THE LUBRICATION APPROXIMATION FOR THIN VISCOUS FILMS: REGULARITY AND LONG TIME BEHAVIOR OF WEAK SOLUTIONS

A.L. BERTOZZI AND M. PUGH

Abstract. We consider the fourth order degenerate diffusion equation

$$h_t = -\nabla \cdot (f(h)\nabla\Delta h)$$

in one space dimension. This equation, derived from a 'lubrication approximation', models surface tension dominated motion of thin viscous films and spreading droplets [14]. The equation with $f(h) = |h|$ also models a thin neck of fluid in the Hele-Shaw cell [9, 10, 22]. In such problems $h(x,t)$ is the local thickness of the the film or neck. This paper will consider the properties of weak solutions which are more relevant to the droplet problem than to Hele-Shaw.

For simplicity we consider periodic boundary conditions with the interpretation of modeling a periodic array of droplets. We consider two problems: The first has initial data $h_0 \geq 0$ and $f(h) = |h|^n$, $0 < n < 3$. We show that there exists a weak *nonnegative* solution for all time and that this solution becomes a strong *positive* solution after some finite time T^* and asymptotically approaches its mean as $t \to \infty$. The weak solution is in a classical sense of distributions for $\frac{3}{8} < n < 3$ and in a weaker sense introduced in [1] for the remaining $0 < n \leq \frac{3}{8}$. Furthermore, the solutions have sufficiently high regularity to just include the unique 'source type' solutions [2] with zero slope at the edge of the support. They do not include any of the less regular solutions with positive slope at the edge of the support. Secondly we consider strictly positive initial data $h_0 \geq m > 0$ and $f(h) = |h|^n$, $0 < n < \infty$. For this problem we show that even if a finite time singularity does occur of the form $h \to 0$, there exists a weak *nonnegative* solution for all time t and that this weak solution becomes strong and *positive* again after some critical time T^*. As in the first problem, we show that the solution approaches its mean as $t \to \infty$. The main technical idea is to introduce new classes of dissipative entropies to prove the existence and higher regularity. We show that these entropies are related to norms of the difference between the solution and its mean to prove the relaxation result.

Date: April 1994.

1. Introduction

We consider weak solutions of the fourth order degenerate diffusion equation

$$h_t = -\nabla \cdot (f(h)\nabla \Delta h) \tag{1}$$

in one space dimension. We consider the case where $f(h) = |h|^p$, $p > 0$; the analysis can be directly extended to the case where $f(h)$ is a sum of such terms. This equation, derived from a 'lubrication approximation', models surface tension dominated motion of thin viscous films and spreading droplets [14]. The equation with $f(h) = |h|$ also models a thin neck of fluid in the Hele-Shaw cell [9, 10, 22]. In such problems h is the thickness of the the film or neck. This paper considers weak solutions that are zero on a set of non-zero measure, hence are much more relevant to the droplet problem than to Hele-Shaw.

We briefly compare this fourth order problem to the well known second order degenerate diffusion equation,

$$u_t = \Delta(u^m), \tag{2}$$

the 'porous media' equation [20]. Some similarities between the fourth and second order cases are that both equations are parabolic and in divergence form with a 'sub-diffusive' nonlinear diffusion coefficient. Furthermore, both equations have 'weak' solutions that are nonnegative. At any deeper level, the similarities between the second and fourth order problems cease to exist. One striking difference is the lack of a maximum principle for the fourth order problem. In particular, analytical results for the lubrication approximation are not due to a maximum principle but due to the dissipation of nonlinear 'entropies' present in these problems. Furthermore, the question of whether initially positive solutions can develop finite time singularities of the form $h \to 0$ has been the subject of recent and ongoing study [6, 4]. The maximum principle prohibits such behavior in the second order case.

1.1. **Elementary properties and exact solutions.**

For simplicity, we consider the 1-D problem with periodic boundary conditions. These boundary conditions have the physical interpretation of modeling a periodic array of spreading droplets. The equation

$$h_t + (f(h)h_{xxx})_x = 0 \tag{3}$$

is derived from a conservation law. We now state some elementary properties that follow from integration by parts for strong solutions. We use these and other properties of strong solutions to prove results for weak solutions. The first property is conservation of mass,

$$\int_{S^1} h(x,t)dx = \int_{S^1} h_0(x)dx.$$

One application of our theorems is a model for the spreading and eventual merging of the droplets to form a film of uniform thickness. We explore this application in greater detail in a companion paper [5]

Second, we have dissipation of surface tension energy,

$$\frac{1}{2}\int_{S^1} |h_x(x,T)|^2\,dx + \iint_{S^1\times[0,T]} f(h)h_{xxx}^2\,dxdt = \int_{S^1} |h_x(x,0)|^2\,dx. \tag{4}$$

In addition, we have the basic entropy dissipation: consider a function $G(y)$ satisfying $G''(y) = 1/f(y)$. The convexity of G and mass conservation allow us to choose G so that $\int_{S^1} G(h(x,t))\,dx \geq 0$ for all t. Integration by parts yields

$$\int_{S^1} G(h(x,T))\,dx + \iint_{Q_T} h_{xx}^2\,dxdt = \int_{S^1} G(h(x,0))\,dx. \tag{5}$$

For $n=0$, the linear problem, the entropy is merely the L^2 norm. Bernis and Friedman first introduced these entropies in [1].

The equation

$$h_t + (|h|^n h_{xxx})_x = 0 \tag{6}$$

possesses a number of interesting exact solutions.

Compactly supported nonnegative **'source type' solutions** exist for all $0 < n < 3$ [2]. They have the scaling form

$$h(x,t) = t^{-\alpha}H(\eta), \quad \eta = xt^{-\alpha} \quad \alpha = 1/(n+4). \tag{7}$$

Where $H(\eta)$ solves the ODE

$$H^n H_{\eta\eta\eta} = \alpha\eta H. \tag{8}$$

For a given n and mass, there is more than one compactly supported symmetric solution to the ODE. However, if we impose the additional constraint that the solution have $H_\eta = 0$ at the edge of the support, we obtain a unique solution. This fact was proven in [2]. They also proved that these "most regular" solutions have the following behavior at the edge of their support: Let $[-a,a]$ denote the support of $H(\eta)$. Then

$$\text{for } 0 < n < 3/2, \quad H(\eta) \sim (a-\eta)^2 \text{ as } \eta \uparrow a, \tag{9}$$

$$\text{for } n = 3/2 \quad H(\eta) \sim (a-\eta)^2 \log(1/(a-\eta))^{2/3} \quad \text{as} \quad \eta \uparrow a, \tag{10}$$

$$\text{for} \quad 3/2 < n < 3 \quad H(\eta) \sim (a-\eta)^{3/n} \quad \text{as} \quad \eta \uparrow a. \tag{11}$$

The less regular solutions have $H(\eta) \sim (a-\eta)$. We note that the existence result we prove in Theorem 1 is for solutions in a regularity class that includes the source type solutions (9–11) and excludes the less regular ones. In a companion paper [5] we present numerical simulations of the weak solutions which suggests rapid convergence onto the above unique source type solutions.

Starov [21] first noted that there are no finite mass 'source-type' solutions for $n=3$. Brenner and Bertozzi [8] addressed the significance of this fact for the physical problem of spreading droplets. The $n=3$ case arises when there is a no-slip boundary condition at the liquid solid interface. The lack of such scaling solutions is consistent with the fact that a no-slip boundary condition leads to infinite energy dissipation at the contact line for spreading drops with a finite contact angle [12, 17].

The non-existence of source type solutions for $n \geq 3$ is due to the structure of the ODE (8) and is in sharp contrast to the source type solutions for the porous media equation (2) which exist for all $m > 1$.

There are also **traveling wave solutions** of the form $h(x,t) = H(x - ct)$ as described in [7]. Again, we see transitions in the allowable behavior at different values of n. We omit many details but it is noteworthy that there are no *advancing front* solutions for $n \geq 3$ and that for $n < 2$ there are compactly supported traveling wave solutions (we believe these are unphysical). Furthermore, for $3/2 < n < 3$ there are advancing front solutions with the simple form

$$h(x,t) = \begin{cases} A(x-ct)^{3/n} & x > ct \\ 0 & \text{otherwise}, \end{cases} \qquad c = (\frac{3}{n} - 2)(\frac{3}{n} - 1)\frac{3}{n}A^n.$$

Finally we remark that there are exact **steady** solutions for all n

$$h(x,t) = \begin{cases} A - Bx^2, & |x| < A/\sqrt{B} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Bernis and Friedman [1] first addressed the existence theory for 'weak' nonnegative solutions. For simplicity they considered a bounded domain with boundary conditions $h_x = h_{xxx} = 0$ at both endpoints. In particular, they introduced the basic entropies (5) to prove positivity of solutions for $n \geq 4$ and nonnegativity of weak solutions for all $n \geq 1$. They considered two notions of 'weak solution', the first a very weak definition (see (20)) in which the integral of the flux term, $f(h)h_{xxx}$, is only over the set where the solution h is positive. For this definition their methods directly prove existence of nonnegative solutions for $n > 0$. The only information needed for this very weak definition is the surface tension dissipation (4). They also proved existence of weak solutions in the sense of distributions (18) for nonnegative initial data for $1 < n < 2$ and for positive (or entropy bounded) data for $n \geq 1$. They construct the weak solutions as the limit of smooth approximate solutions.

The first result of this paper extends the existence results in [1] to prove existence results in a sense of distributions for $\frac{3}{8} < n < 3$. We use several different formulations of distribution solution ((18),(19) and modifications of these). For these distribution solutions and weaker nonnegative solutions in the range $0 < n \leq \frac{3}{8}$ we obtain greater regularity for the solution than obtained in [1]. In particular the regularity class includes the unique most regular source type solutions (9–11) but not the less regular ones or the steady parabola solutions (12).

The second result of this paper concerns the long time behavior of our weak solutions as $t \to \infty$. In particular, we show that they approach their mean in the limit as $t \to \infty$. Since the convergence is in the L^∞ norm, there exists a time T^* after which the nonnegative weak solution becomes a positive strong solution. If we view the solution as describing a periodic array of droplets, this implies that the drops will spread and merge to form a uniform layer in the limit as $t \to \infty$.

The third result concerns the case of all $n > 0$ with positive (or entropy bounded) initial data. For this case we prove that even if the solution develops a singularity in finite time, it can be continued past this time as a nonnegative weak solution that approaches its mean in the infinite time limit. In particular, there exists a critical time T^* after which the solution is guaranteed to be positive and strong again.

The main technical tools are a new class of convex entropies which yield more refined existence results. Moreover, we show that for $0 < n < 2$ the basic entropies (5) are related to the L^2 norm of $h - c$ enabling us to prove a long time relaxation result. For the case $2 \leq n < 3$ we relate the new entropies to L^2 norms of nonlinear functions of h which again allow us to prove the relaxation result. In the case of positive initial data, we obtain this result for all $n > 0$.

We use a regularization introduced in [1]. In a subsequent paper [5] we use the same regularization to numerically simulate the weak solutions. The simulations indicate that the support of the solution has finite speed of propagation and continuous flux, two properties desirable for a physically correct model. Moreover they show rapid convergence of the solution onto the similarity solutions (9–11) before the merging of support.

The paper is organized as follows. Section 2 presents the statements of the theorems for nonnegative initial data. Section 3 reviews the properties of the regularization scheme. Section 4 proves the existence results for nonnegative initial data for $0 < n < 3$. Section 5 proves the long time results for nonnegative data for $0 < n < 3$. Section 6 proves the existence and long time results for strictly positive initial data for all $n > 0$. Section 7 briefly discusses unsolved problems and the ramifications of our results. The appendix contains the proof of an interpolation lemma and the uniform convergence of the regularized diffusion coefficient and its derivatives used in Section 4.

