

$$h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x.$$

Suppose h_{ss} is a non-constant positive periodic steady state with *least* period $X > 0$. (If the least period were a fraction of X then Theorem 1 would imply linear instability at period X.) Translate h_{ss} so that its global minimum is atttained at $x = 0$.

Theorem 3 gives linear instability at period X when $r(y) = \mathcal{B}y^{m-n}$ is convex and non-constant, that is, when $m-n<0$ or $m-n \geq 1$. (Technically, Theorem 3 requires $f, g \in C^2(\mathbb{R})$, but one can modify f, g off the range of h_{ss} to accomplish this.) Thus $0 \leq m-n < 1$ is a necessary condition for linear stability. Is it also sufficient? If not, can we determine precisely which exponents n and m yield stability and which give instability?

First, we simplify the problem by rescaling. When $a=0$, the steady states satisfy the nonlinear oscillator equation

$$h_{\mathrm{ss}}'' + H'(h_{\mathrm{ss}}) = 0 \qquad \text{(found by integrating (3))}. \tag{8}$$

Viewing x as a time variable, the steady states have a conserved quantity $\frac{1}{2}h_{\mathrm{ss}}'(x)^2 + H(h_{\mathrm{ss}}(x))$, where $H'' = r = g/f$. See [26, §2] for related information on this nonlinear oscillator formulation.

In the power law case, $r(y) = \mathcal{B}y^{m-n}$ and so if $m-n \neq -1$ then the oscillator equation for the steady state is

$$h_{\mathrm{ss}}'' + \frac{\mathcal{B}h_{\mathrm{ss}}^q - D}{q} = 0 \tag{9}$$

for some constant D, where

$$q := m-n+1.$$

For $m-n=-1$, the equation is $h_{\mathrm{ss}}'' + \mathcal{B}\log h_{\mathrm{ss}} - D = 0$.

The oscillator equation involves three constants: q, $\mathcal{B}$, and D. We remove $\mathcal{B}$ and D by rescaling h_{ss}: let

$$k(x) = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{1/q} h_{\mathrm{ss}}\left(\left(\frac{D}{\mathcal{B}}\right)^{1/2q} \frac{x}{D^{1/2}}\right), & q \neq 0, \\ e^{-D/\mathcal{B}} h_{\mathrm{ss}}\left(e^{D/2\mathcal{B}} \frac{x}{\mathcal{B}^{1/2}}\right), & q = 0. \end{cases} \tag{10}$$

For $q \neq 0$ this rescaling uses that $D>0$, by [26, §3.1]. This rescaling transforms the steady state equation (9) into

$$k'' + \frac{k^q - 1}{q} = 0, \qquad q \neq 0, \tag{11}$$

$$k'' + \log k = 0, \qquad q = 0. \tag{12}$$

For all q we have

$$k''' + k^{q-1}k' = 0, \tag{13}$$

and k satisfies the rescaled steady state equation $(k^n k''' + k^m k')' = 0$.

Since h_{ss} is non-constant, positive and periodic, $k''(x_0) > 0$ at some point x_0. Evaluating (11–12) at x_0 shows the minimum value of k is less than 1. Also $k'(0) = 0$ since h_{ss} has its

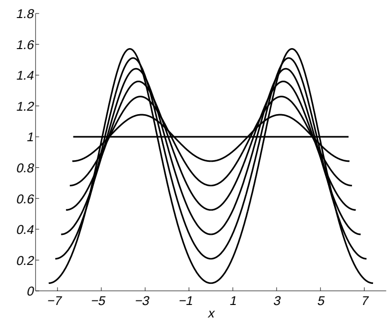

FIGURE 2. Steady states $k_\alpha(x)$, when $q = 3$.

minimum at $x = 0$. Introducing the notation k_α for the solution k that has minimum value $\alpha \in (0, 1)$, we have

$$0 < k_\alpha(0) = \alpha < 1, \qquad k'_\alpha(0) = 0.$$

Every steady state $h_{\rm ss}$ can be rescaled to a k_α. Conversely, for each $q \in \mathbb{R}$ and $\alpha \in (0, 1)$ there exists a unique smooth positive periodic k_α satisfying equation (11–12) with $0 < k_\alpha(0) = \alpha < 1$, $k'_\alpha(0) = 0$ (see [26, Proposition 3.1]). To illustrate, Figure 2 plots the steady states $k_\alpha(x)$ over two periods, for $q = 3$ and seven α values between 0.05 and 1; see [26, §6.1] for details. It is clear from the figure that for $q = 3$ the period and area change with the minimum height α.

In order to state our stability results, we need certain properties of the k_α. The map $(\alpha, x) \mapsto k_\alpha(x)$ is C^∞-smooth for $(\alpha, x) \in (0, 1) \times \mathbb{R}$ by an ODE theorem giving smooth dependence on the initial data [22, Ch. V,§4]. We write

$$P = P(\alpha) \qquad \text{and} \qquad A = A(\alpha)$$

for the least period of k_α and the area of k_α under that period: $A = \int_0^P k_\alpha(x)\,dx$. In Appendix C we prove P and A are smooth functions of α:

Lemma 6. *For each $q \in \mathbb{R}$, the functions $P = P(\alpha)$ and $A = A(\alpha)$ are smooth for $\alpha \in (0, 1)$, with $P \to 2\pi$ and $A \to 2\pi$ as $\alpha \to 1$. When $q > -1$, $P(\alpha)$ and $A(\alpha)$ are continuous at $\alpha = 0$.*

Next, k_α is by definition a linearly unstable periodic steady state of the rescaled evolution $h_t = -\left(h^n h_{xxx}\right)_x - \left(h^m h_x\right)_x$ if and only if $\lambda_1(k_\alpha) < 0$, where by (5) with $a = 0$ we have

$$\lambda_1(k_\alpha) = \min_w \frac{\int_0^P [(w'')^2 - k_\alpha^{q-1}(w')^2]\,dx}{\int_0^P w^2 k_\alpha^{-n}\,dx}; \tag{14}$$

the minimum is taken over $w \in H^2(\mathbb{T}_P) \setminus \{0\}$ with $\int_0^P w k_\alpha^{-n}\,dx = 0$. The rescaling (10) implies $\lambda_1(h_{\rm ss}) = c\lambda_1(k_\alpha)$ for some $c > 0$, and so

$$h_{\rm ss} \text{ is linearly stable if and only if } k_\alpha \text{ is linearly stable.}$$

Thus it suffices below to determine the sign of $\lambda_1(k_\alpha)$.

3.2. **Power law coefficients: linear instability and stability.** For the steady state k_α, $r(y) = g(y)/f(y) = y^{q-1}$ and so Theorem 3 implies:

Theorem 7. *For $q \geq 2$ and $q < 1$, the steady state k_α is linearly unstable at period P for each $\alpha \in (0,1)$. That is, $\lambda_1(k_\alpha) < 0$.*

The remaining cases are $q = 1$ and $1 < q < 2$. For $q = 1$, Goldstein, Pesci, and Shelley [14] proved:

Lemma 8. *Suppose $q = 1$. Then for each $\alpha \in (0,1)$, the steady state k_α is linearly stable at period 2π, that is, $\lambda_1(k_\alpha) = 0$.*

Their paper is not primarily concerned with the linear theory, but with the formation of finite-time singularities for the nonlinear evolution with $m = n = 1$, or $q = 1$.

One can see the stability in Lemma 8 as follows: for $q = 1$ the steady state equation (11) is $k'' + k - 1 = 0$, which has exact solutions

$$k_\alpha(x) = 1 + (\alpha - 1)\cos x. \tag{15}$$

The period is $P = 2\pi$, and so using the Poincaré inequality $\int_0^{2\pi} w''^2\,dx \geq \int_0^{2\pi} w'^2\,dx$ in the Rayleigh quotient (14) implies $\lambda_1(k_\alpha) \geq 0$. This proves the lemma.

The remaining case is $1 < q < 2$. In Section 5 we prove that the quantity

$$E = E(\alpha) := P(\alpha)^{3-q} A(\alpha)^{q-1}$$

characterizes linear stability:

Theorem 9. *Let $1 < q < 2$. For each $\alpha \in (0,1)$, the steady state k_α is linearly stable to P-periodic zero-mean perturbations if and only if $E'(\alpha) \leq 0$.*

The appearance of E as the stability indicator is perfectly natural, since $E^{1/(3-q)}$ is essentially the period (or time) map for a one parameter family of steady states h_{ss} with fixed area and varying minimum height (see Section 6.3). Time maps are known to determine stability for reaction-diffusion equations with Neumann boundary conditions, *e.g.* [7], [39, §4.1], [42, Chapter 24D]. The same general line of reasoning applies here, as we ultimately study a nonlocal reaction-diffusion equation.

Our situation differs from the reaction-diffusion equation case in certain important respects. For example, all the eigenfunctions of our stability problem have mean zero and change sign, since our perturbations are area preserving. For the reaction-diffusion equation with Neumann boundary conditions, the fact that the first eigenfunction does *not* change sign is crucial in the stability analysis. Our stability analysis uses the moral equivalent of

FIGURE 3. (a) Plots of $E(\alpha)$ for $q = 1, 1.125, 1.250, 1.375$. Top curve: $q = 1.375$.
(b) Plots of $E(\alpha)$ for $q = 1.375, 1.500, 1.625, 1.75$. Top curve: $q = 1.375$.

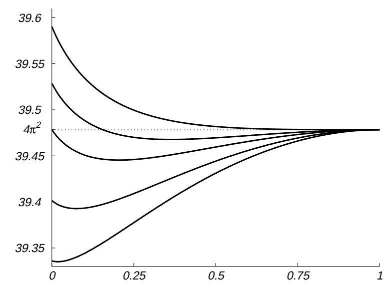

FIGURE 4. Plots of $E(\alpha)$ for $q = 1.75, 1.76, 1.768, 1.78, 1.79$. Top curve: $q = 1.75$.

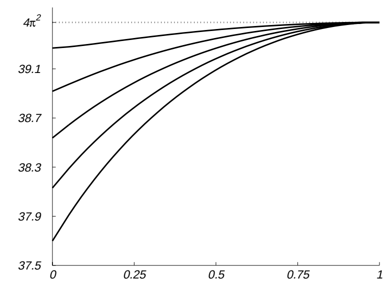

FIGURE 5. Plots of $E(\alpha)$ for $q = 1.8, 1.85, 1.9, 1.95, 2.0$. Top curve: $q = 1.8$.

this fact for functions with zero mean: in Lemma 21(b) we prove that the first eigenfunction, v_1, changes sign only once.

Returning to Theorem 9, we see that analytically establishing the sign of E' when $1 < q < 2$ would complete the linear stability study. We have not been able to do this, but by the numerical methods described in [26, §6.1] we have found: that $E'(\alpha) < 0$ for all α when $1 < q \leq 1.75$, that $E'(\alpha)$ changes sign when $1.75 < q \leq 1.79$, and that $E'(\alpha) > 0$ when $1.80 \leq q < 2$. (More precisely, our numerics suggest a critical exponent between 1.794 and 1.795.) Figures 3, 4 and 5 summarize these findings.

Section 6.3 explains how to interpret these figures as bifurcation diagrams for one parameter families of steady states h_{ss} with fixed area and varying minimum height, where $E' > 0$ characterizes the unstable branches and $E' \leq 0$ the stable ones.

An unavoidable weakness in our definition of linear stability is that it admits 0-eigenvalues. In particular, there is a translational null direction: $u = h'_{\mathrm{ss}}$ is a 0-eigenfunction of the original linearized eigenvalue problem $\mathcal{L}u = -\sigma u$ with zero mean. Normal forms might be used to investigate whether the evolution in the direction of the 0-eigenfunction is towards the steady state or one of its translates, or whether it is away from the positive periodic steady states altogether.

For the linearly stable power-law cases, we can identify all the 0-eigenfunctions of the original problem. Specifically, in Theorem 10 we will identify all smooth X-periodic solutions of $\mathcal{L}u = 0$ with mean value zero. One of the 0-eigenfunctions involves a function κ_α, defined as follows. First rescale k_α to obtain

$$K_\alpha(x) := \frac{P(\alpha)}{A(\alpha)} k_\alpha\left(P(\alpha)x\right).$$

By construction, K_α has period 1 and mean value 1. Now define

$$\kappa_\alpha(x) := \frac{\partial}{\partial\alpha} K_\alpha(x);$$

κ_α is well-defined and smooth because P and A depend smoothly on α by Lemma 6 while $k_\alpha(x)$ is jointly smooth in (α, x). We often write $\kappa = \kappa_\alpha$, suppressing the α-dependence. Notice κ_α is even in x, has period 1, and has mean value zero:

$$\int_0^1 \kappa_\alpha(x)\,dx = \frac{\partial}{\partial\alpha}\int_0^1 K_\alpha(x)\,dx = \frac{\partial}{\partial\alpha}(1) = 0.$$

See Section 5.4 for more properties of κ_α.

In Section 6.1, we prove the following theorem classifying all null directions.

Theorem 10. *Let $h_{\rm ss} \in C^4(\mathbb{T}_X)$ be a non-constant positive periodic steady state of (1) with power law coefficients and $a = 0$. Translate $h_{\rm ss}$ to put its minimum at $x = 0$, so that $h_{\rm ss}$ rescales to k_α as in Section 3.1.*
(a) If $1 < q < 2$ and $E'(\alpha) < 0$ then the 0-eigenspace of $\mathcal{L}$ is spanned by $h'_{\rm ss}(x)$.
(b) If $1 < q < 2$ and $E'(\alpha) = 0$ then the 0-eigenspace of $\mathcal{L}$ is spanned by $h'_{\rm ss}(x)$ and $\kappa_\alpha(x/X)$.
(c) If $q = 1$ then the 0-eigenspace of $\mathcal{L}$ is spanned by $\sin(\sqrt{\mathcal{B}}x)$ and $\cos(\sqrt{\mathcal{B}}x)$.

3.3. **Power law coefficients: consequences for properties of steady states.** We now present some properties of the steady states for the power law case that follow from our linear instability results.

Using the linear instability result for power-law coefficients (Theorem 7), we prove $E(\alpha)$ is monotonic when q is *not* in the range $1 \le q < 2$.

Theorem 11. *Let $q \ge 2$ or $q < 1$. Then $E'(\alpha) > 0$ for all $\alpha \in (0, 1)$.*

See Section 5 for the proof. Our earlier paper, [26, Theorem 7.5], proved monotonicity properties for $A(\alpha)$ and $P(\alpha)$ and yielded $E'(\alpha) > 0$ for $-\frac{1}{2} \le q < 1$ and $3 \le q \le 4.54$.

The above monotonicity of E as a function of α is important in answering the question of whether one can specify *a priori* both the period and area of a positive periodic steady

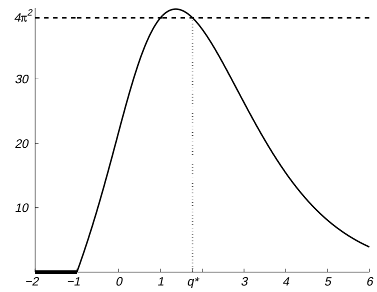

FIGURE 6. Plot of $E_0(q)$ together with a line at height $4\pi^2$, intersecting at 1 and $q^* \approx 1.768$.

state and whether there can be two different positive periodic steady states with the same period and area. More precisely, fix the physical parameters m and n and the Bond number $\mathcal{B} > 0$. Given positive numbers $P_{\rm ss}$ and $A_{\rm ss}$, does a constant D exist such that the steady state equation $h''_{\rm ss} + \mathcal{B} h^q_{\rm ss} = D$ has a positive periodic solution $h_{\rm ss}$ with period $P_{\rm ss}$ and area $A_{\rm ss}$? If so, is $h_{\rm ss}$ unique up to translation?