Before proceeding further, we discuss briefly the physical problem of thin films and contact lines.

1.2. **Thin Films, Contact Lines and the Lubrication Approximation.** The lubrication approximation for a thin film of liquid on a solid surface yields a fourth order degenerate diffusion equation for the film height [14]. In one space dimension, it is

$$h_t + ((|h|^3 + b_p|h|^p)h_{xxx})_x = 0 \tag{13}$$

where the parameter b_p fixes a 'slip' length [15] and the specific slip model determines the power p. The derivation uses the Stokes equation for steady viscous flow combined with a depth averaging of the fluid velocity in the direction perpendicular to the surface. A slip boundary condition on the liquid solid interface,

$$\lambda(h)\frac{\mathrm{v}}{\underset{\cdot}{\mathrm{z}}} = v, \quad \lambda(h) \sim h^{p-2}, \tag{14}$$

determines the power of p in the equation above [14]. For the purposes of this paper we consider p in the range $0 < p < 3$. We remark that the paper [15] considers $p = 0, 1,$ and

2. The case $p = 1$ models a thin film spreading on a slightly porous surface [19]. In this porous case, the parameter $0 < b \ll 1$ determines a microscopic length scale correlated with the porosity of the surface. The case $p = 2$ is also a well known 'slip model' (see e.g. [13]). Dussan [11] suggested that the choice of slip model is far less important that the slip length for the macroscopic dynamics of the drop. Nonetheless correct use of the various slip models requires a good understanding of the mathematics.

The boundary of the support of the weak solution physically corresponds to the *contact line* in the spreading droplet problem. In addition to a 'slip' law (14) Greenspan [14] suggested the need for a 'constitutive law' for the motion of the edge of the drop. Recent studies use such a relation [13, 15, 16]. The constitutive laws are motivated by the order of the equation (hence a need for an extra boundary condition) and are based on experimental observation. It is unclear that the problem remains well-posed or even solvable when constitutive laws are imposed at the edge of the support.

We propose an alternative to constitutive laws in the case of complete wetting. We view this as a free boundary problem in which the motion of the boundary is determined by global properties of the solution, as in the porous media equation. Rather than enforcing a specific constitutive law, we advocate letting a regulatization scheme pick out a solution. This has the advantage of guaranteeing a true solution to the problem. An obvious concern is that, since we have no proof of uniqueness, the problem may not be well posed and other regularization schemes might converge to other solutions. We conjecture that within a higher regularity class well-posedness results exist for this problem. It is interesting to note that this regularity implies a *zero local contact angle* for the weak nonnegative solutions with $0 < n < 3$. We address these issues in more detail in [5].

When h is small, the slip term in (13) dominates the behavior of the solution. Hence the results for the case $f(h) = |h|^n$ apply to the equation (13) with $p = n$.

2. Evolution From Nonnegative Initial Data

We consider solutions of the equation

$$h_t + (|h|^n h_{xxx})_x = 0 \tag{15}$$

on the circle, S^1, with periodic boundary conditions. This choice is made for simplicity, however it has the physical interpretation of modeling a periodic array of droplets.

The case $n = 0$ is the fourth order linear heat equation. It is elementary to show that for this problem there are global strong solutions that decay to their average as $t \to \infty$.

However, the case $p = 0$ is the nondegenerate linear case which does not preserve positivity of the solution. Hence, we do not consider this a viable model.

We also remark that the 1-D equation with $f(h) = |h|$ is models thin necks in the Hele-Shaw cell [9, 10, 22] However, our the weak solutions do not make sense physically for that problem. One simple consequence of the entropy argument here which does apply to Hele-Shaw is that given positive initial data, if no singularity occurs by the critical time T^*, then the solution will stay strong for all time and decay exponentially fast to its mean.

the triple contact point of the air/liquid, air/solid, and solid/liquid interfaces, each of which has its own local interfacial energy

However, unlike the second order linear heat equation, the linear fourth order equation does not preserve positivity. For example, with periodic boundary conditions on $[-1, 1]$ the initial condition

$$h_0(x) = 0.8 - \cos(\pi x) + 0.25\cos(2\pi x) \tag{16}$$

yields a solution that is initially positive but which is negative at the origin for a finite interval of time: $h(0,t) < 0$ for $t \in (t_1, t_2)$. Hence is it quite remarkable that for sufficiently strong nonlinearity, the equation preserves positivity. This section addresses the case of nonnegative intial data for $0 < n < 3$. In section 6 we prove analogous results for the case of strictly positive intial data and $0 < n < \infty$.

Let

$$Q_T = S^1 \times (0,T), \quad P(h) = \{(x,t) \in \bar{Q}_T \mid h(x,t) > 0\}. \tag{17}$$

The test functions, ϕ, are in $C_0^\infty((0,T); C^\infty(S^1))$. We introduce the following definitions of weak solution:

The strongest formulation uses two integration by parts

$$\iint_{Q_T} h\phi_t - \iint_{Q_T} f(h)h_{xx}\phi_{xx} - \iint_{Q_T} f'(h)h_x h_{xx}\phi_x = 0. \tag{18}$$

A second formulation uses a third integration by parts

$$\iint_{Q_T} h\phi_t + \frac{3}{2}\iint_{Q_T} f'(h)h_x^2\phi_{xx} + \frac{1}{2}\iint_{Q_T} f''(h)h_x^3\phi_x + \iint_{Q_T} f(h)h_x\phi_{xxx} = 0 \tag{19}$$

and a final, weakest, version has only one derivative on the test function, but integrates the flux over the set P, where $h > 0$:

$$\iint_{Q_T} h\phi_t + \iint_P f(h)h_{xxx}\phi_x = 0. \tag{20}$$

Versions (18) and (20) were introduced in [1]. We consider all three versions here.

The main result for nonnegative initial data is

Theorem 1. *Given any nonnegative initial condition $h_0 \in H^1(S^1)$, $h_0 \geq 0$ we have the following results*

Case 1: Given $f(h) = h^n$, $1 < n < 2$, $s < \min(2-n, \frac{1}{2})$, and a time interval $(0,T)$ there exists $h(x,t) \geq 0$ $h \in L^\infty\big(0,T; H^1(S^1)\big) \cap L^2\big(0,T; H^2(S^1)\big)$ and h satisfies the equation in the following sense:

$$\iint_{Q_T} h\phi_t - \iint_{Q_T} f(h)h_{xx}\phi_{xx} - \iint_{Q_T} f'(h)h_x h_{xx}\phi_x = 0 \tag{21}$$

Numerical results indicate that for the nonlinear lubrication approximation for all $n \leq 0.6$ finite time singularities occur at $x = 0$ with the initial condition (16) [4]. They have a similar blowup structure as the solutions described in [6] with different boundary conditions. In particular the fourth derivative blows up for all $n > 0$ and the third derivative blows up for all $n > \frac{1}{2}$.

Moreover,

$$h(x,0) = h_0(x) \qquad\qquad \forall x \in S^1$$

$$h_x(\cdot,t) \to h_{0x} \qquad\qquad \text{strongly in } L^2(S^1) \text{ as } t \to 0.$$

Furthermore, given $\alpha \geq \frac{1}{2} - s/4$*,* h *has the additional regularity*

$$h^{1-s/2} \in L^2(0,T;H^2(S^1))$$

and

$$(h^\alpha)_x \in L^4(Q_T).$$

Moreover, there exist positive A *and* c *such that for all* $t \in [0,T]$*,*

$$\|h(\cdot,t) - \overline{h}\|_{L^\infty} \leq Ae^{-ct}, \tag{22}$$

Where $\overline{h}$ *is the mean value of* h*.* A *depends only on* $|h_0|_{H^1}$*,* $\overline{h}$*,* $|S^1|$*,* n*, and* c*, the rate of decay, depends only on* n*, and* $\overline{h}$*. In particular, if* h_0 *is nonzero, there exists a time* T^* *after which the solution is a positive strong solution.*

Case 1A: If $f(h) = h^n$*,* $\frac{3}{8} < n \leq 1$ *the above is true if we replace the equation (21) with a solution in the sense*

$$\iint_{Q_T} h\phi_t - \iint_{Q_T} f(h)h_{xx}\phi_{xx} - \iint_{Q_T} nh^{n-\alpha}(\frac{h^\alpha}{\alpha})_x h_{xx}\phi_x = 0 \tag{23}$$

and choose α *in the above so that* $n > \alpha \geq \frac{1}{2} - \frac{s}{4}$*.*

Case 2: If $f(h) = h^n$*,* $2 < n < 3$*, given any* $0 < r < 1$ *satisfying* $0 < 2 + r - n < 1$ *there exists* $h \geq 0$ *such that on any time interval* $h \in L^\infty\big(0,T;H^1(S^1)\big)$ *and* h *satisfies the equation in the following sense (19):*

$$\iint_{Q_T} h\phi_t + \frac{3}{2}\iint_{Q_T} f'(h)h_x^2\phi_{xx} + \frac{1}{2}\iint_{Q_T} f''(h)h_x^3\phi_x + \iint_{Q_T} f(h)h_x\phi_{xxx} = 0.$$

The initial data is achieved as above. Furthermore, h *has the additional regularity*

$$h^{1+r/2} \in L^2\big(0,T;H^2(S^1)\big)$$

and

$$(h^\alpha)_x \in L^4(Q_T) \quad \forall \alpha \geq r/4 + 1/2.$$

For all $t \in [0,T]$*,*

$$\|h(x,t) - \overline{h}\|_{L^\infty} \leq Ae^{-ct} \tag{24}$$

A *and* c *have the same dependence as above. In particular, there exists a time* T^* *after which the solution is a positive strong solution.*

The statement for $n = 2$ is as in Case 2 with the minor change in the form of the equation. The details are presented in Section 4. We state and prove theorems for these cases in section 4.

The basic existence result for $1 < n < 2$ is proved in Bernis and Friedman [1] using a different regularization. The other existence results, the additional regularity, and the asymptotic behavior (22) and (24) of the solutions is new.

In addition, we prove the following for $0 < n \le 3/8$.

Theorem 2. *Given any nonnegative initial condition in $h_0 \in H^1(S^1)$, $h_0 \ge 0$, $f(h) = h^n$, $3/8 \ge n > 0$, and any time interval $[0,T]$, there exists $h \ge 0$*

$$h \in L^\infty\big(0,T; H^1(S^1)\big) \cap L^2\big(0,T; H^2(S^1)\big),$$

satisfying the equation in the following sense (20):

$$\iint_{Q_T} h\phi_t + \iint_P f(h) h_{xxx}\phi_x = 0.$$

where

$$h^n h_{xxx} \in L^2(P(h)).$$

The initial data is achieved in the sense described in case 1 above. Furthermore, for any $0 < s < \frac{1}{2}$ and $\alpha \ge \frac{1}{2} - \frac{s}{4}$, there exists an h satisfying the above with the additional regularity

$$h^{1-s/2} \in L^2(0,T; H^2(S^1))$$

$$(h^\alpha)_x \in L^4(Q_T).$$

Moreover, there exist positive A and c such that for all $t \in [0,T]$,

$$\|h(\cdot,t) - \overline{h}\|_{L^\infty} \le Ae^{-ct}. \tag{25}$$

A and c have the same dependence as above. In particular, there exists a time T^* after which the solution is a positive, strong solution.

Significant Remark:

Note that the additional regularity inherited by the weak solutions for $0 < n < 3$ is in *exact* agreement with the regularity of the 'zero contact angle' nonnegative source type solutions (9–11). That is, if we assume that the limiting solution $h(x,t)$ has support compactly contained in S^1 and and $h(x) \sim x^\beta$ at the edge of the support for all t on some interval $[0,T]$, then the regularity constraints demand that

$$\beta \ge 2 \quad 0 < n < 3/2 \tag{26}$$

$$\beta \ge 3/n \quad 3/2 < n < 3. \tag{27}$$

Hence the similarity solutions (9–11) just fit into our regularity class.

Outline of Proof: First we regularize the initial data and the equation to obtain an approximate solution that is a strong, smooth solution for all time. The choice of regularization comes from [1]. In section 3 we discuss the regularization and some basic apriori bounds associated with the solution.

The existence results require passing to limit in the regularization parameter. In section 4 we present the details, including new nonlinear estimates constructed specifically for this problem.

In section 5 we show that the various weak solutions constructed in the previous section all have the indicated long time behavior. The key tool is a lemma which shows that certain nonlinear entropies are equivalent to the L^2 norm of the difference between the solution and its average. The convexity of the entropy function allows us to prove such an estimate.

In a companion paper [5] we show numerical simulations of the regularized equations for several different values of the regularization parameter. The numerics indicate the solutions have very nice properties including finite speed of propagation of the support and, for early times, rapid convergence onto the similarity solutions described in (9–11).

3. Regularized problem

The regularization involves altering the equation and lifting the intial data. That is, we bound the initial data for the regularized problem away from zero by

$$h_{\epsilon 0}(x) = h_0(x) + \delta(\epsilon). \tag{28}$$

In addition, we regularize the equation by considering

$$h_{\epsilon t} + (f_\epsilon(h_\epsilon)h_{\epsilon xxx})_x = 0$$
$$f_\epsilon(h_\epsilon) = \frac{h_\epsilon^4 f(h_\epsilon)}{\epsilon f(h_\epsilon) + h_\epsilon^4}.$$

Note that f_ϵ is still degenerate however for $n < 4$, $f_\epsilon \sim h^4/\epsilon$ as $h \to 0$. Bernis and Friedman [1] proved that this approximate problem has global, positive, smooth solutions:

Theorem 3. *(Global existence of smooth positive solutions for the regularized problem* [1]*) Let $h_0 \in H^1(S^1)$, $h_0 \geq 0$. Given an initial condition*

$$h_{\epsilon 0}(x) = h_0(x) + \delta(\epsilon)$$

there exists a unique positive solution to the regularized equation

$$h_{\epsilon t} + (f_\epsilon(h_\epsilon)h_{\epsilon xxx})_x = 0$$
$$f_\epsilon(h_\epsilon) = \frac{h_\epsilon^4 f(h_\epsilon)}{\epsilon f(h_\epsilon) + h_\epsilon^4}.$$

We omit many details as they are presented in [1]. The main points are that

- Classical parabolic Schauder estimates guarantee existence of a smooth solution up to a time σ.
- In this short time of existence, the smooth solution satisfies

$$\int_{S^1} h_{\epsilon_x}^2(x,t)dx + \int_0^t \int_{S^1} f_\epsilon(h_\epsilon)h_{\epsilon_{xxx}}^2 dx = \int_{S^1} h_{\epsilon_x}^2(x,0)dx \tag{29}$$

  guaranteeing an upper bound, independent of t, for $\int_{S^1} h_{\epsilon_x}^2$ for any time $t < \sigma$.
  The Sobolev inequality and (29) provide an a priori bound for the Hölder norm $|h_\epsilon|_{C^{1/2}(S^1)}$.

$$|h_\epsilon(x_1,t) - h_\epsilon(x_2,t)| \leq C|x_1 - x_2|^{1/2} \quad \forall t < \sigma, \tag{30}$$
$$|h_\epsilon(\cdot,t)|_{L^\infty(S^1)} \leq C \quad \forall t < \sigma. \tag{31}$$

One can use these to prove

$$|h_\epsilon(x,t_1) - h_\epsilon(x,t_2)| \le C|t_1 - t_2|^{1/8} \tag{32}$$

where C depends only on the H^1 norm of the initial data.

- Furthermore, on any interval of existence of a smooth solution, we have the basic entropy
$$\int_{S^1} G_\epsilon(h_\epsilon) = \int_{S^1} \big[\frac{\epsilon}{6h_\epsilon^2} + G_0(h_\epsilon) + c\big]\, dx,$$
where
$$G_0''(y) = 1/f(y) \qquad\qquad \text{for } y > 0.$$
Integration by parts yields that
$$\int_{S^1} G_\epsilon(h_\epsilon(x,t))dx + \int_0^t \int_{S^1} h_{\epsilon xx}^{\,2} = \int_{S^1} G_\epsilon(h_\epsilon(x,0))dx \tag{33}$$
for any $t < \sigma$. The constant c is chosen to make $\int_{S^1} G_\epsilon(h_\epsilon) \ge 0$ implying an apriori bound for $\int_{S^1} \epsilon/h_\epsilon^2$. This provides an a priori pointwise *lower* bound for the solution $h_\epsilon(x,t)$. Indeed, let $\delta(t)$ be the minimum value of $h_\epsilon(x,t)$, occurring at a point $x_0(t)$. Then Hölder continuity implies $h_\epsilon(x,t) \le \delta(t) + C|x - x_0|^{1/2}$ for all $x \in S^1$. Hence
$$\begin{aligned} C \ge \int_{S^1} \frac{\epsilon}{h_\epsilon^2} &\ge \int_{S^1} \frac{\epsilon}{(\delta(t) + C|x - x_0(t)|^{1/2})^2} \\ &\ge 2\epsilon \int_0^{1/\delta^2} \frac{1}{(1 + C(y)^{1/2})^2} dy \\ &\ge 2C_1\epsilon |\log\delta|. \end{aligned}$$
Therefore
$$\delta(t) \ge \exp -(C_2/\epsilon).$$
This bound is quite crude. The entropy (45) described in the next section provides an algebraic in ϵ lower bound for the minimum height.
- Hence we have an a priori bound for the minimum in terms of the H^1 norm of the initial data and ϵ and for the maximum of the solution depending only on the H^1 norm of the initial data. I.e., the solution is uniformly parabolic on $[0,\sigma]$ and can be continued to any time T.

Moreover, (30–32) imply that $\{h_\epsilon\}$ is a uniformly bounded equicontinuous family of functions on Q_T. The Arzela-Ascoli theorem guarantees that a subsequence converges pointwise uniformly to a limit, h.

We summarize the apriori bounds independent of ϵ:
surface energy dissipation:

$$\int_{S^1} h_{\epsilon x}^{\,2}(x,t)\; dx + \int_0^t \int_{S^1} f_\epsilon(h_\epsilon) h_{\epsilon xxx}^{\,2} = \int_{S^1} h_{\epsilon x}^{\,2}(x,0)\; dx, \tag{34}$$

conservation of mass,

$$\int_{S^1} h_\epsilon(x,T)dx = \int_{S^1} h_\epsilon(x,0)dx, \tag{35}$$

and the basic entropy dissipation

$$\int_{S^1} G_\epsilon(h_\epsilon(x,T))\,dx + \int_0^T \int_{S^1} h_{\epsilon xx}^2 = \int_{S^1} G_\epsilon(h_{\epsilon 0}(x))\,dx. \tag{36}$$

Additional constraints on the initial data guarantee an apriori bound for $\int_{S^1} G_\epsilon(h_{\epsilon 0})$. In particular for $0 < n < 2$ we require that $\delta(\epsilon) \leq c\epsilon^{1/2}$ and for $n \geq 2$ the initial data cannot be zero on a set of positive measure. This is the key reason for the upper bound of $n < 2$ in theorem 1, Case 1.

4. Existence of Weak Solutions

Recall the regularized equation and initial data from the previous section:

$$h_{\epsilon 0}(x) = h_0(x) + \delta(\epsilon) \tag{37}$$

$$h_{\epsilon t} + (f_\epsilon(h_\epsilon)h_{\epsilon xxx})_x = 0 \tag{38}$$

$$f_\epsilon(h_\epsilon) = \frac{h_\epsilon^4 f(h_\epsilon)}{\epsilon f(h_\epsilon) + h_\epsilon^4} \tag{39}$$

We now show that with a good choice of $\delta(\epsilon)$ we can pass to the limit, proving the existence part of Theorems 1 and 2.

Recall that surface energy dissipation implies

$$\int_{S^1} h_{\epsilon x}^2(x,t)\,dx \leq \int_{S^1} h_{\epsilon 0x}^2\,dx \leq C.$$

The Sobolev embedding theorem implies there exists an $M < \infty$ such that

$$|h_\epsilon(x,t)| \leq M. \qquad \forall x,t.$$

Moreover $h_\epsilon > 0$.

Proposition 4. *Given $1 < n < 2$, $h_0 \geq 0$, and $h_0 \in H^1(S^1)$, let h_ϵ be the unique smooth solution to the regularized problem (37–39) with*

$$\delta(\epsilon) = \epsilon^\theta, \theta < 2/5. \tag{40}$$

Then on any time interval $[0,T]$, a subsequence h_ϵ converges pointwise uniformly, weakly in

$$L^2(0,T;H^2(S^1)) \cap L^\infty(0,T;H^1(S^1))$$

to a solution h in the sense of distributions (18):

$$\iint_{Q_T} h\phi_t = \iint_{Q_T} f(h)h_{xx}\phi_{xx} + \iint_{Q_T} f'(h)h_x h_{xx}\phi_x.$$

Furthermore, given any $0 < s < \min(2-n, \frac{1}{2})$ *and* $\alpha \geq \frac{1}{2} - \frac{s}{4}$, *there exists a solution with the additional regularity:*

$$h^{1-s/2} \in L^2(0,T; H^2(S^1)), \tag{41}$$

$$(h^\alpha)_x \in L^4(Q_T) \quad . \tag{42}$$

This existence result was proved in [1] using a different regularization. However we use the regularization (39) in the long time behavior results. For this reason, we only sketch the proof of existence. The additional regularity (41–42) is new. It follows from the existence of additional entropies, and its proof is presented in full.

Proof. Since the regularized solution, h_ϵ, is smooth it satisfies the following: For any test function $\phi \in C_0^\infty((0,T); C^\infty(S^1))$

$$\iint_{Q_T} h_\epsilon \phi_t = \iint_{Q_T} f_\epsilon(h_\epsilon) h_{\epsilon xx} \phi_{xx} + \iint_{Q_T} f'_\epsilon(h_\epsilon) h_{\epsilon x} h_{\epsilon xx} \phi_x. \tag{43}$$

We introduce the entropy

$$\int_{S^1} G_\epsilon(h_{\epsilon 0}) = \int_{S^1} \frac{\epsilon}{6 h_{\epsilon 0}^2} + \int_{S^1} G_0(h_{\epsilon 0}) \tag{44}$$

$$G_0''(y) = \frac{1}{f(y)} \quad \text{for } y > 0.$$

The constraint (40) on $\delta(\epsilon)$ provides a uniform bound for

$$\int_{S^1} \frac{\epsilon}{6 h_{\epsilon 0}^2} \leq C.$$

As G_0 is only determined up to two constants of integration, Jensen's inequality allows us to choose them so that $0 \leq \int_{S^1} G_0(h_\epsilon)$. Moreover, since $n < 2$, G_0 is a bounded function of h on the interval $0 \leq h \leq M$, providing a uniform bound

$$0 \leq \int_{S^1} G_0(h_{\epsilon 0}) \leq C.$$

Combining these yields an a priori bound for the entropy of the initial data. As a result, $\iint_{Q_T} h_{\epsilon xx}^2$ is bounded a priori since

$$\int_{S^1} G_\epsilon(h_\epsilon(x,T)) + \iint_{Q_T} h_{\epsilon xx}^2 = \int_{S^1} G_\epsilon(h_{\epsilon 0}) \leq C.$$

Weak compactness implies that a subsequence converges weakly in $L^2(0,T; H^2(S^1))$ to the limit h. Surface tension dissipation (34) implies that h_ϵ/t is uniformly bounded in $L^2(0,T; H^{-1}(S^1))$. The well-known Lions-Aubin lemma [18] then implies that a subsequence converges *strongly* in $L^2\big(0,T; H^1(S^1)\big)$ to the limit h. We note that this is the only place in this existence section where we explicitly use this compactness argument. In the remaining proofs we prove strong convergence by using the regularity properties of the solution on the set where $h > 0$ combined with the fact that $f(h)$ or a higher derivative vanishes on the set where $h=0$. This idea was used in [1] to prove the very weak existence result in the sense of (20). The additional entropies (45) and (58) allow us to use this idea to prove a weak existence result in a distribution sense.