It turns out that we can answer this question if we understand the closure of the range of E. Until now, we have only considered $0 < \alpha < 1$, corresponding to positive periodic steady states. We need to extend $E(\alpha)$ to $\alpha = 0$. If $q > -1$ then one can take $\alpha = 0, k(0) = 0$, and still have a periodic steady state by [26, §3.1.2], and $E(0)$ can be calculated in terms of a Beta function of q by [26, eq. (3.13)]. For $q \leq -1$, however, one cannot take $\alpha = 0$ and have a periodic steady state [26, §2.2]. But $\lim_{\alpha\to 0} E(\alpha) = 0$ by Appendix C.2 and so we define $E(0) = 0$ when $q \leq -1$. Figure 6 shows a plot of $E_0(q) := E(0)$ as a function of q. The following theorem answers these questions of unique specification by period and area, for most q-values:

Theorem 12. *Let $q \geq 2$ or $q < 1$, and take $P_{\rm ss}$, $A_{\rm ss} > 0$. Then a non-constant positive periodic steady state $h_{\rm ss}$ with least period $P_{\rm ss}$ and with area $A_{\rm ss}$ exists for (1) with power law coefficients and $a = 0$ if and only if*

$$E(0) < \mathcal{B} P_{\rm ss}^{3-q} A_{\rm ss}^{q-1} < E(1) = 4\pi^2.$$

This steady state is unique up to translation.

We prove this in Section 6.2, using the monotonicity of $E(\alpha)$ from Theorem 11. In our earlier paper [26, Claim 5.1.3], we proved the theorem for $-\frac{1}{2} \leq q < 1$ and $3 \leq q \leq 4.54$.

If $E' > 0$ when $1.795 < q < 2$ (as suggested by Figure 5) then Theorem 12 extends to these q-values also. If $E' < 0$ when $1 < q \leq 1.75$ (suggested by Figure 3) then Theorem 12 holds but with the inequalities reversed. If E is nonmonotonic when $1.75 < q < 1.79$ (indicated by Figure 4) then Theorem 12 fails, for as its proof indicates, there can exist two distinct steady states with the same period and area. In that case, our results suggest the steady state with the lower minimum value is linearly stable and the other one is linearly unstable, by Figure 4, Theorem 9 and the numerical observation in Section 6.3 that the minimum value of $h_{\rm ss}$ increases with α when the area is fixed.

In Theorem 12 one might also ask if there can be a non-constant positive periodic steady state $h_{\rm ss}$ whose period equals $P_{\rm ss}/j$ for some integer $j \geq 2$, and which has correspondingly reduced area $A_{\rm ss}/j$ per period. For $q \geq 2$ or $q < 1$, the theorem shows this is possible precisely when $j^{-2}\mathcal{B}P_{\rm ss}^{3-q}A_{\rm ss}^{q-1} \in (E_0(q), 4\pi^2)$. Referring to Figure 6, we see this condition holds for all large j when $q \leq -1$, since $E_0(q) = 0$. When $q > -1$, it will hold for at most finitely many j. (Of course, any such positive periodic steady state would be linearly unstable at period $P_{\rm ss}$, by Theorem 1.)

We close this section of results for power-law coefficients with the aesthetically pleasing result that the mean value A/P of the steady state k_α varies monotonically with the minimum height α.

Theorem 13. *For $0 < \alpha < 1$,*

$$\left(\frac{A}{P}\right)'(\alpha) \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \quad \textit{and} \quad \frac{A}{P} \begin{cases} < 1 \\ = 1 \\ > 1 \end{cases} \quad \textit{when} \quad \begin{cases} q > 1, \\ q = 1, \\ q < 1, \end{cases}$$

with $A/P \to 1$ as $\alpha \to 1$.

The theorem is proved in Section 6.4.

4. Proofs of Theorems 1–5

4.1. **Proof of Theorem 1.** Translate $h_{\rm ss}$ so that it has a global minimum at $x = 0$ with this minimum recurring at $X/j, 2X/j, \ldots, X$. Define a trial function by truncating and vertically translating $h_{\rm ss}$: let

$$w(x) = \left\{ \begin{array}{ll} h_{\rm ss}(x) & \text{on } [0, X/j] \\ h_{\rm ss}(0) & \text{on } (X/j, X) \end{array} \right\} - c$$

where the constant c is chosen to ensure w has weighted mean value $\int_0^X wf(h_{\rm ss})^{-1}\,dx = 0$. This implies $c \neq h_{\rm ss}(0)$. Extend w to be X-periodic on $\mathbb{R}$ and notice $w \in H^2(\mathbb{T}_X)$ since $h_{\rm ss}' = 0$ at $x = 0, X/j$. With this trial function, the numerator of the Rayleigh quotient in

(5) is

$$\begin{aligned}
&\int_0^X \left[w''^2 - r(h_{ss})w'^2 + aw^2 f(h_{ss})^{-1}\right] dx \\
&= \int_0^{X/j} \left\{h_{ss}''^2 - r(h_{ss})h_{ss}'^2 + ah_{ss}^2 f(h_{ss})^{-1} + a\left[-2ch_{ss} + c^2\right] f(h_{ss})^{-1}\right\} dx \\
&\qquad + a\int_{X/j}^X w^2 f(h_{ss})^{-1}\, dx \\
&= \int_0^{X/j} \left\{\left[(h_{ss}''' + r(h_{ss})h_{ss}')' + af(h_{ss})^{-1}h_{ss}\right] h_{ss} + a\left[-2ch_{ss} + c^2\right] f(h_{ss})^{-1}\right\} dx \\
&\qquad + a\int_{X/j}^X w^2 f(h_{ss})^{-1}\, dx \qquad \text{by integration by parts} \\
&= \int_0^{X/j} a\left[-2ch_{ss} + c^2\right] f(h_{ss})^{-1}\, dx + a\int_{X/j}^X w^2 f(h_{ss})^{-1}\, dx, \\
&\qquad \text{by the steady state equations (2) and (3)} \\
&= \left\{ \begin{array}{cl} 0 & \text{if } a = 0 \\ ac^2 X/j + a(h_{ss}(0) - c)^2 (X - X/j) & \text{if } a < 0 \end{array} \right\}
\end{aligned}$$

where in the last step we used that if $a < 0$ then $f \equiv 1$ by assumption and $\int_0^{X/j} h_{ss}\, dx = 0$ by integrating the steady state equation (2). From this, we conclude that $\lambda_1(h_{ss}) < 0$ when $a < 0$, and $\lambda_1(h_{ss}) \leq 0$ when $a = 0$.

It remains to prove $\lambda_1(h_{ss}) < 0$ when $a = 0$. Suppose instead $\lambda_1(h_{ss}) = 0$ and $a = 0$. Then our trial function w is a minimizer for the Rayleigh quotient and so it is C^4-smooth and satisfies the Euler–Lagrange equation $\mathcal{I}w = 0$: $w'''' + (r(h_{ss})w')' = 0$. However, w is constant on the interval $(X/j, X)$, and so by the uniqueness theorem for linear ODEs with continuous coefficients, w must equal the same constant on $(0, X/j)$ also, contradicting the hypothesis that h_{ss} is non-constant. Hence $\lambda_1(h_{ss}) < 0$, finishing the proof.

This argument breaks down when the steady state has the same period as the perturbations (*i.e.* when $j = 1$) because then we do not know w is constant on an interval. It also breaks down when $a > 0$.

4.2. **Proof of Lemma 2.** Multiplying the equation $(f(h_{ss})h_{ss}''')' + (g(h_{ss})h_{ss}')' + ah_{ss} = 0$ by $-h_{ss}''$ and integrating by parts gives

$$\begin{aligned}
0 &= \int_0^X \left[f(h_{ss}){h_{ss}'''}^2 + g(h_{ss})h_{ss}'h_{ss}''' + a{h_{ss}'}^2\right] dx \\
&= \int_0^X \left[f(h_{ss})\left(h_{ss}''' + \frac{1}{2}\frac{g(h_{ss})}{f(h_{ss})}h_{ss}'\right)^2 + \left(a - \frac{1}{4}\frac{g(h_{ss})^2}{f(h_{ss})}\right){h_{ss}'}^2\right] dx,
\end{aligned}$$

by completing the square. The lemma follows, since $f > 0$.

4.3. **Proof of Theorem 3.** We take as our trial function $w = h'_{\rm ss} \not\equiv 0$. This belongs to $H^2(\mathbb{T}_X)$ and has $\int_0^X w f(h_{\rm ss})^{-1}\,dx = 0$ as required, and

$$\begin{aligned}
&\text{numerator of the Rayleigh quotient for } \lambda_1(h_{\rm ss}) \text{ in (5)}\\
&= \int_0^X \left[h'''^2_{\rm ss} - r(h_{\rm ss}) h''^2_{\rm ss} + a f(h_{\rm ss})^{-1} h'^2_{\rm ss} \right] dx \\
&= \int_0^X h''_{\rm ss} \left[-h''''_{\rm ss} - r(h_{\rm ss}) h''_{\rm ss} - a f(h_{\rm ss})^{-1} h_{\rm ss} \right] dx, \\
&\qquad \text{where we used that } f(h_{\rm ss})^{-1} \equiv 1 \text{ if } a \neq 0 \\
&= \int_0^X h''_{\rm ss} r'(h_{\rm ss}) h'^2_{\rm ss} dx \qquad \text{by the steady state equation (2)} \\
&= \frac{1}{3} \int_0^X r'(h_{\rm ss}) \left[h'^3_{\rm ss} \right]' dx = -\frac{1}{3} \int_0^X r''(h_{\rm ss}) h'^4_{\rm ss}\, dx \leq 0
\end{aligned}$$

because $r'' \geq 0$ by assumption. Hence $\lambda_1(h_{\rm ss}) \leq 0$.

If $\lambda_1(h_{\rm ss}) < 0$ then we are done, so suppose $\lambda_1(h_{\rm ss}) = 0$. Then $w = h'_{\rm ss}$ is a minimizer for the Rayleigh quotient (5) and so it satisfies the Euler–Lagrange equation $\mathcal{I} w = 0$:

$$\begin{aligned}
0 = \mathcal{I} h'_{\rm ss} &= h'''''_{\rm ss} + (r(h_{\rm ss}) h''_{\rm ss})' + a f(h_{\rm ss})^{-1} h'_{\rm ss} \\
&= -\left(r'(h_{\rm ss}) h'^2_{\rm ss} \right)'
\end{aligned}$$

by the steady state equations (2) and (3), and using that if $a \neq 0$ then $f \equiv 1$. Hence $r'(h_{\rm ss}) h'^2_{\rm ss}$ is constant. Evaluating at a minimum point of $h_{\rm ss}$ shows this constant is zero and so $r'(h_{\rm ss}) h'_{\rm ss} \equiv 0$. This means $r(h_{\rm ss}(x))$ is constant, contradicting a hypothesis of the theorem and completing the proof.

In the above we used the fifth derivative of $h_{\rm ss}$, which exists and is continuous by bootstrapping: the lower-order terms in the steady state equation (2) are all C^1-smooth by the hypotheses in this theorem.

4.4. **Proof of Theorem 4.** Define a trial function by truncating and vertically translating the steady state $h_{\rm ss}$ so that it equals zero at the endpoints:

$$w(x) = \begin{cases} h_{\rm ss}(x) - h_{\rm ss}(b) & \text{if } 0 \leq x \leq b, \\ 0 & \text{if } b < x < X. \end{cases}$$

Clearly $w \in H^1_0(0, X)$, but also $w \in H^2(0, X)$ since $h'_{\rm ss}(b) = 0$. For w to be a valid trial function, we must first argue $w \not\equiv 0$. If $w \equiv 0$ then $h_{\rm ss} \equiv h_{\rm ss}(b)$ on $(0, b)$ and so $a h_{\rm ss}(b) = 0$ by the steady state equation (2). Hence $y_1 \equiv h_{\rm ss}(b)$ solves the linear ODE $(f(h_{\rm ss}) y''' + g(h_{\rm ss}) y')' + a y = 0$ on $(0, X)$, where $f > 0$. By (2), $y_2 = h_{\rm ss}$ is another solution of this equation, and since $y_1 \equiv y_2$ on $(0, b)$, uniqueness for linear ODEs with continuous coefficients implies $y_1 \equiv y_2$ everywhere. This proves $h_{\rm ss}$ is constant, a contradiction.

We now evaluate the numerator of the Rayleigh quotient in (7) as

$$\begin{aligned}
&\int_0^X \left[w''^2 - r(h_{\rm ss})w'^2 + aw^2 f(h_{\rm ss})^{-1}\right] dx \\
&\quad = \int_0^b \left\{\left[h_{\rm ss}''^2 - r(h_{\rm ss})h_{\rm ss}'^2 + ah_{\rm ss}^2 f(h_{\rm ss})^{-1}\right] + a\left[-2h_{\rm ss}(b)h_{\rm ss} + h_{\rm ss}(b)^2\right] f(h_{\rm ss})^{-1}\right\} dx \\
&\quad = \int_0^b \left\{\left[h_{\rm ss}'''' + (r(h_{\rm ss})h_{\rm ss}')' + ah_{\rm ss} f(h_{\rm ss})^{-1}\right] h_{\rm ss} + a\left[-2h_{\rm ss}(b)h_{\rm ss} + h_{\rm ss}(b)^2\right] f(h_{\rm ss})^{-1}\right\} dx \\
&\quad = \int_0^b a\left[-2h_{\rm ss}(b)h_{\rm ss} + h_{\rm ss}(b)^2\right] f(h_{\rm ss})^{-1}\, dx, \quad \text{by the steady state equations (2) and (3)} \\
&\quad = \left\{ \begin{array}{cl} 0 & \text{if } a = 0 \\ ah_{\rm ss}(b)^2 b & \text{if } a < 0 \end{array} \right\} \\
&\quad \le 0
\end{aligned}$$

where in the second-to-last step we used that if $a \neq 0$ then $f \equiv 1$ by assumption and $\int_0^b h_{\rm ss}\, dx = 0$ by integrating the steady state equation (2). We conclude $\nu_1(h_{\rm ss}) \le 0$.

If $\nu_1(h_{\rm ss}) < 0$ then we are done, so suppose $\nu_1(h_{\rm ss}) = 0$. Then w is a minimizer, and so it satisfies the natural boundary condition $w''(0) = 0$. By construction, w, w', and w''' equal zero at $x = 0$. Since w and its first three derivatives vanish at $x = 0$, the uniqueness theorem for the linear equation (6) satisfied by the minimizer w now implies $w \equiv 0$. This contradiction completes the proof.

4.5. **Proof of Theorem 5.** Follow the proof of Theorem 3 almost verbatim. Observe that $w = h_{\rm ss}'$ belongs to $H^2(0, X) \cap H_0^1(0, X)$, since the Neumann boundary conditions give $w = h_{\rm ss}' = 0$ at the endpoints. Thus we can use w as a trial function in the Rayleigh quotient (7) for $\nu_1(h_{\rm ss})$.

At the end of the proof, show $r'(h_{\rm ss})h_{\rm ss}'^2 \equiv 0$ by evaluating at $x = 0$ and using the Neumann boundary conditions.

5. Proof of Theorems 9 and 11

We start the proof by reducing to a second order eigenvalue problem connected with a nonlocal reaction-diffusion equation. We then rescale to period 1 and show that stability is characterized by the sign of $E'(\alpha)$.

Second order problem. Recalling the general set-up of Section 2 and the Rayleigh quotient (5) for λ_1, assume $a = 0$ and define a new Rayleigh quotient and minimization problem by

$$\tau_1(h_{\rm ss}) = \min_u R[u] := \min_u \frac{\int_0^X \left[(u')^2 - r(h_{\rm ss})u^2\right] dx}{\int_0^X u^2\, dx}, \tag{16}$$

where the minimum is taken over $u \in H^1(\mathbb{T}_X) \setminus \{0\}$ with u having mean value zero. In Appendix A we prove the minimum for $\tau_1(h_{\rm ss})$ is attained, and that each minimizing function

u_1 is C^2-smooth and satisfies the Euler–Lagrange equation

$$u_1'' + r(h_{\rm ss})u_1 + \tau_1(h_{\rm ss})u_1 = \gamma, \qquad \int_0^X u_1\,dx = 0, \tag{17}$$

for some constant γ. Integrating over a period gives $\gamma = \int_0^X r(h_{\rm ss})u\,dx/X$, making this a *nonlocal* reaction-diffusion equation. We refer the reader interested in stability of general nonlocal reaction-diffusion equations to Freitas [12].

Notice $\lambda_1(h_{\rm ss}) < 0$ if and only if $\tau_1(h_{\rm ss}) < 0$, by comparing the formulas (5) and (16) for λ_1 and τ_1 and using trial functions w and u related by $w' = u$ with $\int_0^X wf(h_{\rm ss})^{-1}\,dx = 0$. Thus linear stability is determined by τ_1. But the minimizers for τ_1 may be qualitatively different from those of λ_1: for example, with $a = 0, f \equiv 1, g \equiv r \equiv 9$ and $X = 2\pi$ we find $u(x) = \sin x$ and $\cos x$ are minimizers for τ_1, whereas $w(x) = \sin 2x$ and $\cos 2x$ are minimizers for λ_1.

Characterizing stability in terms of the eigenvalue τ_1 of the second order problem is very natural from another perspective as well. In a companion paper [27] we study a Liapunov function (see the conclusions §7) that is dissipated in time by positive solutions of the evolution equation. If one perturbs a steady state in some direction u then the first variation of this Liapunov function at a steady state is 0. The second variation in the direction u is precisely the numerator of the Rayleigh quotient for τ_1, so that if $\tau_1 < 0$ then there are arbitrarily small perturbations of the steady state that decrease the Liapunov function. A positive solution with such perturbed initial data cannot relax to the steady state: the steady state is not asymptotically stable. Such a Liapunov function approach to stability for the Cahn–Hilliard equation goes back at least to Carr, Gurtin, and Slemrod [6].

Notice $u = h_{\rm ss}'$ is an eigenfunction of (17) with zero eigenvalue, $\tau = 0$, by (3). This eigenfunction corresponds to an infinitesimal translation of the steady state. It follows that $\tau_1(h_{\rm ss}) \leq 0$, and so linear stability is equivalent to $\tau_1 = 0$ while instability requires $\tau_1 < 0$.

Rescaling to period 1. Let $q \in \mathbb{R}, \alpha \in (0,1)$, and now consider power law coefficients as in Theorems 9 and 11. We are studying the rescaled problem, that is, studying the stability of k_α rather than $h_{\rm ss}$, and so $X = P = P(\alpha)$ and $r(k_\alpha) = k_\alpha^{q-1}$. Recall $E = P^{3-q}A^{q-1}$ and that

$$K_\alpha(x) = \frac{P(\alpha)}{A(\alpha)}k_\alpha(P(\alpha)x), \qquad \kappa_\alpha(x) = \frac{\partial}{\partial\alpha}K_\alpha(x).$$

By writing $v(x) = u(P(\alpha)x)$ we see $\tau_1(k_\alpha) = P(\alpha)^{-2}\mu_1(\alpha)$, where

$$\mu_1(\alpha) := \min\left\{\frac{\int_0^1 [(v')^2 - E(\alpha)K_\alpha^{q-1}v^2]\,dx}{\int_0^1 v^2\,dx} : v \in H^1(\mathbb{T}_1)\setminus\{0\}, \int_0^1 v(x)\,dx = 0\right\}. \tag{18}$$

Since $\lambda_1(k_\alpha) < 0$ if and only if $\tau_1(k_\alpha) < 0$, if and only if $\mu_1(\alpha) < 0$, we see

> k_α is linearly unstable at period $P(\alpha)$ if and only if $\mu_1(\alpha) < 0$, and is linearly stable otherwise.

Note that minimizers for $\mu_1(\alpha)$ satisfy $v'' + E(\alpha)K_\alpha^{q-1}v + \mu_1(\alpha)v = \gamma$ for some constant γ.

The following propositions relate the sign of μ_1 to the sign of E'; these relations prove Theorems 9 and 11.

Proposition 14. *Let $q > 1$. For each $\alpha \in (0,1)$, if $E'(\alpha) > 0$ then $\mu_1(\alpha) < 0$.*

Proposition 15. *Let $q \in \mathbb{R}$. For each $\alpha \in (0,1)$, if $\mu_1(\alpha) < 0$ then $E'(\alpha) > 0$.*

Propositions 14 and 15 imply for $q > 1$ that k is linearly stable ($\mu_1(\alpha) \geq 0$) if and only if $E'(\alpha) \leq 0$. This proves Theorem 9. When $q \geq 2$ or $q < 1$, k is linearly unstable by Theorem 7, hence $\mu_1(\alpha) < 0$ and so $E'(\alpha) > 0$ by Proposition 15, proving Theorem 11.

Proposition 14 is proved in Section 5.3 by using κ_α as a trial function for $\mu_1(\alpha)$. Proposition 15 is proved in Section 5.4 by adapting the time map method for proving stability of *local* reaction-diffusion equations with Neumann boundary conditions.

In the rest of Section 5, we will not use the α-dependence of k_α, K_α, and κ_α. For this reason we will suppress the α-subscripts.

5.1. **Differential equations used in proving Propositions 14 and 15.** We first collect differential equations involving k, K, and κ. We recall that k is positive, periodic and non-constant, and satisfies $k(0) = \alpha, k'(0) = 0$, and

$$k'' + \frac{k^q - 1}{q} = 0 \qquad \text{when } q \neq 0, \tag{19}$$

with the term $(k^q - 1)/q$ being replaced by $\log k$ when $q = 0$. Here k' denotes k_x. Integrating (19), one finds $\int_0^P k^q \, dx/P = 1$ when $q \neq 0$ and $\int_0^P \log k \, dx/P = 0$ when $q = 0$. It follows by Jensen's inequality that

$$\frac{A}{P} \begin{cases} \leq 1 & \text{when } q > 1, \\ = 1 & \text{when } q = 1, \\ \geq 1 & \text{when } q < 1. \end{cases} \tag{20}$$

Multiplying (19) by k' and integrating yields

$$\frac{1}{2}(k')^2 + H(k) = H(\alpha) \tag{21}$$

where for $y > 0$,

$$H(y) := \begin{cases} \frac{1}{q}\left[\frac{y^{q+1}}{q+1} - y\right], & q \neq 0, -1, \\ y \log y - y, & q = 0, \\ y - \log y, & q = -1. \end{cases} \tag{22}$$

Note $H(y)$ is strictly convex, $H'' > 0$, with its minimum at $y = 1$. We see from (21) that the maximum of $k = k_\alpha$ is the number $\beta > 1$ that solves $H(\beta) = H(\alpha)$; β will be used later.

For future reference, note that

$$(q+1)H(y) \to -1 \qquad \text{as } q \to -1 \text{ with } y \text{ fixed}, \tag{23}$$

even though $(q+1)H(y)=0$ for all y when $q=-1$.

Rescaling equation (19), K satisfies a differential equation:

$$K'' + E\frac{K^q - (P/A)^q}{q} = 0 \qquad \text{when } q \neq 0, \tag{24}$$

with $K'' + E\left[\log K - \log(P/A)\right] = 0$ when $q = 0$.

Rescaling equation (21), K also satisfies

$$\frac{1}{2}(K')^2 + E\frac{1}{q}\left[\frac{K^{q+1}}{q+1} - \left(\frac{P}{A}\right)^q K\right] = E\left(\frac{P}{A}\right)^{q+1} H(\alpha) \qquad \text{when } q \neq 0, -1, \tag{25}$$

with

$$\frac{1}{2}(K')^2 + EK\left(\log K - 1 - \log(P/A)\right) = E\left(\frac{P}{A}\right) H(\alpha) \qquad \text{when } q = 0$$

and

$$\frac{1}{2}(K')^2 + E\left[\left(\frac{P}{A}\right)^{-1} K - \log K + \log(P/A)\right] = EH(\alpha) \qquad \text{when } q = -1.$$

Next we find equations for κ. Differentiating (24) with respect to α gives

$$\kappa'' + EK^{q-1}\kappa - E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'(\alpha) + E'(\alpha)\frac{K^q - (P/A)^q}{q} = 0 \qquad \text{when } q \neq 0. \tag{26}$$

When $q = 0$, the same formula holds but with $[K^q - (P/A)^q]/q$ on the lefthand side replaced by $[\log K - \log(P/A)]$. In (25)–(26) and in the rest of this section, $K', K'', \kappa', \kappa''$ denote x-derivatives, but otherwise $'$ denotes an α-derivative.

Differentiating (25) with respect to α, for $q \neq -1, 0$ we have

$$\begin{aligned} K'\kappa' \quad + \quad & E\frac{K^q - (P/A)^q}{q}\kappa + E'(\alpha)\left(\frac{P}{A}\right)^{q+1} H(k) - E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'(\alpha)K \\ = \quad & E'(\alpha)\left(\frac{P}{A}\right)^{q+1} H(\alpha) + E\left(\frac{P}{A}\right)^{q}\left(\frac{P}{A}\right)'(\alpha)\left[(q+1)H(\alpha)\right] + E\left(\frac{P}{A}\right)^{q+1} H'(\alpha), \end{aligned} \tag{27}$$

where k is evaluated at Px. When $q = 0$, the same formula holds but with $[K^q - (P/A)^q]/q$ on the lefthand side replaced by $[\log K - \log(P/A)]$. When $q = -1$, the same formula holds but with $(q+1)H(\alpha)$ on the righthand side replaced by -1 (which is consistent with the $q \to -1$ limiting behavior in (23)).

5.2. **Derivatives of P, A, E and integrals of K.** Next we establish formulas for the α-derivatives of $P = P(\alpha)$, $A = \int_0^P k\,dx$, and of $\int_0^1 K^{q+1}\,dx$. These are used in the proofs of Propositions 14 and 15. We also obtain formulas for the α-derivatives of $E = P^{3-q}A^{q-1}$ and A/P (the mean value of k).

We first multiply (19) by k and integrate, giving

$$-\int_0^P (k')^2\,dx + \frac{1}{q}\int_0^P k^{q+1}\,dx - \frac{1}{q}A = 0 \qquad \text{when } q \neq 0, \tag{28}$$

or $-\int_0^P (k')^2\,dx + \int_0^P k\log k\,dx = 0$ when $q = 0$. Further, integrating (21) yields

$$\int_0^P (k')^2\,dx + 2\int_0^P G(k)\,dx - \frac{2}{q}A = 2H(\alpha)P \qquad \text{when } q \neq 0, \tag{29}$$

where

$$G(y) := \begin{cases} \frac{y^{q+1}}{q(q+1)}, & q \neq 0, -1, \\ y\log y - y, & q = 0, \\ -\log y, & q = -1. \end{cases}$$

When $q = 0$ one omits the term $-\frac{2}{q}A$ from (29).

Adding (28) and (29), we find

$$\int_0^P G(k)\,dx = \frac{1}{q+3}\left[\frac{3}{q}A + 2H(\alpha)P\right] \qquad \text{when } q \neq -3, -1, 0. \tag{30}$$

When $q = 0$ the same formula holds except the term $\frac{3}{q}A$ is replaced by $-A$. When $q = -1$ one adds $P/2$ to the righthand side of (30). When $q = -3$ we do not get a formula for $\int_0^P G(k)\,dx$, but adding (28) and (29) yields an exact relation between the period and area:

$$A = 2H(\alpha)P \qquad \text{when } q = -3, \tag{31}$$

or $A = [\alpha^{-2} + 2\alpha]P/3$.

We use identities (28) and (29) to solve for $\int k'^2$:

$$0 < \int_0^P (k')^2\,dx = \frac{2}{q+3}\left[A + (q+1)H(\alpha)P\right] \qquad \text{when } q \neq -3, -1. \tag{32}$$

Notice $0 < \int_0^P (k')^2\,dx = A - P$ when $q = -1$, directly from (28), and $0 = A + (q+1)H(\alpha)P$ when $q = -3$, by (31). Hence

$$A + (q+1)H(\alpha)P \begin{cases} > 0 & \text{when } q > -3, \\ = 0 & \text{when } q = -3, \\ < 0 & \text{when } q < -3, \end{cases} \tag{33}$$

where for $q = -1$ we replace $(q+1)H(\alpha)$ with -1 on the lefthand side.

By differentiating (32) with respect to α,

$$\frac{d}{d\alpha}\int_0^P (k')^2\,dx = \frac{2}{q+3}\left[A' + (q+1)H(\alpha)P' + (q+1)H'(\alpha)P\right] \qquad \text{when } q \neq -3, -1. \tag{34}$$

However we also know directly from the nonlinear oscillator formulation (21) that for all q,

$$\begin{aligned}
\frac{d}{d\alpha}\int_0^P (k')^2\,dx &= 2\frac{d}{d\alpha}\int_0^{P/2}\left(\frac{dk}{dx}\right)^2 dx \\
&= 2\frac{d}{d\alpha}\int_\alpha^\beta \frac{dk}{dx}\,dk \\
&= 2\frac{d}{d\alpha}\int_\alpha^\beta \sqrt{2H(\alpha)-2H(y)}\,dy \qquad \text{by (21)} \\
&= 2\int_\alpha^\beta \frac{H'(\alpha)}{\sqrt{2H(\alpha)-2H(y)}}\,dy \qquad \text{using that } H(\beta)=H(\alpha) \\
&= 2H'(\alpha)\cdot\int_\alpha^\beta \frac{dx}{dk}\,dk = H'(\alpha)P.
\end{aligned}$$

Because $H'(\alpha) < 0$ for $\alpha \in (0,1)$, the last equality implies

$$\frac{d}{d\alpha}\int_0^P (k')^2\,dx = H'(\alpha)P < 0. \tag{35}$$

Equating equations (34) and (35), we solve for $A'(\alpha)$:

Lemma 16. *For all $q \in \mathbb{R}\setminus\{-1\}$,*

$$A' = -(q+1)H(\alpha)P' - \frac{q-1}{2}H'(\alpha)P.$$

For $q=-1$, we replace $(q+1)H(\alpha)$ on the righthand side with -1.

For $q=-3$, Lemma 16 follows from differentiating (31). For $q=-1$ it follows from differentiating $\int_0^P (k')^2\,dx = A-P$.