We now argue that h is a weak solution in the sense (18). Passing to the limit requires $f_\epsilon'(h_\epsilon)$ to converge *uniformly* on Q_T to $f'(h)$. Since h_ϵ converges uniformly to h on Q_T, it suffices to show that $f'(x)$ is a continuous function of x and $f_\epsilon'(x)$ converges uniformly on $[0,M]$ to $f'(x)$. This is true for $n>1$ and is proved in the appendix. Similarly, $f_\epsilon(h_\epsilon)$ converges uniformly on Q_T to $f(h)$.

Hence the limit $h(x,t)$ solves the equation in the sense (18). In particular, the nonlinear terms converge to the desired limits. For example,

$$\iint_{Q_T} f_\epsilon'(h_\epsilon) h_{\epsilon x} h_{\epsilon xx}\phi_x \rightarrow \iint_{Q_T} f'(h) h_x h_{xx}\phi_x$$

since $h_{\epsilon xx}$ converges weakly in L^2 to h_{xx}, $h_{\epsilon x}$ converges strongly in L^2 to h_x, and $f_\epsilon'(h_\epsilon)$ converges uniformly to $f'(h)$.

For the additional regularity, we introduce the following entropy:

$$G_{n\epsilon}^{-s}(y) = \frac{1}{(2+s)(3+s)}\frac{\epsilon}{y^{2+s}} + \frac{1}{(2-n-s)(1-n-s)} y^{2-n-s} + c \tag{45}$$

chosen so that $(G_{n\epsilon}^{-s})''(y) = \frac{1}{y^s f_\epsilon(y)}$. Integration by parts yields

$$\frac{d}{dt}\int_{S^1} G_{n\epsilon}^{-s}(h_\epsilon) = -\int_{S^1} h_\epsilon^{-s} h_{\epsilon xx}^2 + \frac{s(s+1)}{3}\int_{S^1} h_\epsilon^{-s-2} h_{\epsilon x}^4. \tag{46}$$

Note that for $0\le s$

$$\frac{s(s+1)}{3}\int_{S^1} h_\epsilon^{-s-2}h_{\epsilon x}^4 = s\int_{S^1} h_\epsilon^{-s-1} h_{\epsilon x}^2 (h_\epsilon)_{xx} \le s\sqrt{\int_{S^1} h_\epsilon^{-s} h_{\epsilon xx}^2}\sqrt{\int_{S^1} h_\epsilon^{-s-2} h_{\epsilon x}^4}$$

so that

$$\int_{S^1} h_\epsilon^{-s-2} h_{\epsilon x}^4 \le \frac{9}{(s+1)^2}\int_{S^1} h_\epsilon^{-s} h_{\epsilon xx}^2.$$

If $s<\frac{1}{2}$,

$$\frac{d}{dt}\int_{S^1} G_{n\epsilon}^{-s}(h_\epsilon) = -\int_{S^1} h_\epsilon^{-s} h_{\epsilon xx}^2 + \frac{s(s+1)}{3}\int_{S^1} h_\epsilon^{-s-2} h_{\epsilon x}^4 \tag{47}$$

$$\le \left(-1+\frac{3s}{s+1}\right)\int_{S^1} h_\epsilon^{-s} h_{\epsilon xx}^2 = -C(s)\int_{S^1} h_\epsilon^{-s} h_{\epsilon xx}^2 < 0. \tag{48}$$

The constant $C(s)$ decreases to zero as s increases to $\frac{1}{2}$.

As before, Jensen's inequality implies that the constant c in (45) can be chosen so that $0 \le \int_{S^1} G_{n\epsilon}^{-s}(h_\epsilon)$. Since $0\le s<\frac{1}{2}$, the constraint (40) on $\delta(\epsilon)$, implies

$$\int_{S^1} \frac{\epsilon}{h_{\epsilon 0}^{2+s}} \le C.$$

Moreover, y^{2-n-s} is a bounded function of y on $[0, M]$, implying

$$\int_{S^1} h_{\epsilon 0}^{2-n-s} \leq C.$$

Therefore we have an a priori bound for the entropy of the initial data, $\int_{S^1} G_{n\epsilon}^{-s}(h_{\epsilon 0})$.

Hence (46) implies there is a C, independent of ϵ so that

$$0 < \iint_{Q_T} h_\epsilon^{-s} h_{\epsilon xx}^{\ 2} - \frac{s(s+1)}{3} \iint_{Q_T} h_\epsilon^{-s-2} h_{\epsilon x}^{\ 4} \leq C. \tag{49}$$

Moreover, (48) implies

$$C(s) \iint_{Q_T} h_\epsilon^{-s} h_{\epsilon xx}^{\ 2} \leq C. \tag{50}$$

For smooth $h(x)$ bounded away from zero,

$$\int_{S^1} (h^{1-s/2})_{xx}^2 = (1-s/2)^2 \left(\int_{S^1} h^{-s} h_{xx}^2 + [(\frac{s^2}{4} - \frac{s}{3}(s+1)] \int_{S^1} h_x^4 h^{-s-2} \right). \tag{51}$$

Combining (49), (50), and (51), yields the a priori bound

$$\iint_{Q_T} (h_\epsilon^{1-s/2})_{xx}^2 \leq C_1(s).$$

Thus there is a subsequence so that

$$h_\epsilon^{1-\frac{s}{2}} \text{ converges weakly in } L^2(0,T; H^2(S^1)).$$

Furthermore, the uniform bound in $L^\infty(0,T; H^1(S^1))$ implies via standard parabolic arguments (see e.g. [1] sec. 2) that $\{h_\epsilon\}$ is a uniformly bounded, equicontinuous family of functions on Q_T, hence it has a subsequence ϵ' converging uniformly to a limit h. Hence $h_\epsilon^{1-s/2}$, $h_\epsilon^{\frac{1}{2}-s/4}$ converge to $h^{1-s/2}$ and $h^{\frac{1}{2}-s/4}$. By the definition of distribution derivative, the weak limits of $(h_\epsilon^{1/2-s/4})_x$ and $(h_\epsilon^{1-s/2})_{xx}$ are $(h^{1/2-s/4})_x$ and $(h^{1-s/2})_{xx}$ respectively.

The weak solution, h, inherits the a priori bounds

$$\iint_{Q_T} (h^{1-\frac{s}{2}})_{xx}^2 \leq C \quad \iint_{Q_T} (h^{\frac{1}{2}-\frac{s}{4}})_x^4 \leq C \quad \iint_{Q_T} h_{xx}^2 \leq C.$$

Moreover, $||h(\cdot,t)||_\infty \leq M$, implies

$$\iint_{Q_T} (h^\alpha)_x^4 \leq C \iint_{Q_T} (h^{\frac{1}{2}-\frac{s}{4}})_x^4 \leq C$$

for all $\alpha \geq \frac{1}{2} - \frac{s}{4}$. □

For $\frac{3}{8} < n \leq 1$, we do not have uniform convergence of $f_\epsilon'(x)$ to $f'(x)$ on $(0, M]$. However, we prove in the appendix for $0 < n < 1$ that given any $\alpha < n$, $h^{1-\alpha} f_\epsilon'(h)$ converges uniformly to $nh^{n-\alpha}$. This, combined with the additional regularity above, yields the following theorem:

Proposition 5. *Given* $\frac{3}{8} < n \leq 1$, $h_0 \geq 0$, *and* $h_0 \in H^1(S^1)$, *let* h_ϵ *be the unique smooth solution to the regularized problem (37–39) with*

$$\delta(\epsilon) = \epsilon^\theta, \theta < 2/5. \tag{52}$$

Then on any time interval $[0, T]$, *given* $0 < s < \frac{1}{2}$, *and* $n > \alpha \geq \frac{1}{2} - \frac{s}{4}$, *there exists a subsequence* h_ϵ *converging pointwise uniformly, and weakly in*

$$L^2(0, T; H^2(S^1)) \cap L^\infty(0, T; H^1(S^1))$$

to a solution h *which satisfies the additional regularity conditions,*

$$h^{1-s/2} \in L^2(0, T; H^2(S^1)), \tag{53}$$

$$(h^\alpha)_x \in L^4(Q_T). \tag{54}$$

and satisfies the equation in the following sense

$$\iint_{Q_T} h\phi_t = \iint_{Q_T} f(h) h_{xx}\phi_{xx} + \iint_{Q_T} h^{n-\alpha}\left(\frac{h^\alpha}{\alpha}\right)_x h_{xx}\phi_x.$$

Proof. Recall the entropy introduced in the previous proof:

$$G_{1\epsilon}^{-s}(y) = \frac{1}{(2+s)(3+s)}\frac{\epsilon}{y^{2+s}} - \frac{1}{s(1-s)} y^{1-s}.$$

Exactly as before, (52) implies that the entropy of the initial data is uniformly bounded, yielding a priori bounds for the following:

$$\iint_{Q_T} h_\epsilon^{-s-2} h_{\epsilon x}^{4} \qquad\qquad \iint_{Q_T} \left(h_\epsilon^{1-\frac{s}{2}}\right)_{xx}^2$$

Similarly the entropy $G_\epsilon(y)$ (44) provides an a priori bound of

$$\iint_{Q_T} h_{\epsilon xx}^{2}.$$

Exactly as before,

$$h_\epsilon \rightharpoonup h \quad \text{weakly in } L^2(0, T; H^2(S^2))$$

and

$$h_\epsilon^{1-\frac{s}{2}} \rightharpoonup h^{1-\frac{s}{2}} \quad \text{weakly in } L^2(0, T; H^2(S^2)).$$

To prove that h is a weak solution, we need the following lemma to prove convergence of the nonlinear tems:

Lemma 6. *Let* $0 < s < \frac{1}{2}$, $\frac{3}{8} < n \leq 1$, $n > \alpha \geq \frac{1}{2} - \frac{s}{4}$. *The limit* h *satisfies*

$$h^{1-s/2} \in L^2(0, T; H^2(S^1)), \quad (h_\epsilon^\alpha)_x \in L^4(Q_T).$$

Let $\Omega \subset\subset Q_T$ *be compactly contained in* Q_T. *Then*

$$f_\epsilon'(h_\epsilon) h_{\epsilon x} \to h^{n-\alpha}\left(\frac{h^\alpha}{\alpha}\right)_x \qquad \textit{strongly in } L^2(\Omega).$$

Proof. Fix $\mu > 0$.