We use Lemma 16 to classify the monotonicity of the mean, A/P:

Lemma 17. *For all $q \in \mathbb{R}\setminus\{-1\}$,*

$$\left(\frac{A}{P}\right)' = -\frac{P'}{P^2}\left[A+(q+1)H(\alpha)P\right] - \frac{q-1}{2}H'(\alpha).$$

For $q=-1$, we replace $(q+1)H(\alpha)$ on the righthand side with -1.

Therefore

$$\left(\frac{A}{P}\right)'(\alpha)\begin{cases} >0 & \text{when } q>1, \\ <0 & \text{when } q\le -3 \text{ or } -\frac{1}{2}\le q<1. \end{cases}$$

Proof. The formula for $(A/P)'$ follows directly from Lemma 16. The monotonicity claims for A/P follow in a straightforward manner by combining this formula for $(A/P)'$ with formula

(33), the fact that $H'(\alpha) < 0$, and the inequalities for $P'(\alpha)$

$$P'(\alpha) \begin{cases} < 0 & \text{when } q > 1, \\ = 0 & \text{when } q = 1, \\ > 0 & \text{when } -\frac{1}{2} < q < 1, \\ = 0 & \text{when } q = -\frac{1}{2}, \\ < 0 & \text{when } q < -\frac{1}{2}. \end{cases} \tag{36}$$

taken from [26, Proposition 7.3]. □

Lemmas 16 and 17 then yield information on the monotonicity of E:

Lemma 18. *For all $q \in \mathbb{R} \setminus \{-1\}$,*

$$E' = -\left(\frac{A}{P}\right)^{q-2} \left\{ P'\left[(q-3)A + (q-1)(q+1)H(\alpha)P\right] + \frac{1}{2}(q-1)^2 H'(\alpha) P^2 \right\}.$$

For $q = -1$, we replace $(q+1)H(\alpha)$ on the righthand side with -1.

Also $E'(\alpha) > 0$ when $-\frac{1}{2} \leq q < 1$.

Proof. The formula for E' follows from differentiating $E = P^{3-q}A^{q-1}$ and using the formula for A' in Lemma 16. For $-\frac{1}{2} \leq q < 1$ we know $P' \geq 0$ by (36) and $(A/P)' < 0$ by Lemma 17. Differentiating $E = P^2(A/P)^{q-1}$ yields $E' > 0$. □

We now find formulas for $\int_0^1 K^{q+1}\,dx$ and its derivative. From (30),

$$\int_0^1 \frac{K^{q+1}}{q(q+1)}\,dx = \frac{1}{q+3}\left(\frac{P}{A}\right)^q \left[\frac{3}{q} + 2H(\alpha)\frac{P}{A}\right] \qquad \text{when } q \neq -3, -1, 0.$$

Differentiating and using Lemma 17, we obtain

$$\frac{d}{d\alpha}\int_0^1 G(K)\,dx = \frac{1}{q+3}\left(\frac{P}{A}\right)^{q-1} \left\{ \frac{1}{A}\left[3A + 2(q+1)H(\alpha)P\right]\left(\frac{P}{A}\right)' + 2H'(\alpha)\left(\frac{P}{A}\right)^2 \right\} \tag{37}$$

for $q \neq -3, -1, 0$. For $q = 0$ the same formula holds. For $q = -1$ it holds with $(q+1)H(\alpha)$ on the righthand side replaced by -1.

Lemma 19. *For all $q > 1$,*

$$\frac{d}{d\alpha}\int_0^1 G(K)\,dx < 0.$$

Proof. Since $H'(\alpha) < 0$, and $(P/A)' < 0$ by Lemma 17, it suffices observe that on the righthand side of (37), $[3A + 2(q+1)H(\alpha)P] > A > 0$ by (33). □

5.3. **Proof of Proposition 14.** Assume $E'(\alpha) > 0$. For the trial function $v = \kappa$, the numerator of the Rayleigh quotient (18) is

$$\begin{aligned}\int_0^1 \left[(\kappa')^2 - EK^{q-1}\kappa^2\right] dx &= \int_0^1 \kappa\left[-\kappa'' - EK^{q-1}\kappa\right] dx \\ &= \int_0^1 \kappa\left[E'(\alpha)\frac{K^q - (P/A)^q}{q} - E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'(\alpha)\right] dx \qquad \text{by (26)} \\ &= E'(\alpha)\int_0^1 \frac{K^q}{q}\kappa\, dx \qquad \text{since } \int_0^1 \kappa\, dx = 0 \text{ has mean value zero} \\ &= E'(\alpha)\frac{d}{d\alpha}\int_0^1 G(K)\, dx < 0,\end{aligned}$$

by Lemma 19. Therefore $\mu_1(\alpha) < 0$, as desired.

5.4. **Proof of Proposition 15.** We will use two lemmas, proved later in the section. The first lemma concerns the monotonicity properties of $\kappa = \kappa_\alpha$:

Lemma 20. *Let $q \in \mathbb{R}$. Take $0 < \alpha < 1$ and assume $E'(\alpha) \leq 0$. Then $\kappa'(x) < 0$ for all $0 < x < \frac{1}{2}$.*

The second lemma concerns the functions at which the minimum is achieved for $\mu_1(\alpha)$, in (18). If v_1 is such a function then it is smooth (see Appendix A) and satisfies the Euler–Lagrange equation

$$v_1'' + EK^{q-1}v_1 + \mu_1(\alpha)v_1 = \gamma \tag{38}$$

for some constant γ. By construction, $v_1 \not\equiv 0$ has mean value zero and is 1-periodic in x. We can assume either $v_1(0) > 0$ or else $v_1(0) = 0$ and $v_1''(0) \geq 0$, by replacing v_1 with $-v_1$ if need be.

Lemma 21. *Let $q \in \mathbb{R}$ and let v_1 be as above.*
(a) If $\mu_1(\alpha) < 0$ then v_1 is even.
(b) If v_1 is even then there exists a point $x_v \in (0, \frac{1}{2})$ such that $v_1 > 0$ on $(0, x_v)$ and $v_1 < 0$ on $(x_v, \frac{1}{2})$, except perhaps at finitely many points at which $v_1 = 0$.
(c) Suppose v_1 is even. If $q > 1$ then $\gamma < 0$, and if $q < 1$ then $\gamma > 0$.

Part (b) says the minimizer v_1 changes sign just once per half-period; this rather natural result may well exist elsewhere in the literature. Also, the 'exceptional' points in part (b) do not actually occur (although we will not use this fact). For example, when $q > 1$ we know $\gamma < 0$ by part (c), and so equation (38) implies $v_1'' < 0$ wherever $v_1 = 0$. Hence $v_1 > 0$ on $(0, x_v)$. Also, a decreasing rearrangement of v_1 on $(x_v, \frac{1}{2}]$ (and an increasing rearrangement on $[\frac{1}{2}, 1 - x_v))$ would lower the Rayleigh quotient in (18) since K^{q-1} is increasing on $(0, \frac{1}{2})$. Hence v_1 must be decreasing on $(x_v, \frac{1}{2})$, since it minimizes the Rayleigh quotient. Therefore $v_1 < 0$ on $(x_v, \frac{1}{2})$.

We first use Lemmas 20 and 21 to prove Proposition 15, and then we prove the lemmas.

Proof of Proposition 15. For $q \neq 0$,

$$\begin{aligned}
\mu_1(\alpha)\int_{-\frac{1}{2}}^{\frac{1}{2}} v_1\kappa\,dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\gamma - v_1'' - EK^{q-1}v_1\right]\kappa\,dx \quad \text{by (38)} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[-\kappa'' - EK^{q-1}\kappa\right] v_1\,dx \\
&\qquad \text{by integration by parts and since } \int_{-1/2}^{1/2}\kappa\,dx = 0 \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[E'(\alpha)\frac{K^q - (P/A)^q}{q} - E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'(\alpha)\right] v_1\,dx \quad \text{by (26)} \\
&= E'(\alpha)\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{K^q}{q} v_1\,dx \quad \text{since } v_1 \text{ has mean value zero.} \qquad (39)
\end{aligned}$$

For $q = 0$ the argument is the same, with K^q/q replaced by $\log K$.

Lemma 21(a) applies since $\mu_1(\alpha) < 0$ by assumption. Thus v_1 is even and the point x_v exists by Lemma 21(b). Hence for $q \neq 0$,

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{K^q}{q} v_1\,dx = 2\int_0^{\frac{1}{2}} \frac{K^q}{q} v_1\,dx &= 2\int_0^{x_v} \frac{K^q}{q} v_1\,dx + 2\int_{x_v}^{\frac{1}{2}} \frac{K^q}{q} v_1\,dx \\
&< 2\,\frac{K(x_v)^q}{q}\int_0^{x_v} v_1\,dx + 2\,\frac{K(x_v)^q}{q}\int_{x_v}^{\frac{1}{2}} v_1\,dx = 0. \qquad (40)
\end{aligned}$$

Here we used that K^q/q is strictly increasing on $(0, \frac{1}{2})$ and is even, and that v_1 has mean value zero and is even. The same argument holds for $q = 0$ with K^q/q replaced by $\log K$.

Because $\mu_1(\alpha)$ and $\int (K^q/q)v_1\,dx$ are both negative, it follows from (39) that

$$E'(\alpha) \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} v_1\kappa\,dx \quad \text{have the same sign.} \qquad (41)$$

We now prove $E'(\alpha) > 0$. Suppose instead $E'(\alpha) \leq 0$, so that $\int_{-1/2}^{1/2} v_1\kappa\,dx \leq 0$ by (41). Then Lemma 20 applies and $\kappa' < 0$ on $(0, \frac{1}{2})$, so that

$$\begin{aligned}
0 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \kappa v_1\,dx = 2\int_0^{\frac{1}{2}} \kappa v_1\,dx &= 2\int_0^{x_v} \kappa v_1\,dx + 2\int_{x_v}^{\frac{1}{2}} \kappa v_1\,dx \\
&> 2\,\kappa(x_v)\int_0^{x_v} v_1\,dx + 2\,\kappa(x_v)\int_{x_v}^{\frac{1}{2}} v_1\,dx = 0,
\end{aligned}$$

a contradiction. Above, we used that κ and v_1 are even, and that v_1 has mean value zero. This proves $E'(\alpha) > 0$, as desired. □

Proof of Lemma 20.

For $-\frac{1}{2} \leq q < 1$ we know $E'(\alpha) > 0$ by Lemma 18, and so there is nothing to prove since the hypothesis of Lemma 20 cannot hold. For $q = 1$ the steady states are known explicitly: $k(x) = 1 + (\alpha - 1)\cos x$ and $P = 2\pi, A = 2\pi$ by (15), so that $K(x) = 1 + (\alpha - 1)\cos(2\pi x)$ and $\kappa(x) = \cos(2\pi x)$. Hence $\kappa'(x) < 0$ for $0 < x < \frac{1}{2}$.

The cases $q < -\frac{1}{2}$ and $q > 1$ remain.

Write $x^* \in (0, \frac{1}{2})$ for the point where $K(x^*) = P/A$, in other words where $k(Px^*) = 1$. Then $K < (P/A)$ on $[0, x^*)$ and $K > (P/A)$ on $(x^*, \frac{1}{2}]$. We will prove that if $x_0 \in [0, \frac{1}{2}]$ and $\kappa'(x_0) = 0$, then

$$\kappa'' \frac{K^q - (P/A)^q}{q} > 0 \qquad \text{at } x_0. \tag{42}$$

This implies $\kappa'(x^*) \neq 0$, because the lefthand side of (42) equals zero at x^*. Also, (42) implies

$$\begin{aligned} x \in [0, x^*), \quad \kappa'(x) = 0 \quad &\Longrightarrow \quad \kappa''(x) < 0, \\ x \in (x^*, 1/2], \quad \kappa'(x) = 0 \quad &\Longrightarrow \quad \kappa''(x) > 0. \end{aligned}$$

Since $\kappa'(x) = 0$ at $x = 0$ and $x = \frac{1}{2}$ (by evenness and periodicity of κ), we conclude $\kappa' < 0$ on $(0, \frac{1}{2})$, proving the lemma. So we have only to prove (42).

Let $x_0 \in [0, \frac{1}{2}]$ be such that $\kappa'(x_0) = 0$. Evaluating identity (27) at x_0, the first term vanishes. Also, the last term in (27) is negative, since $H'(\alpha) < 0$. Therefore at the point x_0,

$$\begin{aligned} E\frac{K^q - (P/A)^q}{q}\kappa \quad < \quad & E'(\alpha)\left(\frac{P}{A}\right)^{q+1}[H(\alpha) - H(k)] \\ & + E\left(\frac{P}{A}\right)^q \left(\frac{P}{A}\right)'(\alpha)\left[(q+1)H(\alpha)\right] + E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'(\alpha)K. \end{aligned} \tag{43}$$

Multiplying (26) by $[K^q - (P/A)^q]/q$, and evaluating at the point x_0,

$$\begin{aligned} & \kappa'' \frac{K^q - (P/A)^q}{q} \\ = \quad & -E\frac{K^q - (P/A)^q}{q}\kappa K^{q-1} + E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'(\alpha)\frac{K^q - (P/A)^q}{q} - E'(\alpha)\left(\frac{K^q - (P/A)^q}{q}\right)^2 \\ > \quad & -E'(\alpha)\left(\frac{P}{A}\right)^{q+1}[H(\alpha) - H(k)]\,K^{q-1} - E\left(\frac{P}{A}\right)^q\left(\frac{P}{A}\right)'[(q+1)H(\alpha)]\,K^{q-1} \\ & -E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'K^q + E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)'\frac{K^q - (P/A)^q}{q} - E'(\alpha)\left(\frac{K^q - (P/A)^q}{q}\right)^2 \end{aligned}$$

by using (43) to estimate the first term. Hence

$$\begin{aligned} & \kappa'' \frac{K^q - (P/A)^q}{q} \\ & > -E'(\alpha)\left(\frac{P}{A}\right)^{2q}\left([H(\alpha) - H(k)]\,k^{q-1} + \left(\frac{k^q - 1}{q}\right)^2\right) + E\left(\frac{P}{A}\right)^{2q-1}\left(\frac{P}{A}\right)'c(k) \end{aligned} \tag{44}$$

at x_0, where we have introduced the function

$$c(y) = -(q+1)H(\alpha)y^{q-1} - y^q + \frac{y^q - 1}{q} \qquad \text{for } y > 0.$$

We continue here the convention that $(q+1)H(\alpha) = -1$ when $q = -1$.

Case 1: $q > 1$. In this case $c(y)$ is a concave function of y^{q-1}, attaining its maximum at $y_0 = -(q+1)H(\alpha)$. Notice $0 < y_0 < 1$ since

$$0 = H(0) > H(\alpha) > H(1) = -1/(q+1).$$

Thus $c(y_0) = (y_0^q - 1)/q < 0$ and so $c(k) < 0$ in (44). Now, $(P/A)'(\alpha) < 0$ by Lemma 17, $E'(\alpha) \leq 0$ by assumption in Lemma 20, and $H(\alpha) \geq H(k)$ by construction (recall from (22) and the associated comments that H is convex with $H(\alpha) = H(\beta)$, where β is the maximum value of k). Therefore from (44) we deduce (42).

Case 2: $q \leq -1$. Since $q < 0$, $c(y)$ is a convex function of y^{q-1}, attaining its minimum at $y_0 = -(q+1)H(\alpha)$. Notice $y_0 > 1$ when $q < -1$ because

$$H(\alpha) > H(1) = -\frac{1}{q+1} > 0,$$

and $y_0 = 1$ when $q = -1$ because then $(q+1)H(\alpha) = -1$ by convention. Thus $c(y_0) = (y_0^q - 1)/q \geq 0$, and so $c(k) \geq 0$ in (44). Differentiating the relation $(P/A)^{1-q} = EP^{-2}$ with respect to α and recalling $P' < 0$ (by (36), valid for $q < -\frac{1}{2}$) we deduce

$$\left(\frac{P}{A}\right)' > (1-q)^{-1} \left(\frac{P}{A}\right)^q E'(\alpha)P^{-2}.$$

Using this inequality to estimate $(P/A)'$ in (44) yields that at x_0,

$$\kappa'' \frac{K^q - (P/A)^q}{q} > -E'(\alpha) \left(\frac{P}{A}\right)^{2q} \left([H(\alpha) - H(k)]\, k^{q-1} + \left(\frac{k^q - 1}{q}\right)^2 - (1-q)^{-1} c(k) \right).$$

Since $E'(\alpha) \leq 0$ by assumption, it remains to prove

$$[H(\alpha) - H(y)]\, y^{q-1} + \left(\frac{y^q - 1}{q}\right)^2 - (1-q)^{-1} c(y) \geq 0 \qquad \text{whenever } \alpha \leq y \leq \beta, \tag{45}$$

for this implies inequality (42).

The y-derivative of the lefthand side of (45) equals $2\,[H(y) - H(\alpha)]\, y^{q-2}$, which is negative when $\alpha < y < \beta$. So it suffices to check (45) at the endpoint $y = \beta$. Evaluating at $y = \beta$ and using $H(\alpha) = H(\beta)$, (45) becomes

$$\frac{1}{1-q} \left(\frac{\beta^q - 1}{q}\right)^2 \geq 0,$$

which is true since $q \leq -1$. Thus (45) holds, implying (42) as desired.

Case 3: $-1 < q < -\frac{1}{2}$. If $c(k(Px_0)) \geq 0$ then the argument given for the $q \leq -1$ case applies, yielding (45) and then (42). So we assume $c(k(Px_0)) < 0$. If $(P/A)'(\alpha) \leq 0$ then

(44) implies (42), since $E'(\alpha) \le 0$ and $H(\alpha) \ge H(k)$. So we assume $c(k(Px_0)) < 0$ and $(P/A)'(\alpha) > 0$.

We evaluate (27) at x_0, where $\kappa'(x_0) = 0$. Since $H(\alpha) < H(0) = 0$,

$$E\frac{K^q - (P/A)^q}{q}\kappa < E\left(\frac{P}{A}\right)^{q-1}\left(\frac{P}{A}\right)' K + E\left(\frac{P}{A}\right)^{q+1} H'(\alpha) \qquad \text{at } x_0.$$

After multiplying (26) by $[K^q - (P/A)^q]/q$, we therefore obtain that

$$\kappa''\frac{K^q - (P/A)^q}{q} > E\left(\frac{P}{A}\right)^{2q-1}\left(\frac{P}{A}\right)'\left[-k^q + \frac{k^q-1}{q}\right] - E\left(\frac{P}{A}\right)^{2q} H'(\alpha)k^{q-1} \tag{46}$$

at x_0. To help with the remaining estimates, we now show

$$\left(\frac{P}{A}\right)' < -\left(\frac{P}{A}\right) H'(\alpha). \tag{47}$$

Indeed

$$\left(\frac{P}{A}\right)' < \frac{q-1}{2}\left(\frac{P}{A}\right)^2 H'(\alpha),$$

by the formula for $(A/P)'$ in Lemma 17 together with the observations that $P' < 0$ by (36) and $[A + (q+1)H(\alpha)P] > 0$ by (33). Also $(P/A) \le 1$ by (20) while $(q-1)/2 > -1$ and $H'(\alpha) < 0$, implying (47).

In (46) we have $[-k^q + (k^q-1)/q] < c(k) < 0$, and so we can apply (47) to the righthand side of (46), resulting in

$$\kappa''\frac{K^q - (P/A)^q}{q} > E\left(\frac{P}{A}\right)^{2q} H'(\alpha)\left[k^q - \frac{k^q-1}{q} - k^{q-1}\right].$$

On the righthand side, $[k^q - (k^q-1)/q - k^{q-1}] \le 0$ because $y \mapsto [y^q - (y^q-1)/q - y^{q-1}]$ is a concave function of y^{q-1} with maximum value 0, attained at $y = 1$. Finally, $H'(\alpha) < 0$ and so (42) holds, completing the proof. □

Proof of Lemma 21.
Proof of Lemma 21(a): We prove a more general statement. Assume u is C^2-smooth with least period $X > 0$, and that u satisfies $u'' + r(h_{ss})u + \tau u = \gamma$ for some real constants τ and γ. As usual, $r = g/f$ with $f, g \in C^1(\mathbb{R}), f > 0$, and $h_{ss} \in C^4(\mathbb{R})$ is a non-constant X-periodic steady state of (1) with $a = 0$. Assume h_{ss} has been translated to have its minimum at $x = 0$. We now prove that if u is not even then $\tau \ge 0$. This will prove part (a) of the lemma, because (38) implies that $u(x) := v_1(x/P)$ satisfies

$$u'' + k_\alpha^{q-1}u + \frac{\mu_1(\alpha)}{P^2}u = \frac{\gamma}{P^2}.$$

First, note that h_{ss} is symmetric about every point at which $h'_{ss} = 0$, by uniqueness for the second order oscillator ODE (8). Therefore $h'_{ss} > 0$ on $(0, X/2)$, since otherwise h_{ss} would have period less than X.

Assume u is not even, hence the odd function $u_o(x) = u(x) - u(-x)$ is not identically zero. By the evenness of $h_{\rm ss}$, u_o satisfies $u_o'' + r(h_{\rm ss})u_o + \tau u_o = 0$, which is a homogeneous linear equation. Since $u_o(0) = 0$, one must have $u_o'(0) \neq 0$ because otherwise $u_o \equiv 0$ by the uniqueness theorem for ODEs. Also $u_o(X/2) = 0$ by the oddness and periodicity of u_o, and so there is a point $b \in (0, X/2]$ with $u_o(b) = 0$ and $u_o \neq 0$ between 0 and b. Suppose $u_o > 0$ between 0 and b (otherwise consider $-u_o$). Then

$$\begin{aligned}
\tau \int_0^b u_o h_{\rm ss}' \, dx &= -\int_0^b \left[u_o'' + r(h_{\rm ss})u_o\right] h_{\rm ss}' \, dx \quad \text{since } u_o'' + r(h_{\rm ss})u_o + \tau u_o = 0 \\
&= -u_o'(b)h_{\rm ss}'(b) - \int_0^b \left[h_{\rm ss}''' + r(h_{\rm ss})h_{\rm ss}'\right] u_o \, dx \\
&\qquad\qquad\qquad\qquad \text{by integration by parts and } h_{\rm ss}'(0) = 0 \\
&= -u_o'(b)h_{\rm ss}'(b) \qquad \text{by (3)} \\
&\geq 0,
\end{aligned}$$

because $u_o'(b) \leq 0$ and $h_{\rm ss}'(b) \geq 0$. Since u_o and $h_{\rm ss}'$ are positive on $(0, b)$ it follows that $\tau \geq 0$.

Proof of Lemma 21(b): Assume v_1 is even. Since v_1 has mean value zero, it vanishes at some point in $(0, \frac{1}{2})$. Furthermore, v_1 has only finitely many zeros in $(0, \frac{1}{2})$ because otherwise there would be an accumulation point of zeros of v_1, and v_1, v_1' and v_1'' would all vanish there, implying $\gamma = 0$ in (38) and hence $v_1 \equiv 0$ by the uniqueness theorem for ODEs.

Let x_1 be the smallest zero of v_1 in $(0, \frac{1}{2})$ and recall that either $v_1(0) > 0$ or else $v_1(0) = 0$ and $v_1''(0) \geq 0$, by the construction of v_1 before the Lemma. If either $v_1(0) > 0$ or $v_1(0) = 0$ and $v''(0) > 0$ then $v_1 > 0$ on $(0, x_1)$. If $v_1(0) = 0, v_1''(0) = 0$, then $v_1 \equiv 0$ by the uniqueness theorem since also $v_1'(0) = 0$ by evenness; thus this case cannot occur. Hence $v_1 > 0$ on $(0, x_1)$.

Let x_2 be the largest zero of v_1 in $(0, \frac{1}{2})$. Then $x_1 \leq x_2 < \frac{1}{2}$ and $v_1 \neq 0$ on $(x_2, \frac{1}{2})$. If $x_1 = x_2$ then $v_1 < 0$ on $(x_1, \frac{1}{2})$ (since v_1 has mean value 0) and the conclusion of the lemma holds with $x_v = x_1$.

Assume that $x_1 < x_2$. We define the sets

$$\begin{aligned}
\mathcal{P} &= \{x \in (x_1, x_2) : v_1(x) > 0\}, \\
\mathcal{N} &= \{x \in (x_1, x_2) : v_1(x) < 0\}.
\end{aligned}$$

If $\mathcal{N}$ is empty then $v_1 > 0$ on $(0, x_2)$ except perhaps at finitely many points, and so $v_1 < 0$ on $(x_2, \frac{1}{2})$ (since v_1 has mean value zero): thus the conclusion of the lemma holds with $x_v = x_2$. So assume $\mathcal{N} \neq \emptyset$. There are three cases to consider.

Case (i): $v_1 > 0$ on $(x_2, \frac{1}{2})$. We find a contradiction by constructing a new minimizer v for $\mu_1(\alpha)$ that vanishes on an open set. Hence Case (i) cannot occur.

Define

$$A = -\mathcal{N}, \qquad B = -\mathcal{P} \cup (-x_1, x_1) \cup \mathcal{P}, \qquad C = \mathcal{N}.$$

In this case,

$$-\int_{A\cup B\cup C} v_1\,dx = -2\int_0^{x_2} v_1\,dx = 2\int_{x_2}^{\frac{1}{2}} v_1\,dx > 0,$$

where we used that v_1 has mean value zero and is even. Defining

$$I_A = \int_A |v_1|\,dx, \qquad I_B = \int_B |v_1|\,dx, \qquad I_C = \int_C |v_1|\,dx,$$

one has:

(48) $\quad A, B, C$ are nonempty and $I_A, I_B, I_C > 0$, with $I_A = I_C$ and $I_A - I_B + I_C \geq 0$.

To construct the new minimizer, we introduce numbers

$$\begin{aligned} a &= \sqrt{I_B}\left[\sqrt{I_B} - \sqrt{I_A - I_B + I_C}\right]\Big/ (I_A + I_C), \\ c &= \sqrt{I_B}\left[\sqrt{I_B} + \sqrt{I_A - I_B + I_C}\right]\Big/ (I_A + I_C), \end{aligned}$$

which have been chosen to solve $aI_A - I_B + cI_C = 0$ and $a^2 I_A - I_B + c^2 I_C = 0$.

The proposed minimizer v is

$$v(x) = \begin{cases} av_1(x) & \text{for } x \in A \\ v_1(x) & \text{for } x \in B \\ cv_1(x) & \text{for } x \in C \\ 0 & \text{for } x \notin A \cup B \cup C, |x| \leq \frac{1}{2} \end{cases}$$

with v extended to have period 1. Certainly $v \in H^1(\mathbb{T}_1)$, and it has mean value zero since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} v\,dx = a\int_A v_1\,dx + \int_B v_1\,dx + c\int_C v_1\,dx = -aI_A + I_B - cI_C = 0.$$

By construction, $v \not\equiv 0$ because $v_1 \neq 0$ on B.

To show v is a minimizer for $\mu_1(\alpha)$, first multiply (38) by v_1 and integrate over A, B, C to obtain

$$\begin{aligned} \mu_1(\alpha)\int_A (av_1)^2\,dx &= \int_A \left[(av_1')^2 - EK^{q-1}(av_1)^2\right] dx - \gamma a^2 I_A, \\ \mu_1(\alpha)\int_B (v_1)^2\,dx &= \int_B \left[(v_1')^2 - EK^{q-1}(v_1)^2\right] dx + \gamma I_B, \\ \mu_1(\alpha)\int_C (cv_1)^2 dx &= \int_C \left[(cv_1')^2 - EK^{q-1}(cv_1)^2\right] dx - \gamma c^2 I_C. \end{aligned} \tag{49}$$

Here we have used that $v_1 = 0$ on $\partial A, \partial B, \partial C$. Adding these three equations,

$$\mu_1(\alpha)\int_{-\frac{1}{2}}^{\frac{1}{2}} v^2\,dx = \int_{-\frac{1}{2}}^{1/2} \left[(v')^2 - EK^{q-1}v^2\right] dx \tag{50}$$

because $a^2 I_A - I_B + c^2 I_C = 0$.

By equation (50), v is a minimizer for $\mu_1(\alpha)$. This is impossible because all minimizers for $\mu_1(\alpha)$ are smooth and have only finitely many zeros, whereas $v \equiv 0$ on $(x_2, \frac{1}{2})$. This contradiction eliminates Case (i).

Case (ii): $v_1 < 0$ on $(x_2, \frac{1}{2})$ and $\int_{\mathcal{N}} |v_1|\, dx \geq \int_{\mathcal{P}} |v_1|\, dx$. We assume $\mathcal{P}$ is nonempty, since otherwise we would have $v_1 < 0$ on $(x_1, \frac{1}{2})$ except for finitely many points, and the lemma would hold with $x_v = x_1$. Define

$$A = -\mathcal{N}, \qquad B = -\mathcal{P} \cup \mathcal{P}, \qquad C = \mathcal{N},$$

so that $I_A - I_B + I_C = 2\int_{\mathcal{N}} |v_1|\, dx - 2\int_{\mathcal{P}} |v_1|\, dx \geq 0$ by assumption. Then (48) holds once more, and we find a contradiction as in Case (i).

Case (iii): $v_1 < 0$ on $(x_2, \frac{1}{2})$ and $\int_{\mathcal{N}} |v_1|\, dx < \int_{\mathcal{P}} |v_1|\, dx$. In this case $\mathcal{P}$ *must* be nonempty. Define

$$A = -\mathcal{P}, \qquad B = -\mathcal{N} \cup \mathcal{N}, \qquad C = \mathcal{P},$$

so that $I_A - I_B + I_C = 2\int_{\mathcal{P}} |v_1|\, dx - 2\int_{\mathcal{N}} |v_1|\, dx > 0$ by assumption. Then (48) holds once more, and we find a contradiction as in Case (i). (The only difference is that the terms $-\gamma a^2 I_A, +\gamma I_B, -\gamma c^2 I_C$ in the equations (49) all change sign.)

This finishes the proof of Lemma 21(b).