$$\iint_\Omega \left(f'_\epsilon(h_\epsilon)h_{\epsilon x} - h^{n-\alpha}(\frac{h^\alpha}{\alpha})_x \right)^2$$
$$= \iint_{\Omega\cap\{h>\mu\}} \left(f'_\epsilon(h_\epsilon)h_{\epsilon x} - h^{n-\alpha}(\frac{h^\alpha}{\alpha})_x \right)^2$$
$$+ \iint_{\Omega\cap\{h\le\mu\}} \left(f'_\epsilon(h_\epsilon)h_{\epsilon x} - h^{n-\alpha}(\frac{h^\alpha}{\alpha})_x \right)^2$$

By the regularity theory of uniformly parabolic equations, h is smooth in $\Omega \cap \{h \ge \mu\}$, and h_ϵ and its derivatives converge uniformly to h and its derivatives on this set. Hence, by taking ϵ to zero,

$$\iint_{\Omega\cap\{h>\mu\}} \left(f'_\epsilon(h_\epsilon)h_{\epsilon x} - h^{n-\alpha}(\frac{h^\alpha}{\alpha})_x \right)^2 \to 0$$

For the second integral, we expand out the square and bound each term. One such term is

$$\iint_{\Omega\cap\{h\le\mu\}} (f'_\epsilon(h_\epsilon)h_{\epsilon x})^2 \le C \iint_{h\le\mu} (h_\epsilon^{1-\alpha} f'_\epsilon(h_\epsilon))^2\,(h_\epsilon^\alpha)_x^2$$
$$\le C \left[\sup_{\{h\le\mu\}} (h_\epsilon^{1-\alpha} f'_\epsilon(h_\epsilon)) \right]^2 \iint_{Q_T} (h_\epsilon^\alpha)_x^2$$
$$\le C \left[\sup_{\{h\le\mu\}} (h_\epsilon^{1-\alpha} f'_\epsilon(h_\epsilon)) \right]^2 .$$

Here we use the fact that since $\alpha \ge 1/2 - s/4$, $(h_\epsilon^\alpha)_x$ is uniformly bounded in $L^4(Q_T)$. We also use the fact (proved in the appendix) that $y^{1-\alpha} f'_\epsilon(y)$ converges uniformly on $[0, M]$ to $ny^{n-\alpha}$. As before, this and the uniform convergence of h_ϵ to h imply $h_\epsilon^{1-\alpha} f'_\epsilon(h_\epsilon)$ converges uniformly on Q_T to $h^{n-\alpha}$. Therefore, by taking ϵ small,

$$\left[\sup_{\{h\le\mu\}} (h_\epsilon^{1-\alpha} f'_\epsilon(h_\epsilon)) \right]^2 \le C\mu^{2(n-\alpha)}.$$

The other two terms from the integral over $\Omega \cap \{h \le \mu\}$ are bounded in the exact same manner. By taking $\mu \to 0$, we have the result. □

This lemma implies

$$\iint_{Q_T} f'_\epsilon(h_\epsilon)h_{\epsilon x}h_{\epsilon xx}\phi_x \to \frac{1}{1-s/2} \iint_{Q_T} h^{n-\alpha}(\frac{h^\alpha}{\alpha})_x h_{xx}\phi_x$$

since any test function ϕ has support Ω compactly contained in Q_T. Convergence of the remaining terms follows in the same way. The additional regularity results (53), (54) follow as in the proof for $1 < n < 2$. □

We now consider the case $2 \le n < 3$. For n fixed, we take r satisfying

$$0 < r < 1, \tag{55}$$

$$0 < 2 + r - n < 1. \tag{56}$$

Proposition 7. *Let $2 < n < 3$ and r be as defined above, $h_0 \ge 0$, and $h_0 \in H^1(S^1)$. Take h_ϵ to be the unique smooth solution to the regularized problem (37–39) with*

$$\delta(\epsilon) = \epsilon^\theta, \theta < \frac{1}{2}. \tag{57}$$

Then there exists a subsequence h_ϵ that converges pointwise uniformly and weakly in $L^\infty(0, T; H^1(S^1))$ to a limit h. The limit h is a weak solution in the following sense:

$$\iint_{Q_T} h\phi_t = -\frac{1}{2}\iint_{Q_T} f''(h)h_x^3\phi_x - \frac{3}{2}\iint_{Q_T} f'(h)h_x^2\phi_{xx} - \iint_{Q_T} f(h)h_x\phi_{xxx}.$$

Furthermore the solution has the additional regularity

$$h^{1+\frac{r}{2}} \in L^2(0, T; H^2(S^1))$$

$$(h^\alpha)_x \in L^4(Q_T) \quad \textit{for all} \quad \alpha \ge \frac{r}{4} + \frac{1}{2}.$$

Proof. We introduce the entropy $G^r_{n\epsilon}(y)$ chosen so that $G^{r\,\prime\prime}_{n\epsilon}(y) = \frac{y^r}{f(y)}$ for $y > 0$.

$$G^r_{n\epsilon}(y) = \frac{y^{r-n+2}}{(2-n+r)(1-n+r)} + c + \frac{\epsilon y^{r-2}}{(r-2)(r-3)} \tag{58}$$

where c is chosen so that $\int_{S^1} G^r_{n\epsilon}(h_\epsilon) \ge 0$.

As before, we can integrate by parts:

$$\frac{d}{dt}\int_{S^1} G^r_{n\epsilon}(h_\epsilon) = -\int_{S^1} h_\epsilon^r h_{\epsilon xx}^{\,2} + \frac{1}{3}r(r-1)\int_{S^1} h_\epsilon^{r-2} h_{\epsilon x}^{\,4}.$$

Recall that for a smooth function $h > 0$,

$$\int_{S^1} (h^{1+r/2})_{xx}^2 = (1 + r/2)^2 \left(\int_{S^1} h^r h_{xx}^2 + \{\frac{r^2}{4} - \frac{r}{3}(r-1)\}\int_{S^1} h_x^4 h^{r-2}\right). \tag{59}$$

Combining these, we find for $0 \le r \le 1$,

$$\frac{d}{dt}\int_{S^1} G^r_{n\epsilon}(h_\epsilon) \le -C_r \int_{S^1} (h_\epsilon^{1+\frac{r}{2}})_{xx}^2 \qquad \text{where } C_r = \tfrac{16(1-r)}{(4-r)(r+2)^2}$$

hence

$$\int_{S^1} G^r_{n\epsilon}(h_\epsilon(x, T))\,dx + C_r \iint_{Q_T} (h_\epsilon^{1+\frac{r}{2}})_{xx}^2\,dx\,dt \le \int_{S^1} G^r_{n\epsilon}(h_{\epsilon 0}(x))\,dx. \tag{60}$$

As before, the constraint (57) on $\delta(\epsilon)$, and $r - n + 2 > 0$ provide an a priori bound of the entropy of the initial data $\int_{S^1} G^r_{n\epsilon}(h_{\epsilon 0}) \le C$.

This in turn, implies that the following are bounded uniformly in ϵ:

$$\iint_{Q_T} (h_\epsilon^{1+\frac{r}{2}})_{xx}^2, \qquad \iint_{Q_T} h_\epsilon^{r-2}(h_\epsilon)_x^4 = \frac{1}{(\frac{r}{4}+\frac{1}{2})^4}\iint_{Q_T} \left(h_\epsilon^{r/4+1/2}\right)_x^4.$$

These bounds imply that for fixed r, there exists a subsequence so that

$$(h_\epsilon^{1+\frac{r}{2}})_{xx} \rightharpoonup (h^{1+\frac{r}{2}})_{xx} \qquad \text{in } L^2(0,T;L^2(S^1)), \tag{61}$$

$$(h_\epsilon^\alpha)_x \rightharpoonup (h^\alpha)_x \qquad \text{in } L^4(0,T;L^4(S^1)) \;\; \forall \alpha \geq \frac{r}{4}+\frac{1}{2}, \tag{62}$$

$$h_{\epsilon x} \rightharpoonup h_x \qquad \text{in } L^\infty(0,T;L^2(S^1)). \tag{63}$$

We need to show convergence of all the nonlinear terms. We present the argument for $\iint_{Q_T} f_\epsilon''(h_\epsilon)h_{\epsilon x}^3\phi_x$. The others are simpler to show and follow analogously.

Lemma 8. *Let $\Omega \subset\subset Q_T$ be compactly contained in Q_T. Then for $2 < n < 3$,*

$$f_\epsilon''(h_\epsilon)h_{\epsilon x}^2 \to f''(h)h_x^2 \;\; \textit{strongly in } L^2(\Omega).$$

Proof. We use the fact that $f_\epsilon''(y) \to f''(y) = n(n-1)y^{n-2}$ uniformly on $[0,M]$ as $\epsilon \to 0$ for $n > 2$. The details are otherwise identical to proof of Lemma 6. In the appendix, we prove the uniform convergence of $f_\epsilon''(y)$. □

As before, this lemma implies

$$\iint_{Q_T} f_\epsilon''(h_\epsilon)h_{\epsilon x}^3\phi_x \to \iint_{Q_T} f''(h)h_x^3\phi_x.$$

since ϕ has support Ω compact in Q_T and $h_{\epsilon x}$ converges weakly in $L^2(Q_T)$. □

Proposition 9. *For $n = 2$, $0 < r < 1$, $h_0 \geq 0$, and $h_0 \in H^1(S^1)$, let h_ϵ be the unique smooth solution to the regularized problem (37–39) with*

$$\delta(\epsilon) = \epsilon^\theta, \theta < 1/2 \tag{64}$$

Then on any time interval $[0,T]$, for any $1 > \alpha > \frac{r}{4}+\frac{1}{2}$, a subsequence h_ϵ converges pointwise uniformly, and weakly in $L^\infty(0,T;H^1(S^1))$ to a limit h, and the limit h is a weak solution in the following sense:

$$\iint_{Q_T} h\phi_t = -\frac{1}{\alpha}\iint_{Q_T} h^{1-\alpha}(\frac{h^\alpha}{\alpha})_x h_x^2\phi_x - \frac{3}{2}\iint_{Q_T} f'(h)h_x^2\phi_{xx} - \iint_{Q_T} f(h)h_x\phi_{xxx}.$$

Furthermore, the solution has the additional regularity

$$h^{1+\frac{r}{2}} \in L^2(0,T;H^2(S^1)).$$

The proof is idential to that for $3/8 < n \leq 1$ and is left to the reader. The following lemma is needed:

Lemma 10. *Let $1 > \alpha > \frac{r}{4}+\frac{1}{2}$, n=2. Let $\Omega \subset\subset Q_T$. Then*

$$f_\epsilon''(h_\epsilon)h_{\epsilon x}^2 \to 2(h^{1-\alpha}(\frac{h^\alpha}{\alpha})_x)^2 \;\; \textit{strongly in } L^2(\Omega).$$

Outline of proof: The proof is identical to the proof of Lemma 6, and is left to the reader. It uses the fact that for $n = 2$, and $\beta > 0$, $y^\beta f''_\epsilon(y)$ converges uniformly on $[0, M]$ to $2y^s$ and the additional a priori bound

$$\iint_{Q_T} (h_\epsilon^\alpha)_x^4 \leq C.$$

for $\alpha \geq \frac{r}{4} + \frac{1}{2}$. □

Proposition 11. *Given $0 < n$, $h_0 \geq 0, h_0 \in H^1(S^1)$ let h_ϵ be the unique smooth solution to the regularized problem (37–39) with*

$$\delta(\epsilon) = \epsilon^\theta, \theta < 2/5. \tag{65}$$

Then a subsequence h_ϵ converges pointwise uniformly, weakly in $L^2(0,T;H^2(S^1))$ and $L^\infty(0,T;H^1(S^1))$ to h where,

$$h \in L^2(0,T;H^2(S^1)) \cap L^\infty(0,T;H^1(S^1)).$$

Moreover, h is a weak solution in the following sense:

$$\iint_{Q_T} h\phi_t = -\iint_P f(h)h_{xxx}\phi_x$$

where

$$f(h)h_{xxx} \in L^2(P(h)).$$

The constraint on θ is not used here but is necessary for the long time behavior proved in the next section. The proof of this proposition is identical to that used in [1] for $n > 1$ and is left to the reader.