Proof of Lemma 21(c): Suppose v_1 is even and $q > 1$. The point x_v exists by part (b). Integrating the eigenfunction equation (38) and using that v_1 has mean value zero gives

$$\begin{aligned}
\gamma = E\int_{-\frac{1}{2}}^{\frac{1}{2}} K^{q-1} v_1\, dx &= 2E\int_0^{\frac{1}{2}} K^{q-1} v_1\, dx \\
&= 2E\int_0^{x_v} K^{q-1} v_1\, dx + 2E\int_{x_v}^{\frac{1}{2}} K^{q-1} v_1\, dx \\
&< 2E\, K(x_v)^{q-1}\int_0^{x_v} v_1\, dx + 2E\, K(x_v)^{q-1}\int_{x_v}^{\frac{1}{2}} v_1\, dx = 0.
\end{aligned}$$

Here we used that K^{q-1} is strictly increasing on $(0, \frac{1}{2})$ (since $q > 1$) and is even, and that v_1 has mean value zero and is even.

One argues similarly if $q < 1$. Since K^{q-1} is strictly decreasing, one derives $\gamma > 0$. $\square$

6. Proofs of Theorems 10, 12 and 13

6.1. **Proof of Theorem 10.** Let $\alpha \in (0, 1)$. We suppress the α-dependence of k_α and κ_α, writing $k = k_\alpha$ and $\kappa = \kappa_\alpha$. After rescaling as in Section 3.1, we see we can take $h_{\mathrm{ss}} = k, X = P, \mathcal{B} = 1, r(y) = y^{q-1}$.

We first show the 0-eigenfunctions solve

$$u'' + k^{q-1} u = \gamma \tag{51}$$

for some constant γ. For this, integrate $\mathcal{L}u = 0$ from the definition (4) to get $(u'' + k^{q-1}u)' = c/k^n$ for some constant c. Integrating over a period from 0 to P shows $c = 0$, and (51) follows. Conversely, if (51) holds then clearly u satisfies $\mathcal{L}u = 0$.

Before completing the proof, we remark that part (c) of the theorem, for $q = 1$, is really the same as part (b). Indeed when $q = 1$, (15) gives $P = 2\pi = A$. Thus $E(\alpha) = P^{3-1}A^{1-1} = 4\pi^2$ for all α and $E'(\alpha) = 0$. Also (15) shows $\kappa(x) = \cos(2\pi x)$, so that $\cos(x) = \kappa(x/P)$ while $\sin(x)$ is a multiple of $k'(x)$. Thus part (c) reduces to part (b).

Part (a). Suppose $1 < q < 2$ and $E'(\alpha) < 0$. Certainly k' solves (51), since $k''' + k^{q-1}k' = 0$ by (13). Let $u(x)$ be another 0-eigenfunction; $u \not\equiv 0$ is smooth, P-periodic, has mean value zero, and $u'' + k^{q-1}u = \gamma$ for some constant γ. We now prove u is a multiple of k'. We do this by splitting u into its odd and even parts,

$$u_o(x) = \frac{u(x) - u(-x)}{2} \quad \text{and} \quad u_e(x) = \frac{u(x) + u(-x)}{2},$$

and showing u_o is a multiple of k' and $u_e \equiv 0$.

Because k is even, both $y_1 = k'$ and $y_2 = u_o$ solve the second order homogeneous linear ODE $y'' + k^{q-1}y = 0$ with $y(0) = 0$. By construction, $y_1'(0) = k''(0) = -(\alpha^q - 1)/q \neq 0$. It follows from uniqueness for ODEs that $y_2 = [y_2'(0)/y_1'(0)]y_1$, since the two sides of the equation solve the same ODE and have the same value and slope at $x = 0$. That is, u_o is a multiple of k'.

Suppose $u_e(x)$ is not identically zero. It has mean value zero and is a zero eigenfunction: $u_e'' + k^{q-1}u_e = \gamma$. Rescaling u_e to a function $v_1(x) = u_e(Px)$ of period 1, we see $v_1'' + EK^{q-1}v_1 = \gamma_1$ where $\gamma_1 = \gamma P^2$. (Recall that $E = P^{3-q}A^{q-1}$ and $K(x) = Pk(Px)/A$.) Also notice $\mu_1(\alpha) = 0$ by Proposition 15, since $E'(\alpha) < 0$. Thus $v_1 \not\equiv 0$ is even and 1-periodic and satisfies (38); we can suppose either $v_1(0) > 0$ or else $v_1(0) = 0, v_1''(0) \geq 0$, after multiplying by -1 if necessary. Because $E'(\alpha) < 0$ and $\mu_1(\alpha) = 0$, from (39) we get $\int_{-1/2}^{1/2} K^q v_1\,dx = 0$. However, Lemma 21(b) applies, and so $\int_{-1/2}^{1/2} K^q v_1\,dx < 0$ by (40). This contradiction shows $u_e(x)$ must be identically zero, completing the proof of (a).

Part (b). Suppose $1 < q < 2$ and $E'(\alpha) = 0$. As before k' solves (51), but now so does $\kappa(x/P)$, as one sees by putting $E'(\alpha) = 0$ into (26).

Let $u(x)$ be another 0-eigenfunction with zero mean. We want to show $u(x)$ is a linear combination of $k'(x)$ and $\kappa(x/P)$. By the proof of part (a), $u_o(x)$ is a multiple of $k'(x)$, and so it suffices to show $u_e(x)$ is a multiple of $\kappa(x/P)$. That is, we want v_1 to be a multiple of κ.

For this, recall from above that $v_1'' + EK^{q-1}v_1 = \gamma_1$. We also know

$$\kappa'' + EK^{q-1}\kappa = E(P/A)^{q-1}(P/A)'(\alpha) \neq 0$$

by (26) with $E'(\alpha)=0$, and by Lemma 17. Thus for some constant c the function $\tilde{v}_1 = v_1 - c\kappa$ satisfies $\tilde{v}_1'' + EK^{q-1}\tilde{v}_1 = 0 =: \gamma_2$. Here $\tilde{v}_1$ is even and 1-periodic and has mean value zero, by construction, and we can ensure either $\tilde{v}_1(0) > 0$ or else $v_1(0) = 0, v_1''(0) \geq 0$ by multiplying by -1 if necessary.

Suppose $\tilde{v}_1 \not\equiv 0$. Observe $\mu_1(\alpha) = 0$ by Proposition 15, since $E'(\alpha) = 0$, so that $\tilde{v}_1$ solves (38). Lemma 21(c) applies to $\tilde{v}_1$ and so $\gamma_2 < 0$, a contradiction. Hence $\tilde{v}_1 \equiv 0$ and so $v_1 = c\kappa$ is a multiple of κ, proving (b).

Part (c). If $q = 1$ then (51) is $u'' + u = \gamma$. Integrating from 0 to $P = 2\pi$ proves $\gamma = 0$, since u has mean value zero. Hence $u'' + u = 0$, and so the 0-eigenspace is spanned by $\sin x$ and $\cos x$.

6.2. **Proof of Theorem 12.** Assume by a translation that h_{ss} has its minimum at $x = 0$.

If P_{ss} and A_{ss} are the period and area of the original steady state h_{ss}, and P and A are the period and area of the rescaled steady state k_α, then the rescaling (10) implies

$$P = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{1/2q} D^{1/2} P_{\mathrm{ss}}, & q \neq 0, \\ e^{-D/2\mathcal{B}} \mathcal{B}^{1/2} P_{\mathrm{ss}}, & q = 0, \end{cases} \quad \text{and} \quad A = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{3/2q} D^{1/2} A_{\mathrm{ss}}, & q \neq 0, \\ e^{-3D/2\mathcal{B}} \mathcal{B}^{1/2} A_{\mathrm{ss}}, & q = 0. \end{cases} \tag{52}$$

For later reference, notice that the rescaling (10) can be written as

$$h_{\mathrm{ss}}(x) = \frac{A_{\mathrm{ss}}}{A} \frac{P}{P_{\mathrm{ss}}}\, k\left(\frac{P}{P_{\mathrm{ss}}} x\right). \tag{53}$$

From (52) we see E is essentially invariant:

$$\mathcal{B} P_{\mathrm{ss}}^{3-q} A_{\mathrm{ss}}^{q-1} = P(\alpha)^{3-q} A(\alpha)^{q-1} = E(\alpha). \tag{54}$$

It follows that if there is a steady state h_{ss} with period P_{ss} and area A_{ss} then $\mathcal{B} P_{\mathrm{ss}}^{3-q} A_{\mathrm{ss}}^{q-1}$ lies in the range of E. The converse follows by choosing α with $\mathcal{B} P_{\mathrm{ss}}^{3-q} A_{\mathrm{ss}}^{q-1} = E(\alpha)$ and determining D from (52), then determining h_{ss} from (10).

By Theorem 11, E is strictly increasing for $q < 1$ and $q \geq 2$, and so the desired non-constant positive periodic steady state exists if and only if $E(0) < \mathcal{B} P_{\mathrm{ss}}^{3-q} A_{\mathrm{ss}}^{q-1} < E(1)$. Here $E(1) = 4\pi^2$ since $P(1) = A(1) = 2\pi$ by Lemma 6.

Uniqueness of this steady state (up to translation) follows from the strict monotonicity of $E(\alpha)$, which allows only one choice of α in the converse direction above.

6.3. **Bifurcation diagram interpretation of $E(\alpha)$.** The plots of $E(\alpha)$ in Figures 3, 4 and 5 can be interpreted as bifurcation diagrams for a family of steady states that all have the same area but have varying minimum heights and hence periods.

To see this, start by fixing the value of A_{ss}, say $A_{\mathrm{ss}} = 1$, and assuming $q \neq 3$. For each $\alpha \in (0,1)$, solve for D in the 'A' equation of (52), determining D in terms of $A = A(\alpha)$. Using D, construct h_{ss} from the rescaling (10). This gives a family of steady states h_{ss} parametrized by $\alpha \in (0,1)$, each having area 1 and attaining its minimum at $x = 0$.

For the bifurcation diagram interpretation of the Figures, we show that $E(\alpha)$ corresponds in some fashion to the period of $h_{\rm ss}$, and that α corresponds to the minimum height. First, (54) with $A_{\rm ss} = 1$ gives

$$P_{\rm ss} = [E(\alpha)/\mathcal{B}]^{1/(3-q)},$$

and so the period is proportional to a power of $E(\alpha)$, giving the desired first correspondence. Second, the minimum height of $h_{\rm ss}$ is $(A_{\rm ss}P/AP_{\rm ss})\alpha$ by (53), and so

$$\text{minimum height of } h_{\rm ss} = \mathcal{B}^{1/(3-q)}\alpha A(\alpha)^{2/(q-3)}$$

by (54). The map $\alpha \mapsto \alpha A(\alpha)^{2/(q-3)}$ is strictly increasing when $q < 1$, since $A' < 0$ by [26, Proposition 7.4]. Numerical evidence suggests it is also strictly increasing when $q > 1$. So it appears the minimum height of $h_{\rm ss}$ is monotonically related to α.

Turning the graph of $E(\alpha)$ on its side, then, we essentially get a plot of the minimum height of $h_{\rm ss}$ against its period. If $E' > 0$ on a branch of this diagram then the corresponding steady states are linearly unstable, by Theorem 9, while if a branch has $E' \leq 0$ then the steady states are linearly stable.

Incidentally, the figures can also be viewed as bifurcation diagrams for a family of steady states with fixed period but with varying minimum heights and areas. For this interpretation, E corresponds to the area under $h_{\rm ss}$ and α to its minimum height .

6.4. **Proof of Theorem 13.** The $q = 1$ case is immediate since $P \equiv 2\pi, A \equiv 2\pi$ by (15).

For all q, $A/P \to 1$ as $\alpha \to 1$, by Lemma 6. Thus it suffices to show $(A/P)'(\alpha)$ is positive when $q > 1$ and is negative when $q < 1$.

It is enough to consider $q < -\frac{1}{2}$, because for $q \geq -\frac{1}{2}$, Lemma 17 applies. Now, $P'(\alpha) < 0$ when $q < -\frac{1}{2}$ by (36), and $E'(\alpha) > 0$ by Theorem 11. Writing $(A/P)^{1-q}$ as $E^{-1}P^2$ and differentiating, we deduce $(A/P)'(\alpha) < 0$ as desired.

7. Conclusions and Future Directions

The underlying question motivating this paper is:

> If you perturb a periodic (or Neumann) steady state $h_{\rm ss}$ of $h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x - ah$ without changing its area, might the subsequent solution relax back to the steady state?

We have not addressed this question directly, but we have considered the *linear* stability question in two situations, proving linear instability when the perturbations have longer period than the steady state, and when the perturbations have the same or shorter period and g/f is convex. We also have proved more precise linear stability results when the coefficients are power laws, $f(y) = y^n$ and $g(y) = \mathcal{B}y^m$ with g/f non-convex.

We considered smooth steady states only. In [26] we described conditions on g/f that would allow existence of compactly supported 'droplet' steady states of thin film equations. These droplet steady states have equal acute contact angles at the ends of the support, and generally fail to be smooth at the contact line. Other conditions on g/f can lead to steady states with non-acute contact angles [32]. It would be interesting to study the linear stability of these droplet steady states. In this direction, the computations of Goldstein, Pesci and Shelley [14] with $f(y) = y$ and $g(y) = \mathcal{B}y$ suggest that droplet steady states are asymptotically stable with large basins of attraction. Both Grün [19] and Oron and Bankoff [34] computed a van der Waals model that has net repulsion at long scales and net attraction at short scales: $f(y) = y^3$ and $g(y) = 1/y - \varepsilon/y^2$ with $\varepsilon \ll 1$. Considering a wide range of initial data and ε, they found robust long-time convergence to configurations that look like droplets connected by a thin film. The thin film connecting the 'droplets' diminishes as $\varepsilon \to 0$, suggesting that droplet steady states are asymptotically stable with large basins of attraction when $\varepsilon = 0$. Our preliminary numerical simulations of solutions to the long-wave evolution equation with other coefficients f and g reveal the same phenomenon.

Nonlinear stability of the steady states remains an open problem in general, though there are some results when $f \equiv 1$, *e.g.* [31, Theorem 6.5]. The thin film evolution equations have a natural Liapunov function defined by

$$\mathcal{E}(h(\cdot,t)) = \int_0^X \left[\frac{1}{2} h_x(x,t)^2 - H(h(x,t))\right] dx,$$

where H is any function such that $H''(y) = g(y)/f(y)$. For classical solutions, the evolution dissipates $\mathcal{E}(t)$ in time:

$$\frac{d}{dt}\mathcal{E}(h(\cdot,t)) \leq 0, \qquad \text{with} \qquad \frac{d}{dt}\mathcal{E}(h(\cdot,t)) = 0 \Longleftrightarrow h \text{ is a steady state.}$$

Goldstein, Pesci and Shelley [14, §IIIB] used this Liapunov function to prove nonlinear instability of the constant steady state for the case $f(y) = y^n$ and $g(y) = \mathcal{B}y^m$ with $m = n$ and either $2 \leq \mathcal{B} < 4$ or $\mathcal{B} = j^2$ for some integer $j \geq 2$.