It uses the following lemma:

Lemma 12. *Let $\Omega \subset\subset Q_T$. Then $f_\epsilon(h_\epsilon)h_{\epsilon xxx}$ converges* strongly *in $L^2(\Omega)$ to the following function*

$$fl(x) = \begin{cases} f(h)h_{xxx} & h > 0, \\ 0 & h = 0. \end{cases}$$

Proof. Recall that the surface energy dissipation for the regularized problem (37–39) yields the following a priori bound:

$$\iint_{Q_T} f_\epsilon(h_\epsilon)h_{\epsilon xxx}^2 \leq C.$$

Let $\mu > 0$. In the appendix, we prove that for $0 < n$, $f_\epsilon(x)$ converges uniformly on $[0, M]$ to $f(y) = y^n$.

To show the strong convergence in $L^2(\Omega)$ we note that

$$\iint_\Omega (f(h_\epsilon)h_{\epsilon xxx} - fl)^2 = \tag{66}$$

$$\iint_{\Omega \cap \{h > \mu\}} (f(h_\epsilon)h_{\epsilon xxx} - fl)^2 + \iint_{\Omega \cap \{h \leq \mu\}} (f(h_\epsilon)h_{\epsilon xxx} - fl))^2.$$

Again, on $\Omega \cap \{h \geq \mu\}$ $f_\epsilon(h_\epsilon)^{1/2} h_{\epsilon xxx}$ converges uniformly to $f(h)^{1/2} h_{xxx}$ by the uniform parabolicity of the equation. Therefore the first integral goes to zero.

This also yields the regularity result $f(h)^{1/2} h_{xxx} \in L^2(P)$.

To compute the second integral note that

$$\iint_{\Omega \cap \{h \leq \mu\}} (f(h_\epsilon)^2 h_{\epsilon xxx}^2 \leq C \sup_{h \leq \mu} |f_\epsilon(h_\epsilon)| \leq C\mu^n.$$

Moreover,

$$\iint_{\Omega \cap \{h \leq \mu\}} f l^2 = \iint_{\Omega \cap \{0 < h \leq \mu\}} f(h)^2 h_{xxx}^2 \leq \mu^n \iint_P f(h) h_{xxx}^2 \leq C\mu^n.$$

Taking $\mu \to 0$, we have the desired result. □

5. Long Time Behavior of Solutions

We now show that the solutions from the previous section satisfy the long time bounds (22), (24), (25). Recall the regularized problem

$$h_{\epsilon t} = -(f_\epsilon(h_\epsilon) h_{\epsilon xxx})_x, \tag{67}$$

$$f_\epsilon(h_\epsilon) = \frac{h_\epsilon^4 h_\epsilon^n}{\epsilon h_\epsilon^n + h_\epsilon^4}, \tag{68}$$

$$h_{0\epsilon}(x) = h_0(x) + \delta(\epsilon), \quad \delta = \epsilon^\theta, \quad \theta < 2/5. \tag{69}$$

In this section we prove

Proposition 13. *Given $h_0 \in H^1(S^1)$, $h_0 \geq 0$, let h be a uniform limit of the regularization scheme (67–69) on $[0,T]$ with $0 < n < 3$. $\overline{h}$ is the mean of the initial data, $\overline{h} = \frac{1}{|S^1|}\int_{S^1} h_0$ then there exist positive A and c such that*

$$\|h(\cdot,t) - \overline{h}\|_{L^\infty(S^1)} \leq Ae^{-ct}. \tag{70}$$

A is determined by $|h_0|_{H^1}$, $\overline{h}$, n, and $|S^1|$, and c is determined by n and $\overline{h}$. In particular, if h_0 is not identically zero there is a time T^* after which h is a positive strong solution.

To prove this proposition, we first prove the exponential convergence in the L^2 norm and then use an interpolation lemma to get L^∞ convergence. For technical reasons, we prove it in two steps, first for $0 < n < 2$ and then for $1 < n < 3$.

For $0 < n < 2$, we show that the basic entropy $\int_{S^1} G_0(h)$ is equivalent to $\int_{S^1}(h - \overline{h})^2$. A standard Poincaré inequality and Gronwall argument then proves the result.

Lemma 14. *Given $0 < n < 2$ and any $0 < y_0 < M$, there exists positive constants C_{n,y_0} and C_{n,M,y_0} so that the basic entropy,*

$$G_{0,y_0}(y) = \begin{cases} \frac{1}{(1-n)(2-n)} y^{2-n} - \frac{y_0^{1-n}}{(1-n)} y + \frac{y_0^{2-n}}{2-n} & n \neq 1 \\ y \log y - (1 + \log y_0) y + y_0 & n = 1 \end{cases} \tag{71}$$

satisfies

$$G_{0,y_0}(y) \le C_{n,y_0}(y-y_0)^2 \qquad \forall \ \ 0<y, \tag{72}$$

$$C_{n,M,y_0}(y-y_0)^2 \le G_{0,y_0}(y) \quad \forall \ \ 0<y<M. \tag{73}$$

In the above, the integration constants were chosen so that both G_{0,y_0} and its first derivative vanish at y_0. The geometric interpretation of this lemma is that for any fixed $0<y_0<M$, the graph of $G_{0,y_0}(y)$ can be "sandwiched" between two parabolas with the vertex $(y_0,0)$. The proof follows from the convexity of $G_0(y)$ and the boundedness of $G_0(y)$ as y decreases to zero. Since $||h||_\infty \le M$, the lemma proves that $\int G_{0,\overline{h}}(h)$ is equivalent to $\int_{S^1}(h-\overline{h})^2$.

To apply this lemma to the entropy dissipation, we recall

Poincaré's inequality. *Let $h \in C^2(S^1)$ and consider $x_0 \in S^1$. Then*

$$\int_{S^1} (h(x)-h(x_0))^2 dx \le |S^1|^4 \int_{S^1} h_{xx}^2 dx.$$

The regularized entropy

$$G_{\epsilon,\overline{h}+\delta}(y) = G_{0,\overline{h}+\delta}(y) + \frac{\epsilon}{6y^2}$$

satisfies

$$\frac{d}{dt}\int_{S^1} G_{\epsilon,\overline{h}+\delta}(h_\epsilon(x,t))\, dx = -\int_{S^1} h_\epsilon(x,t)_{xx}^2\, dx.$$

Applying Poincaré's inequality and (72),

$$\begin{aligned}\int_{S^1} G_{\epsilon,\overline{h}+\delta}(h_\epsilon(\cdot,T)) - \int_{S^1} G_{\epsilon,\overline{h}+\delta}(h_\epsilon(\cdot,0)) &\le -c\iint_{Q_T} (h_\epsilon - (\overline{h}+\delta))^2 \\ &\le -c \iint_{Q_T} G_{0,\overline{h}+\delta}(h_\epsilon).\end{aligned}$$

Define

$$B_\epsilon = \sup_{t\in[0,T]} \left| \int_{S^1} \frac{\epsilon}{6h_\epsilon(\cdot,t)^2} \right|.$$

Gronwall's lemma, (73), and the a priori bound on the entropy of the initial data imply

$$C_{n,M,y_0}||h_\epsilon(\cdot,t)-(\overline{h}+\delta)||_{L^2}^2 \le \int_{S^1} G_{\epsilon,\overline{h}+\delta}(h_\epsilon(\cdot,t)) \le B_\epsilon + Ae^{-ct}. \tag{74}$$

Furthermore, $B_\epsilon \to 0$ as $\epsilon \to 0$. Indeed, taking $0<s<\min(\frac{1}{2}, 2-n)$, the entropy (45) guarantees an a priori bound independent of ϵ for

$$\int_{S^1} \frac{\epsilon}{h_\epsilon(x,t)^{2+s}} dx \le C.$$

Hence, by Hölder's inequality

$$\int_{S^1} \frac{\epsilon}{h_\epsilon(x,t)^2} dx \le C \left(\int_{S^1} \frac{\epsilon}{h_\epsilon(x,t)^{2+s}} dx \right)^{2/(2+s)} \epsilon^{s/(2+s)} \le C\epsilon^{s/(2+s)}.$$

Taking $\epsilon \to 0$ in (74),

$$||h(\cdot,t) - \overline{h}||_{L^2} \leq Ae^{-ct}. \tag{75}$$

We prove the following in the appendix

Lemma 15. *(interpolation inequality)*
Let $w \in L^1(S^1) \cap C^\alpha(S^1)$, $0 < \alpha < 1$. Let $|\cdot|_\alpha$ denote the Hölder$-\alpha$ seminorm. Then

$$|w|_{L^\infty} \leq (\frac{1+\alpha}{\alpha})^{\alpha/(1+\alpha)} |w|_\alpha^{1/(1+\alpha)} |w|_{L^1}^{\alpha/(1+\alpha)} + (\frac{1+\alpha}{\alpha})|w|_{L^1}/|S^1|. \tag{76}$$

The limit h is uniformly bounded in $C^{1/8,1/2}(Q_T)$. Applying the interpolation inequality with $\alpha = \frac{1}{2}$,

$$|w|_{L^\infty} \leq C|w|_{L^1}^{1/3} \leq C_1|w|_{L^2}^{1/3}. \tag{77}$$

This finishes the proof for $0 < n < 2$, since (75) and (77) imply

$$||h(\cdot,t) - \overline{h}||_{L^\infty} \leq Ae^{-ct}.$$

The rate of decay, c, has bad dependence on n as n increases to 2 since C_{n,s_0} from (72) blows up, forcing c to zero. This is artificial, as a better rate of decay follows from the $1 < n < 3$ result, which we now prove.

The argument is similar to the $0 < n < 2$ case so we leave some of the details to the reader. Recall from the previous argument that on $[0, M]$, the graph of the basic entropy, G_{0,y_0}, was bounded above and below by the graphs of parabolas (73). In the following, we use a similar idea, bounding the entropy $G^r_{0,y_0}(y)$ above and below with the graphs of "nonlinear parabolas" $C(y^{1+\frac{r}{2}} - y_0^{1+\frac{r}{2}})^2$.

Lemma 16. *Given $1 < n < 3$, $0 < r < 1$, $0 < 2-n+r < 1$, $0 < y_0 < M^{1+r/2}$ there exists constants C_{n,y_0} and C_{n,M,y_0} so that the entropy,*

$$G^r_{0,y_0}(y) = \frac{y^{2-n+r}}{(1-n+r)(2-n+r)} - \frac{y_0^{1-n+r}}{(1-n+r)}y + \frac{y_0^{2-n+r}}{2-n+r} \tag{78}$$

satisfies

$$G^r_{0,y_0}(y) \leq C_{n,y_0}(y^{1+r/2} - y_0^{1+r/2})^2 \quad \forall \ \ 0 < y \tag{79}$$

$$C_{n,M,y_0}(y^{1+r/2} - y_0^{1+r/2})^2 \leq G^r_{0,y_0}(y) \quad \forall \ \ 0 < y < M. \tag{80}$$

Again, the integration constants were chosen so that both $G^r_{0,y_0}(y)$ and its first derivative vanish at y_0. As in the case of lemma 14, the lemma has the geometric interpretation of sandwiching the graph of the convex function $G^r_{0,y_0}(y)$ between two 'nonlinear' parabolas.