A good understanding of the Liapunov function near a steady state h_{ss} would be a powerful tool for studying nonlinear stability. Specifically, let $V \subset H^1(\mathbb{T}_X)$ be the affine subset of functions with given mean value $\overline{h_{\mathrm{ss}}}$. If one could prove h_{ss} were a local minimum of $\mathcal{E}$ in V then this might be used to prove the steady state is asymptotically stable. Similarly, if one knew there were functions $v \in V$ arbitrarily close to the steady state such that $\mathcal{E}(v) < \mathcal{E}(h_{\mathrm{ss}})$ then this might be used to prove the steady state is nonlinearly unstable. On another note, there may be more than one steady state with the same period and volume as the initial data. (There is always a *constant* steady state and there is often at least one other.) If one knew the value of $\mathcal{E}$ for all such steady states then one could exclude some of them as possible long-time limits of the solution. Also, if one has two steady states h_{ss} and $\tilde{h}_{ss}$ such that $\mathcal{E}(h_{\rm ss}) < \mathcal{E}(\tilde{h}_{ss})$ then a natural question is whether there is a orbit connecting $\tilde{h}_{ss}$ to $h_{\rm ss}$. Grinfeld and Novick–Cohen [18] have rigorously performed much of this program for the Cahn–Hilliard equation.

Acknowledgments. Laugesen was partially supported by NSF grant number DMS-9970228.

Pugh was partially supported by NSF grant number DMS-9971392, by the MRSEC Program of the NSF under Award Number DMR-9808595, and by the ASCI Flash Center at the University of Chicago under DOE contract B341495. Part of the research was conducted while enjoying the hospitality of the Mathematics Department and the James Franck Institute of the University of Chicago.

Appendix A. Linearization and Spectral theory for periodic boundary conditions

Assume $f, g \in C^1(\mathbb{R})$, $r = g/f$, $a \in \mathbb{R}$, and let $X > 0$ be fixed. Suppose $f > 0$ and that $h_{\rm ss} \in C^4(\mathbb{T}_X)$ is an X-periodic steady state of the evolution equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x - ah. \tag{55}$$

Assume either $a = 0$ (thin film equation), or $a \neq 0$ and $f \equiv 1$ (Cahn–Hilliard-like equation). The steady state equation is

$$0 = (f(h_{\rm ss})h_{{\rm ss}xxx} + g(h_{\rm ss})h_{{\rm ss}x})_x + ah_{\rm ss}.$$

We will use without comment the fact that $f(h_{\rm ss}(x))^{-1}$ is positive, bounded, and bounded away from zero, and that $r(h_{\rm ss}(x))$ is bounded.

Consider a perturbation $\phi(x,t)$ of the steady state, putting $h = h_{\rm ss} + \varepsilon\phi$. Formally expand the evolution equation in orders of ε; at order $O(1)$ the expansion is the steady state equation. At the next order $O(\varepsilon)$, it is (writing $\tilde{h} = h_{\rm ss}$ for notational simplicity)

$$\begin{aligned}
\phi_t &= -\left[f(\tilde{h})\phi_{xxx} + g(\tilde{h})\phi_x + f'(\tilde{h})\phi\tilde{h}_{xxx} + g'(\tilde{h})\phi\tilde{h}_x\right]_x - a\phi \\
&= -\Bigg[f(\tilde{h})\left(\phi_{xxx} + \frac{g(\tilde{h})}{f(\tilde{h})}\phi_x + \frac{g'(\tilde{h})\tilde{h}_x}{f(\tilde{h})}\phi - \frac{g(\tilde{h})f'(\tilde{h})\tilde{h}_x}{f(\tilde{h})^2}\phi\right) \\
&\qquad\qquad + f'(\tilde{h})\phi\tilde{h}_{xxx} + \frac{f'(\tilde{h})}{f(\tilde{h})}g(\tilde{h})\tilde{h}_x\phi\Bigg]_x - a\phi \\
\phi_t &= -\left[f(\tilde{h})\left(\phi_{xx} + r(\tilde{h})\phi\right)_x\right]_x - \left[f'(\tilde{h})\left(\tilde{h}_{xxx} + r(\tilde{h})\tilde{h}_x\right)\phi\right]_x - a\phi.
\end{aligned}$$

This final equation for ϕ_t has two terms in divergence form, but the second term is zero: if $a = 0$ then $\tilde{h}_{xxx} + r(\tilde{h})\tilde{h}_x \equiv 0$ by (3), while if $a \neq 0$ then $f \equiv 1$ hence $f' \equiv 0$. Hence the linearized equation is $\phi_t = \mathcal{L}\phi$ where

$$\mathcal{L}\phi := -\left[f(h_{\rm ss})\left(\phi_{xx} + r(h_{\rm ss})\phi\right)_x\right]_x - a\phi.$$

We reformulate the linearized problem in terms of a symmetric operator, an idea used in various ways by several authors for equations of this type, see *e.g.* [3, 14, 18] and the references therein. To start, we assume that at each time $\phi(x,t)$ has mean value zero over $\mathbb{T}_X$: $\overline{\phi(\cdot,t)} = 0$. We introduce an anti-derivative by $\psi(x,t) = \int_0^x \phi(\xi,t)\,d\xi + c(t)$ where the constant of integration $c(t)$ is chosen to ensure

$$\int_0^X \psi(x,t) f(h_{\mathrm{ss}}(x))^{-1}\,dx = 0 \qquad \text{for all } t. \tag{56}$$

Notice ψ is X-periodic in x since ϕ has mean value zero on $(0,X)$. Substitute $\phi = \psi_x$ into the linearized problem $\phi_t = \mathcal{L}\phi$ to obtain $\psi_{xt} = -\left[f(h_{\mathrm{ss}})\left(\psi_{xxx} + r(h_{\mathrm{ss}})\psi_x\right)_x\right]_x - a\psi_x$. Integrating from 0 to x yields

$$\psi_t = -f(h_{\mathrm{ss}})\left(\psi_{xxx} + r(h_{\mathrm{ss}})\psi_x\right)_x - a\psi + T(t) \tag{57}$$

for some function T. In fact $T \equiv 0$, as we see by dividing (57) by $f(h_{\mathrm{ss}})$, integrating from 0 to X, and using the normalization (56). Thus $\psi_t = -f(h_{\mathrm{ss}})\mathcal{I}\psi$ where $\mathcal{I}$ is the symmetric linear operator

$$\mathcal{I}\psi := \psi_{xxxx} + (r(h_{\mathrm{ss}})\psi_x)_x + af(h_{\mathrm{ss}})^{-1}\psi.$$

For our purposes, the linear evolution $\psi_t = -f(h_{\mathrm{ss}})\mathcal{I}\psi$ will be a satisfactory reformulation of the linear evolution $\phi_t = \mathcal{L}\phi$ if the corresponding eigenproblems have the same eigenvalues λ. The problems are:

$$\mathcal{L}u = -\lambda u, \qquad \int_0^X u\,dx = 0, \tag{58}$$

$$\mathcal{I}w = \lambda f(h_{\mathrm{ss}})^{-1} w, \qquad \int_0^X w f(h_{\mathrm{ss}})^{-1}\,dx = 0, \tag{59}$$

where $u(x)$ and $w(x)$ are X-periodic and depend only on x.

This equivalence is easy to see if the eigenfunction equations hold classically and $u \in C^4(\mathbb{T}_X), w \in C^5(\mathbb{T}_X)$, as follows. Given an eigenfunction $w \not\equiv 0$ of (59) with eigenvalue λ, we know $w \not\equiv$ (const) since $\int_0^X w f(h_{\mathrm{ss}})^{-1}\,dx = 0$. Thus if $u = w'$ then $u \in C^4, u \not\equiv 0, \overline{u} = 0$ and $\mathcal{L}u = -\lambda u$. That is, λ is an eigenvalue of (58). In the other direction, given an eigenfunction $u \not\equiv 0$ of (58) with eigenvalue λ, take the antiderivative w of u and normalize it to satisfy $\int_0^X w f(h_{\mathrm{ss}})^{-1}\,dx = 0$. Obviously $w \not\equiv 0$. Integrating $\mathcal{L}u = -\lambda u$ from 0 to x gives $f(h_{\mathrm{ss}})\mathcal{I}w = \lambda w + c$ for some constant c. In fact $c = 0$, as we see by dividing through by $f(h_{\mathrm{ss}})$, integrating from 0 to X and using the normalization $\int_0^X w f(h_{\mathrm{ss}})^{-1}\,dx = 0$. That is, λ is an eigenvalue of (59). These procedures preserve the multiplicity of eigenvalues in both directions.

In view of this equivalence (for smooth eigenfunctions) of the problems (58) and (59), we call h_{ss} *linearly unstable* if the symmetric weighted eigenproblem (59) has a negative eigenvalue $\lambda < 0$, since then $\mathcal{L}$ has a positive eigenvalue $-\lambda$.

The following theorem shows the spectrum of the reformulated eigenvalue problem is discrete and real. It also proves that if f and g are C^2-smooth then the eigenfunctions are C^5-smooth and hence the reformulated linearized problem (59) is equivalent to the original linearized problem (58).

Theorem 22. *Assume $f, g \in C^1(\mathbb{R})$, $r = g/f$, and $a \in \mathbb{R}$. Let $h_{\rm ss} \in C^4(\mathbb{T}_X)$ be a steady state of the evolution (55). Consider the eigenproblem*

$$w'''' + (r(h_{\rm ss})w')' + af(h_{\rm ss})^{-1}w = \lambda f(h_{\rm ss})^{-1}w \tag{60}$$

on the Hilbert space $W := \{w \in H^2(\mathbb{T}_X) : \int_0^X wf(h_{\rm ss})^{-1}\,dx = 0\}$ with the H^2 inner product.

There is a sequence $w_j \in W$ of weak eigenfunctions for this problem, with corresponding eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$. The first eigenvalue is given by

$$\lambda_1 = \min_{w\in W\setminus\{0\}} \frac{\int_0^X [(w'')^2 - r(h_{\rm ss})(w')^2 + af(h_{\rm ss})^{-1}w^2]\,dx}{\int_0^X w^2 f(h_{\rm ss})^{-1}\,dx}. \tag{61}$$

The w_j form an orthonormal basis of the Hilbert space $\{w \in L^2(\mathbb{T}_X) : \int_0^X wf(h_{\rm ss})^{-1}\,dx = 0\}$ with respect to the weighted inner product $B(w,\tilde{w}) := \int_0^X w\tilde{w}f(h_{\rm ss})^{-1}\,dx$.

The weak eigenfunctions are C^4-smooth and solve equation (60) classically. If $f, g \in C^2(\mathbb{R})$ then the eigenfunctions are C^5-smooth.

Proof. We formally obtain the Rayleigh quotient for the eigenproblem by multiplying equation (60) by w and integrating by parts, getting $R[w] := A(w,w)/B(w,w)$ where

$$A(w,\tilde{w}) := \int_0^X \left[w''\tilde{w}'' - r(h_{\rm ss})w'\tilde{w}' + a\frac{w\tilde{w}}{f(h_{\rm ss})}\right]dx \qquad \text{for } w, \tilde{w} \in H^2(\mathbb{T}_X).$$

Note B is equivalent to the usual inner product on $L^2(\mathbb{T}_X)$, in that it generates an equivalent norm. Similarly, $C(w,\tilde{w}) := B(w,\tilde{w}) + \int_0^X w''\tilde{w}''\,dx$ is equivalent to the usual inner product on $H^2(\mathbb{T}_X)$.

First, we observe that for some number δ, $A + \delta B$ is elliptic: $A(w,w) + \delta B(w,w) \ge \frac{3}{4}C(w,w)$ for all $w \in H^2(\mathbb{T}_X)$. This holds for $\delta = \|r(h_{\rm ss})\|_\infty^2 \|f(h_{\rm ss})\|_\infty + |a| + \frac{3}{4}$, by standard L^∞ bounding and by extracting $\int_0^X \left[\frac{1}{2}w'' + w\|r(h_{\rm ss})\|_\infty\right]^2 dx$ from $A + \delta B$. A Hilbert space spectral result [40, Corollary III.7.D] now proves the theorem, except for its last paragraph and the variational characterization (61) of λ_1.

The existence of a minimizing function for the Rayleigh quotient in (61) follows from the ellipticity estimate and the usual compactness and quadratic form argument (like [13, pp. 213] for the second order case). A minimizer is a weak eigenfunction by the Euler–Lagrange equation, and its eigenvalue must be λ_1 since the Rayleigh quotient is minimal, proving (61).

Now suppose w is a weak eigenfunction with eigenvalue λ, i.e. $A(w,\tilde{w}) = \lambda B(w,\tilde{w})$ for all $\tilde{w} \in W$. Let $w \in H^2(\mathbb{T}_X)$ and choose $c \in \mathbb{R}$ with $\tilde{w} - c \in W$. Then $A(w,\tilde{w}-c) = \lambda B(w,\tilde{w}-c)$

and so $A(w,\tilde{w}) = \lambda B(w,\tilde{w})$, where we have used that $B(w,1) = 0$ for $w \in W$. Hence the weak eigenfunction w satisfies (60) weakly in $H^2(\mathbb{T}_X)$, not just in W.

Finally, we prove the regularity of the weak eigenfunctions. A weak eigenfunction $w \in W$ satisfies $\int_0^X [w''\tilde{w}'' - r(h_{ss})w'\tilde{w}' - (\lambda - a)f(h_{ss})^{-1}w\tilde{w}]\,dx = 0$ for all $\tilde{w} \in H^2(\mathbb{T}_X)$, and hence

$$\int_0^X \left[w''\eta' + \left(-r(h_{ss})w' + \int_0^x \frac{\lambda - a}{f(h_{ss})} w \, d\xi \right) \eta \right] dx = 0 \quad \text{for all zero-mean } \eta = \tilde{w}' \in H^1(\mathbb{T}_X).$$

Hence the weak derivative of w'' is $c - r(h_{ss})w' + \int_0^x (\lambda - a)f(h_{ss})^{-1} w \, d\xi$ for some constant c. Since $w \in H^2(\mathbb{T}_X) \subset C^1(\mathbb{T}_X)$, this weak derivative of w'' is continuous, hence w is C^3 with $w''' = c - r(h_{ss})w' + \int_0^x (\lambda - a)f(h_{ss})^{-1} w \, d\xi$ classically. Since $r \in C^1(\mathbb{R})$, w is C^4 and so equation (60) holds classically. If r is C^2 then w is C^5, since also f is C^1. □

In the $a = 0$ case, we can further reduce the fourth order eigenvalue problem to a second order problem whose lowest eigenvalue has the same sign as the lowest eigenvalue λ_1 of $\mathcal{I}$. This idea goes back at least to Langer [25, §IV]. Define a new Rayleigh quotient and minimization problem by $\tau_1 = \min_u R_\tau[u]$ where $R_\tau[u] := \int_0^X [(u')^2 - r(h_{ss})u^2]\,dx \big/ \int_0^X u^2\,dx$ and the minimum is taken over $u \in H^1(\mathbb{T}_X) \setminus \{0\}$ with zero mean. Existence of a minimizing function u follows as above, using the bilinear form $D(u,\tilde{u}) = \int_0^X [u'\tilde{u}' - r(h_{ss})u\tilde{u}]\,dx$. Each minimizer u_1 of the Rayleigh quotient is a weak solution of the Euler–Lagrange equation $u_1'' + r(h_{ss})u_1 + \tau_1 u_1 = 0$ with respect to $H^1(\mathbb{T}_X) \cap \{\overline{u} = 0\}$: it solves $u_1'' + r(h_{ss})u_1 + \tau_1 u_1 = \gamma$ weakly with respect to $H^1(\mathbb{T}_X)$, for some constant γ. (In fact, $\gamma = \int_0^X r(h_{ss})u_1\,dx$.) This means u_1'' equals a continuous function weakly, hence classically, and so u is C^2-smooth. Thus u_1 solves the equation classically.