We apply Lemma 16 and the Poincaré inequality to the regularized entropy

$$G^r_{\epsilon,y_0}(y) = G^r_{0,y_0}(y) + \frac{\epsilon}{6y^{2-r}}. \tag{81}$$

From section 4 we have (60)

$$\frac{d}{dt}\int_{S^1} G^r_{\epsilon,y_0}(h_\epsilon(x,t))\,dx \le -C_r\int (h_\epsilon^{1+r/2}(x,t))^2_{xx}\,dx, \tag{82}$$

where C_r tends to zero as r tends to 1.

We apply Poincaré's inequality to the function $h_\epsilon^{1+r/2}$ at a point x_0 where $h_\epsilon^{1+r/2}$ equals $(\overline{h}+\delta(\epsilon))^{1+r/2}$. This and (79) imply

$$\int_{S^1} G^r_{\epsilon,\overline{h}+\delta}(\overline{h}(x,T))\,dx - \int_{S^1} G^r_{\epsilon,\overline{h}+\delta}(h_\epsilon(x,0))\,dx \le -c\int_0^T\int_{S^1} G^r_{0,(\overline{h}+\delta)^{1+r/2}}(h_\epsilon(x,t))\,dxdt.$$

Again, define

$$B_\epsilon = \sup_{t\in[0,T]}\left|\int_{S^1}\frac{\epsilon}{6h_\epsilon(x,t)^{2-r}}\,dx\right|.$$

Gronwall's lemma, (80), and the a priori boundedness of the entropy of the initial data imply

$$C_{n,M,(\overline{h}+\delta)^{1+r/2}}||h(\cdot,t)^{1+r/2}-(\overline{h}+\delta)^{1+r/2}||^2_{L^2} \le \int_{S^1} G^r_{0,\overline{h}+\delta}(h_\epsilon(x,t))\,dx \le B_\epsilon + Ae^{-ct}.$$

Furthermore, $B_\epsilon \to 0$ as $\epsilon\to 0$. Indeed, for any $\tilde{r}$ satisfying the conditions of the lemma, $0<\tilde{r}<1$, $0<2-n-\tilde{r}<1$, we have the uniform bound

$$\int_{S^1}\frac{\epsilon}{h_\epsilon(x,t)^{2-\tilde{r}}}dx < C$$

from the $G^{\tilde{r}}_{\epsilon,\overline{h}+\delta}$ entropy dissipation. In particular, we choose $\tilde{r}<r$ satisfying these conditions. Hölder's inequality then implies that

$$\int_{S^1}\frac{\epsilon}{h_\epsilon^{2-r}} \le C\left(\int_{S^1}\frac{\epsilon}{h_\epsilon^{2-\tilde{r}}}\right)^{\frac{2-r}{2-\tilde{r}}}\epsilon^{\frac{r-\tilde{r}}{2-r}} \le C\epsilon^{\frac{r-\tilde{r}}{2-r}}.$$

Taking $\epsilon\to 0$,

$$||h(\cdot,t)^{1+r/2}-\overline{h}^{1+r/2}||_{L^2} \le Ae^{-ct}.$$

The interpolation inequality then gives an L^∞ bound for $h(\cdot,t)^{1+r/2}-\overline{h}^{1+r/2}$ which implies an L^∞ bound for $h(\cdot,t)-\overline{h}$.

The rate of decay, c, decays to zero as r increases to 1. We note that as n increases to 3, the condition $0<2-n+r$ forces r to 1, resulting in a slower rate of decay. This is consistent with the fact that for the $n=3$ case, there are no nonnegative source-type or advancing front exact solutions.

6. Evolution from Positive Initial Data

The preceding parts of this paper consider general nonnegative initial data. For such data, we prove the existence of a weak solution for $0 < n < 3$ that becomes strong in finite time and approaches its mean in the L^∞ norm as $t \to \infty$.

In this section we show that for all $n > 0$, if the initial data strictly positive there exists a nonnegative (possibly weak) solution to the equation that approaches its mean in the infinite time limit.

With strictly positive initial data, there are two possible scenarios. In the first, h remains positive for all time and we have a global strong solution. Bertozzi *et al* showed in [6] that this is precisely what happens for $n \geq 3.5$. The second possibility is that the solution is initially positive but "touches down" somewhere in finite time. Indeed, for sufficiently small n (e.g. $0 < n < 1$) numerical evidence shows that there is initial data that yields such behavior [4]. For such solutions, the following theorem proves the existence of a nonnegative weak continuation of the solution past the singularity time t_c, moreover, after a second critical time, T^*, the solution becomes positive again and converges to its mean.

We have the following result

Theorem 17. *Let $0 < m \leq h_0 \leq M$, $h_0 \in H^1(S^1)$, $n > 0$ and $T > 0$. Given $0 \leq s < \frac{1}{2}$ and $n > \alpha \geq \frac{1}{2} - \frac{s}{4}$ then there exists a weak nonnegative solution to the lubrication approximation in the following sense of distributions*

$$\text{for } n > 1, \iint_{Q_T} h\phi_t dxdt - \iint_{Q_T} nh^{n-1} h_x h_{xx} \phi_x dxdt - \iint_{Q_T} h^n h_{xx} \phi_{xx} dxdt = 0,$$

$$\text{for } \frac{3}{8} < n \leq 1, \ \iint_{Q_T} h\phi_t dxdt - \iint_{Q_T} h^n h_{xx} \phi_{xx} dxdt - \iint_{Q_T} h^{n-\alpha} \left(\frac{h^\alpha}{\alpha}\right)_x h_{xx} \phi_x dxdt = 0$$

$$\text{for } 0 < n < \frac{3}{8}, \ \iint_{Q_T} h\phi_t dxdt + \iint_{P} h^n h_{xxx} \phi_x dxdt = 0$$

for all $\phi \in C_0^\infty(Q_T)$. In all cases the solution has the additional regularity

$$h^{1-s/2} \in L^2(0, T; H^2(S^1)), \tag{83}$$

$$(h^\alpha)_x \in L^4(Q_T). \tag{84}$$

Moreover, in all cases there exist positive A and c so that

$$\|h(\cdot, t) - \bar{h}\|_{L^\infty} \leq Ae^{-ct}. \tag{85}$$

A depends on M, m, $\overline{h}$, and n, and c depends on n and $\overline{h}$. In particular, there exists a critical time T^* after which the solution is strong and positive.

Remarks: Since the initial data is strictly positive the solution will at least exist as a positive smooth solution on a finite time interval $[0, t_c)$. For the case $n = 0$, there exists a unique strong solution for all time that approaches its mean in the infinite time limit.

---

the proof in [6] is an extension of the basic entropy argument used in [1]

However, for this linear equation, as the example in Section 2 shows, the solution does not necessarily preserve positivity. We remind the reader that for $n > 0$ uniqueness is not known, even for nonnegative solutions. For this reason, we cannot exclude the possibility that at the singularity time t_c, there might be more than one way to continue the solution.

We note that the lower bound m on h_0 implies that for any n, the initial entropy $\int_{S^1} G_0(h_0)$ is finite. As was proved in [1], if $\int_{S^1} G_0(h_0) < \infty$ and $n \geq 2$, then at any time t, $h(\cdot, t)$ can only vanish on a set of measure zero. This result can be refined to $n \geq 3/2$ by using the entropy (45). Also, as in [1] the theorem is true with the slightly milder condition of *entropy bounded* initial data.

To prove theorem 17 we consider the regularization

$$h_t = -(f_\epsilon(h) h_{xxx})_x, \tag{86}$$

$$f_\epsilon(h) = \frac{h^{n+4}}{\epsilon h^n + h^4}. \tag{87}$$

Since $0 < m \leq h_0$, we do not lift the initial data. The bound $\int_{S^1} G_0(h_0) < \infty$ allows us to apply the arguments from section 4 to prove the existence result. Indeed, for the case $n \geq 1$, we apply the argument used to prove Proposition 4 for nonnegative data with $(1 < n < 2)$. Furthermore, the solutions inherit the higher regularity proved for the solutions in section 4, and the long time behavior has already been proven for $1 \leq n < 2$. For the case $0 < n < 1$ both the existence and long time behavior follow as in the proof of Theorem 2.

We need to prove the longtime behavior for $2 \leq n < 3.5$. The proof of the longtime behavior (24) in Theorem 1 would suffice for $2 \leq n < 3$, but the rate of decay decreases to zero as n approaches 3. We show in this section that if there is an a priori bound for $\int_{S^1} G_0(h_0)$, this is an artificial effect.

Recall that the basic entropy

$$G_{\epsilon, y_0}(y) = \frac{\epsilon}{6y^2} + G_{0,y_0}(y) \tag{88}$$

satisfies

$$\frac{d}{dt} \int_{S^1} G_{\epsilon, \overline{h}}(h_\epsilon(x,t))\, dx = - \int_{S^1} h_{\epsilon xx}^{\,2}(x,t)\, dx.$$

We now prove a variant of Lemma 14.

Lemma 18. *Given $n \geq 2$ and $0 < y_0 < M$, there exists positive constants C_{n,y_0} and C_{n,M,y_0} so that the basic entropy*

$$G_{0,y_0}(y) = \begin{cases} \frac{1}{(1-n)(2-n)} y^{2-n} - \frac{y_0^{1-n}}{(1-n)} y + \frac{y_0^{2-n}}{2-n} & n \neq 1 \\ y \log y - (1 + \log y_0) y + y_0 & n = 1 \end{cases} \tag{89}$$

satisfies

$$C_{n,M,y_0} (y - y_0)^2 \leq G_{0,y_0}(y) \quad \forall\ 0 < y \leq M \tag{90}$$

$$G_{0,y_0}(y) \leq C_{n,y_0} (y - y_0)^2 \quad \forall\ y_0/2 < y. \tag{91}$$

The geometric interpretation of this lemma is that for any fixed $0 < y_0 < M$, the graph of $G_{0,y_0}(y)$ can be bounded above by a parabola on $y_0/2 < y$ and below by a parabola on $0 < y < M$, where both parabolas have the vertex $(y_0, 0)$. The bound from above does not hold on the full interval $0 < y$ since for $n \geq 2$, $G_{0,y_0}(y)$ blows up at $y = 0$. However in the next lemma we show that an a priori bound on $\int_{S^1} G_{0,\overline{h}}(h_0)$ allows us to prove $\int_{S^1} G_{0,\overline{h}}(h) \sim \int_{S^1}(h(x)-\overline{h})^2\,dx$, which is all that is needed for the long time behavior.