Next we show the number of negative eigenvalues λ_j for the fourth order problem when $a = 0$ is the same as the number of negative eigenvalues τ_j for the second order problem, so that the unstable eigenspaces of the two problems have the same dimension. To prove this equality, suppose $w_1, \ldots, w_j \in H^2(\mathbb{T}_X)$ are linearly independent eigenfunctions of (60) that are pairwise orthogonal with respect to the bilinear form A above and have eigenvalues $\lambda_1 \leq \cdots \leq \lambda_j < 0$. Then $w_1', \ldots, w_j' \in H^1(\mathbb{T}_X)$ are linearly independent functions with mean value zero and $R_\tau[u] < 0$ for all $u \in \mathrm{Span}\{w_1', \ldots, w_j'\} \setminus \{0\}$. Poincaré's minimax characterization [1, p. 97] of the higher eigenvalues implies $\tau_j < 0$. The converse argument works similarly to show that if $\tau_j < 0$ then $\lambda_j < 0$.

Appendix B. Linearization and Spectral theory for Neumann boundary conditions

Now consider steady states and perturbations on the interval $(0, X)$ subject to Neumann boundary conditions: the first and third derivatives are required to vanish at $x = 0, X$. The linearization in Appendix A is unchanged up to the choice of the integration constant in $\psi(x,t)$, which we choose here to be zero: $c(t) = 0$. Then $\psi(0,t) = 0$, and $\psi(X,t) = 0$ since ϕ

has zero mean. Also, ψ_{xx} and ψ_{xxxx} are zero at $x = 0, X$ because ϕ_x and ϕ_{xxx} are zero there. Since $h'_{\rm ss}(0) = 0$, evaluating (57) at $x = 0$ or X yields $T(t) = 0$: the reformulated linearized problem is $\psi_t = -f(h_{\rm ss})\mathcal{I}\psi$.

As before, we regard this as a satisfactory reformulation of the original linearized equation $\phi_t = \mathcal{L}\phi$ if the corresponding eigenproblems have the same eigenvalues ν:

$$\mathcal{L}u = -\nu u, \qquad u'(0) = u'(X) = u'''(0) = u'''(X) = \int_0^X u\,dx = 0, \tag{62}$$

$$\mathcal{I}w = \nu f(h_{\rm ss})^{-1}w, \qquad w(0) = w(X) = w''(0) = w''(X) = 0, \tag{63}$$

where $u(x)$ and $w(x)$ are functions of x only.

The two problems are equivalent for smooth eigenfunctions, as follows. Let $(u, -\nu)$ be an eigenpair for the original problem (62). Then $w := \int_0^x u\,d\xi$ satisfies the reformulated problem (63): w'' and w'''' vanish at $x = 0, X$ since u' and u''' vanish there, $w(0) = 0$ by construction, $w(X) = 0$ since u has zero mean, w is not identically zero since u is not identically zero, and $f(h_{\rm ss})\mathcal{I}w = \nu w + c$ by direct differentiation with $c = 0$ by evaluating at $x = 0$. Similarly, let (w, ν) be an eigenpair for the reformulated problem (63). Then w cannot be constant since if it were constant, $w(0) = 0$ would force it to be identically zero. Then $u := w'$ satisfies the original problem (62): u is not identically zero, $\mathcal{L}u = -\nu u$ by direct calculation, $u'(0) = u'(X) = 0$ from the boundary conditions on w'', $\int_0^X u\ dx = w(X) - w(0) = 0$, and it remains to prove $u'''(0) = u'''(X) = 0$. This follows from evaluating the equation $\mathcal{I}w = \nu f(h_{\rm ss})^{-1}w$ at the endpoints to find that $w'''' = u'''$ vanishes there.

As in Appendix A, this equivalence for smooth eigenfunctions makes it natural to study the reformulated (symmetric) eigenproblem.

Theorem 23. *Assume $f, g \in C^1(\mathbb{R})$, $r = g/f$, and $a \in \mathbb{R}$. Let $h_{\rm ss} \in C^4[0, X]$ be a Neumann steady state of (55). Consider the eigenproblem*

$$w'''' + (r(h_{\rm ss})w')' + af(h_{\rm ss})^{-1}w = \nu f(h_{\rm ss})^{-1}w \tag{64}$$

on the Hilbert space $H^2(0, X) \cap H^1_0(0, X)$ with the H^2 inner product.

There is a sequence $w_j \in H^2(0, X) \cap H^1_0(0, X)$ of weak eigenfunctions for this problem, with corresponding eigenvalues $\nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \to \infty$. The first eigenvalue is given by

$$\nu_1 = \min_{w \in H^2(0,X)\cap H^1_0(0,X)\setminus\{0\}} \frac{\int_0^X \left[(w'')^2 - r(h_{\rm ss})(w')^2 + af(h_{\rm ss})^{-1}w^2\right]\,dx}{\int_0^X w^2 f(h_{\rm ss})^{-1}\,dx}.$$

The w_j form an orthonormal basis of $L^2(0, X)$ with respect to the weighted inner product $B(w, \tilde{w}) := \int_0^X w\tilde{w}f(h_{\rm ss})^{-1}\,dx$.

The weak eigenfunctions are C^4-smooth on $[0, X]$ and solve equation (64) classically, and they satisfy the 'natural boundary conditions': $w''(0) = w''(X) = 0$. If $f, g \in C^2(\mathbb{R})$ then the eigenfunctions are C^5-smooth on $[0, X]$.

The eigenfunctions satisfy $w = 0$ at the endpoints by virtue of belonging to $H^1_0(0, X)$. They satisfy the natural boundary condition $w'' = 0$ also, by the theorem. Evaluating (64) at the endpoints further gives $w'''' = 0$, since $h'_{\mathrm{ss}}(0) = h'_{\mathrm{ss}}(X) = 0$.

Proof. Following the proof of Theorem 22 gives everything except the natural boundary conditions and the regularity at the endpoints of the *closed* interval $[0, X]$.

For regularity at the endpoints, first we observe w' is continuous on $[0, X]$ since $w'(x) = w'(X/2) + \int_{X/2}^{x} w''(\xi)\, d\xi$ and $w'' \in L^2(0, X)$. Next, repeating the arguments in the proof of Theorem 22, w'' has weak derivative equal to $c - r(h_{\mathrm{ss}}(x))w'(x) + \int_{X/2}^{x}(\nu - a)f(h_{\mathrm{ss}})^{-1}w\, d\xi$ for some constant c. This weak derivative is continuous on $[0, X]$, hence w'' is C^1 on $[0, X]$. This proves $w \in C^3[0, X]$ with $w''' = c - r(h_{\mathrm{ss}}(x))w'(x) + \int_{X/2}^{x}(\nu - a)f(h_{\mathrm{ss}})^{-1}w\, d\xi$ classically. Hence since r is C^1-smooth, $w \in C^4[0, X]$. If r is C^2-smooth then $w \in C^5[0, X]$.

We now show the natural boundary conditions are satisfied: $w''(0) = w''(X) = 0$. Let $\eta \in H^2(0, X) \cap H^1_0(0, X)$ be a smooth test function such that $\eta(0) = \eta(X) = \eta'(X) = 0$ and $\eta'(0) = 1$. Since w is a weak eigenfunction, $0 = \int_0^X [w''\eta'' - r(h_{\mathrm{ss}})w'\eta' + (a - \nu)f(h_{\mathrm{ss}})^{-1}w\eta]\, dx$. Integrating by parts and using that w solves the eigenproblem (64) classically,

$$0 = \int_0^X \left[w'''' + (r(h_{\mathrm{ss}})w')' + \frac{a - \nu}{f(h_{\mathrm{ss}})} w \right] \eta\, dx - w''(0)\eta'(0) = -w''(0).$$

A similar argument proves $w''(X) = 0$. □

Like in Appendix A, if $a = 0$ then the stability question can be reduced to a second order eigenproblem for zero-mean functions in $H^1(0, X)$, with a natural Neumann boundary condition on the first derivative.

Appendix C. Smoothness and asymptotics of P and A

C.1. **Proof of Lemma 6.** Recall the formulas [26, §§3.1.1,3.1.2] for the period $P(\alpha)$ and area $A(\alpha)$ of $k_\alpha(x)$:

$$P(\alpha) = \sqrt{2} \int_\alpha^\beta \frac{dy}{\sqrt{H(\alpha) - H(y)}} \qquad \text{and} \qquad A(\alpha) = \sqrt{2} \int_\alpha^\beta \frac{y}{\sqrt{H(\alpha) - H(y)}}\, dy,$$

where $\beta > 1$ is the maximum value of k_α: the unique solution greater than 1 of $H(\beta) = H(\alpha)$. This solution exists and is unique because $H(y)$ is strictly convex and $H'(1) = 0$. The formulas for P and A are valid for $\alpha \in [0, 1)$ when $q > -1$, and for $\alpha \in (0, 1)$ when $q \leq -1$.

We first show $P(\alpha)$ and $A(\alpha)$ are smooth. Consider $\alpha \in (0, 1)$ and observe β is a smooth function of α, by the implicit function theorem. We change variables in $P(\alpha)$ with $z = \sqrt{y - \alpha}$ for $y \in (\alpha, 1)$ and with $z = \sqrt{\beta - y}$ for $y \in (1, \beta)$, yielding

$$P(\alpha) = 2\sqrt{2} \int_0^{\sqrt{1-\alpha}} \sqrt{\frac{z^2}{H(\alpha) - H(\alpha + z^2)}}\, dz + 2\sqrt{2} \int_0^{\sqrt{\beta - 1}} \sqrt{\frac{z^2}{H(\beta) - H(\beta - z^2)}}\, dz. \tag{65}$$

Then $P(\alpha)$ is smooth for $\alpha \in (0,1)$, because the difference quotients $[H(\alpha) - H(\alpha+z^2)]/z^2$ and $[H(\beta) - H(\beta - z^2)]/z^2$ are positive and smooth for $\alpha \in (0,1), |z| < \sqrt{\beta - \alpha}$, by the convexity of H and definition of β. Similarly $A(\alpha)$ is smooth for $\alpha \in (0,1)$.

To prove $P \to 2\pi$ and $A \to 2\pi$ as $\alpha \to 1$, simply change variables in (65) with $w = z^2/(1-\alpha)$ and $w = z^2/(\beta - 1)$ respectively, in the two integrals, and then let $\alpha, \beta \to 1$.

Next consider $q > -1$. We show P and A are continuous at $\alpha = 0$. Since $H(0) = 0$, when $\alpha \to 0$ we obtain

$$P(\alpha) \to 2\sqrt{2}\int_0^1 \sqrt{\frac{z^2}{-H(z^2)}}\, dz + 2\sqrt{2}\int_0^{\sqrt{\beta_0 - 1}} \sqrt{\frac{z^2}{-H(\beta_0 - z^2)}}\, dz = P(0),$$

where β_0 is defined by $H(\beta_0) = H(0) = 0$. This used that the difference quotient $[H(\alpha) - H(\alpha + z^2)]/z^2$ is uniformly bounded below for $\alpha \in (0, \frac{1}{2}), z \in (0, \sqrt{1-\alpha})$ because H is convex with $H'(1) = 0$. Similarly, $[H(\beta) - H(\beta - z^2)]/z^2$ is uniformly bounded below for $\alpha \in (0, \frac{1}{2}), z \in (0, \sqrt{\beta - 1})$. This proves $P(\alpha)$ is continuous at $\alpha = 0$. Similar arguments prove $A(\alpha)$ is continuous at $\alpha = 0$.

C.2. **Asymptotics of** P **and** A**.** As remarked after Theorem 12, $E = P^{3-q}A^{q-1}$ approaches 0 as $\alpha \to 0$, when $q \le -1$. This follows from the following asymptotic formulas.

Lemma 24. *Let $q \le -1$. As $\alpha \to 0$,*

$$\begin{aligned} P(\alpha) &\sim 2\sqrt{2|q|}\left\{ \begin{array}{cl} \log(1/\alpha) & \text{if } q = -1 \\ \alpha^{q+1}/|q+1| & \text{if } q < -1 \end{array} \right\}^{1/2}, \\ A(\alpha) &\sim \frac{4}{3}\sqrt{2|q|}\left\{ \begin{array}{cl} \log(1/\alpha) & \text{if } q = -1 \\ \alpha^{q+1}/|q+1| & \text{if } q < -1 \end{array} \right\}^{3/2}. \end{aligned}$$

Proof. The analogue of (65) for $A(\alpha)$ is

$$\begin{aligned} A(\alpha) &= 2\sqrt{2}\int_0^{\sqrt{1-\alpha}} (\alpha + z^2)\sqrt{\frac{z^2}{H(\alpha) - H(\alpha + z^2)}}\, dz \\ &\quad + 2\sqrt{2}\int_0^{\sqrt{\beta - 1}} (\beta - z^2)\sqrt{\frac{z^2}{H(\beta) - H(\beta - z^2)}}\, dz \qquad (66) \\ &= I + II. \end{aligned}$$

From definition (22), H is strictly convex with $H'(1) = 0$. The convexity implies

$$|I| \le 2\sqrt{2}\int_0^{\sqrt{1-\alpha}} 1\sqrt{\frac{1-\alpha}{H(\alpha) - H(1)}}\, dz.$$

Since $H(\alpha) \to \infty$ as $\alpha \to 0$ (using that $q \le -1$), we see $I \to 0$.

Next, by convexity of H we have an upper bound on II:

$$II \le 2\sqrt{2}\int_0^{\sqrt{\beta-1}} (\beta - z^2)\sqrt{\frac{\beta - 1}{H(\beta) - H(1)}}\, dz = \frac{2}{3}\sqrt{2}(2\beta + 1)(\beta - 1)^{1/2}\sqrt{\frac{\beta - 1}{H(\beta) - H(1)}}.$$

Recalling that $H(\beta) = H(\alpha) \to \infty$ as $\alpha \to 0$, we conclude that $\beta \to \infty$ and so $H(\beta) \sim \beta/|q|$. This upper bound is asymptotic to $(4/3)\sqrt{2|q|}\beta^{3/2}$. As $\alpha \to 0$ we have

$$\left\{ \begin{array}{rl} \log(1/\alpha) & \text{if } q = -1 \\ \alpha^{q+1}/q(q+1) & \text{if } q < -1 \end{array} \right\} \sim H(\alpha) = H(\beta) \sim \frac{\beta}{|q|},$$

and so the upper bound is asymptotic to $(4/3)\sqrt{2|q|}(\log 1/\alpha)^{3/2}$ for $q = -1$, and asymptotic to $(4/3)\sqrt{2|q|}(\alpha^{q+1}/|q+1|)^{3/2}$ for $q < -1$.

Convexity of H also implies a *lower* bound on II:

$$II \geq 2\sqrt{2} \int_0^{\sqrt{\beta-1}} (\beta - z^2) \sqrt{\frac{1}{H'(\beta)}}\, dz = \frac{2}{3}\sqrt{2}(2\beta+1)(\beta-1)^{1/2} \sqrt{\frac{q}{\beta^q - 1}}.$$

The lower bound is asymptotic to $(4/3)\sqrt{2|q|}(\log 1/\alpha)^{3/2}$ when $q = -1$ and asymptotic to $(4/3)\sqrt{2|q|}(\alpha^{q+1}/|q+1|)^{3/2}$ when $q < -1$. This proves the desired asymptotic formula for $A(\alpha)$. The formula for $P(\alpha)$ is proved similarly, using (65) instead of (66).

□

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