Lemma 19. *Given $0 < y_0 < M$ and $0 \leq h < M$ $h \in C^{1/2}(S^1)$ with finite entropy*

$$\int G_{0,y_0}(h) \leq C_{G(y_0)},$$

there exist c_1 and c_2 depending only on y_0, M, the $C^{1/2}$ norm of h, and $C_{G(y_0)}$ so that

$$c_1\|h(\cdot) - y_0\|^2_{L^2(S^1)} \leq \int G_{0,y_0}(h) \leq c_2\|h(\cdot) - y_0\|^2_{L^2(S^1)}.$$

Proof: The lower bound follows directly from the previous lemma. To compute the upper bound note that the interpolation Lemma 15 implies

$$\|h(\cdot) - y_0\|_{L^\infty} \leq C\|h(\cdot) - y_0\|^{1/3}_{L^1},$$

where C depends only on $|h|_{C^{1/2}}$, M, y_0, and $|S^1|$. Hölder's inequality implies then

$$\|h(\cdot) - y_0\|_{L^\infty} \leq C_1\|h(\cdot) - y_0\|^{1/3}_{L^2}.$$

case 1: $\|h(\cdot) - y_0\|^{1/3}_{L^2} \leq y_0/2C_1$
For this case, lemma 15 implies directly that $|h(\cdot) - y_0|_{L^\infty(S^1)} \leq y_0/2$ and hence $y_0/2 \leq h(x)$ for all $x \in S^1$. Lemma 18 then applies, implying

$$\int G_{0,y_0}(h) \leq C_{n,y_0}\|h(\cdot) - y_0\|^2_{L^2(S^1)}.$$

case 2: $\|h - s_0\|^{1/3}_{L^2} > s_0/2C_1$
For this case we have trivially

$$\int G_{0,y_0}(h) \leq \frac{64C_{G(y_0)}C_1^6}{y_0^6}\|h(\cdot) - y_0\|^2_{L^2}. \quad \square$$

The rest of the proof of long time behavior follows exactly as in section 5.

7. Conclusions

This paper presents new results for several classes of weak nonnegative solutions for different values of n for the degenerate diffusion problem $h_t + (|h|^n h_{xxx})_x = 0$. We consider the problem on a bounded domain with periodic boundary conditions.

The main features of these nonnegative weak solutions is that with nonnegative initial data, for $0 < n < 3$ they approach their mean in the limit as $t \to \infty$. For $n > 0$, with positive initial data this result also holds true. Moreover, the solutions have a greater regularity than the exact solutions with 'finite contact angle' discussed in section 1. It is significant that this regularity just includes the known 'source type' similarity solutions with 'zero contact angle'. To prove these results we introduce new families of diffusive entropies. Using the convexity of the entropy functions, we relate entropy dissipation to relaxation of the solution to its mean. We conjecture that well-posedness results exist within this more restricted regularity class.

There are many unsolved problems for such higher order degenerate diffusion equations. For example, what stronger results can one obtain for nonnegative weak solutions when $n \geq 3$? There are no nonnegative source type solutions in this range. There are, however, the steady parabola solutions. We conjecture that there there is a regularity class for $n \geq 3$ for which distribution solutions exist but do not have increasing support. Another unsolved problem is the question of uniqueness for the weak solutions for $n > 0$. This is a difficult problem due to the lack of a comparison principle.

We address some of these issues as well as a comparison of these solutions to the physical problem of droplet spreading in a companion paper [5]. In that work we exhibit numerical computation of the weak solutions via the regularization scheme used here. In particular the numerics show that the approximate solutions are converging to a solution with support that has finite speed of propagation. Even more interesting is that, before the support covers the entire domain, the solution converges rapidly onto the similarity solution (9-11).

There are also a number of questions associated with finite time singularities. For a detailed discussion we refer the reader to [6, 4]. There are very few rigorous results for the singularity problem. However, numerics show that for n sufficiently small, even without forcing, finite time singularities occur. Section 6 of this paper proves that one can continue the solution after the singularity time as a nonnegative weak solution and that moreover there exists a time after which the solution becomes a positive strong solution.

Acknowledgements. We thank Peter Constantin for suggesting the inclusion of Section 6 and for useful conversations concerning the theory. We thank Michael Brenner for useful conversations regarding the subtle physical problem of moving contact lines. We thank Francisco Bernis for pointing out that the very weak existence result in Theorem 2 follows from the method in [1]. We credit Leo Kadanoff with originally discovering the entropy (45) used to prove lack of finite time singularities for $n \geq 3.5$. We also thank him for useful conversations and general encouragement. We also thank Todd Dupont. AB also thanks C. Kenig and P. Olla. MP also thanks M. Shelley.

Both authors are supported by NSF postdoctoral fellowships. AB is also partially supported by the DOE and MP by NSF grant # DMS-9305996.

8. Appendix

8.1. Interpolation Lemma.

Lemma 20. *(interpolation inequality)*

A review article addressing some of these issues is [3].

Let $w \in L^1(S^1) \cap C^\alpha(S^1)$, $0 < \alpha < 1$. Let $|\cdot|_\alpha$ denote the Hölder$-\alpha$ seminorm. Then

$$|w|_{L^\infty} \leq (\frac{1+\alpha}{\alpha})^{\alpha/(1+\alpha)} |w|_\alpha^{1/(1+\alpha)} |w|_{L^1}^{\alpha/(1+\alpha)} + (\frac{1+\alpha}{\alpha})|w|_{L^1}/|S^1|. \tag{92}$$

Proof. Since S^1 is compact, there exists x_0 so that $|w(x_0)| = |w|_{L^\infty}$. For such a point

$$|w|_{L^\infty} - w(x) = |w(x_0) - w(x)| \leq |w|_\alpha |x - x_0|^\alpha.$$

Hence

$$w(x) \geq |w|_{L^\infty} - |w|_\alpha |x - x_0|^\alpha.$$

Note that

$$|w|_{L^\infty} - |w|_\alpha |x - x_0|^\alpha \geq 0 \qquad \text{for } \; |x - x_0| < (\frac{|w|_{L^\infty}}{|w|_\alpha})^{1/\alpha}.$$

Let

$$b = \min\big[(\frac{|w|_{L^\infty}}{|w|_\alpha})^{1/\alpha}, |S^1|\big].$$

Without loss of generality $x_0 = 0$. Hence

$$|w|_{L^1} \geq \int_{0<x<b} (|w|_{L^\infty} - |w|_\alpha |x|^\alpha) dx \tag{93}$$

$$= b(|w|_{L^\infty} - \frac{|w|_\alpha b^\alpha}{1+\alpha}). \tag{94}$$

Since $b \leq \left(\frac{|w|_{L^\infty}}{|w|_\alpha}\right)^{1/\alpha}$

$$|w|_{L^\infty} - \frac{|w|_\alpha b^\alpha}{1+\alpha} \geq |w|_{L^\infty}(\frac{\alpha}{1+\alpha}).$$

Plugging this into (94) gives

$$|w|_{L^\infty} \leq (\frac{1+\alpha}{\alpha})\frac{|w|_{L^1}}{b}.$$

The definition of b then gives the desired result. □

8.2. Convergence of f_ϵ and its Derivatives.

Lemma 21. *(Uniform convergence of f_ϵ for $n > 0$) Let*

$$f_\epsilon(y) = \frac{y^{n+4}}{\epsilon y^n + y^4}.$$

Then for $n > 0$, f_ϵ converges uniformly on $[0, M]$ to y^n as $\epsilon \to 0$.

Proof. In the following, we assume $\epsilon < 1$.

$$f_\epsilon(y) - y^n = -\frac{\epsilon y^{2n}}{\epsilon y^n + y^4}.$$

Fix ϵ and $y \neq 0$. Take b with $y = \epsilon^b$. Therefore

$$|f_\epsilon(y) - y^n| = \frac{\epsilon^{2nb+1}}{\epsilon^{nb+1} + \epsilon^{4b}}.$$

Assume $n < 4$.

case 1: $b \geq \frac{1}{4-n}$

$$|f_\epsilon(y) - y^n| \leq \epsilon^{2nb+1-nb-1} = \epsilon^{\frac{n}{4-n}} \epsilon^{nb-\frac{n}{4-n}} \leq \epsilon^{\frac{n}{4-n}}.$$

In the above, we used that $nb - \frac{n}{4-n} \geq 0$ for $b \geq \frac{1}{4-n}$.

case 2: $b < \frac{1}{4-n}$

$$|f_\epsilon(y) - y^n| \leq \epsilon^{2nb+1-4b} = \epsilon^{\frac{n}{4-n}} \epsilon^{2nb+1-4b-\frac{n}{4-n}} \leq \epsilon^{\frac{n}{4-n}}.$$

In the above we used that $2nb + 1 - 4b - \frac{n}{4-n} > 0$ for $b < \frac{1}{4-n}$.

This proves that for $0 < n < 4$, $f_\epsilon(y) - y^n \leq \epsilon^{\frac{n}{4-n}}$.

For $n \geq 4$, we observe that since $2n - 4 > 0$,

$$|f_\epsilon(y) - y^n| \leq \epsilon y^{2n-4} \leq C\epsilon$$

proving uniform convergence on $[0, M]$. $\square$

Lemma 22. *(Uniform convergence of f'_ϵ for $n > 1$) Let*

$$f_\epsilon(y) = \frac{y^{n+4}}{\epsilon y^n + y^4}.$$

Then for $n > 1$, f'_ϵ converges uniformly on $[0, M]$ to ny^{n-1} as $\epsilon \to 0$.

Proof. In the following, we assume $\epsilon < 1$.

$$f'_\epsilon(h) = \frac{ny^{n+7}}{(\epsilon y^n + y^4)^2} + \frac{4\epsilon y^{2n+3}}{(\epsilon y^n + y^4)^2} = T_1 + T_2.$$

For $n < 5/2$, T_2 has a maximum determined by

$$0 = T'_2(y) = 4\frac{3\epsilon^2 y^{3n+2} + \epsilon(2n-5)y^{2n+6}}{(\epsilon y^n + y^4)^3}$$

which occurs at $y = \left(\frac{3\epsilon}{5-2n}\right)^{\frac{1}{4-n}}$. Hence for $n < 5/2$

$$|T_2(y)| \leq C\epsilon^{\frac{n-1}{4-n}}.$$

For $n \geq 5/2$, $2n + 3 \geq 8$. Recalling that $0 \leq y \leq M$, $|T_2(y)| \leq C\epsilon y^{2n+3-8} \leq C\epsilon$. This proves that for $n > 1$, T_2 converges uniformly to zero.

We now prove that $T_1 - ny^{n-1}$ converges unformly to zero.

$$T_1 - ny^{n-1} = ny^{n-1}\Big[\frac{-2\epsilon y^{4+n} - \epsilon^2 y^{2n}}{(\epsilon y^n + y^4)^2}\Big].$$

Note that the term inside $[\cdot]$ is bounded independently of y and ϵ. Hence for $y < \epsilon^{1/4}$

$$|T_1 - ny^{n-1}| \leq C\epsilon^{(n-1)/4}.$$

For $y \geq \epsilon^{1/4}$ and $n < 4$,

$$\left| ny^{n-1}\left(\frac{-2\epsilon y^{4+n}}{(\epsilon y^n + y^4)^2}\right)\right| \leq Cy^{n-1}\frac{\epsilon y^{4+n}}{y^8}$$

Recalling $y^8 = y^{4+n}y^{4-n} \geq y^{n+4}\epsilon^{\frac{4-n}{4}}$,

$$\leq Cy^{n-1}\epsilon\epsilon^{\frac{n-4}{4}} \leq C\epsilon^{\frac{n}{4}}.$$

Similarly,

$$\left| ny^{n-1}\left(\frac{-\epsilon^2 y^{2n}}{(\epsilon y^n + y^4)^2}\right)\right| \leq C\epsilon^{\frac{2n}{4}}.$$

Assuming $\epsilon < 1$, for $n < 4$ and $y \geq \epsilon^{1/4}$,

$$|T_1(y) - ny^{n-1}| \leq C\epsilon^{\frac{n}{4}}.$$

If $n \geq 4$,

$$|T_1(y) - ny^{n-1}| \leq C\epsilon.$$

Therefore, for $1 < n < 4$,

$$|f_\epsilon'(y) - ny^{n-1}| \leq C\epsilon^{\frac{n-1}{4}}.$$

For $n \geq 4$, $|f_\epsilon'(y) - ny^{n-1}| \leq C\epsilon$, proving that for $n > 1$, f_ϵ' converges uniformly on $[0, M]$ to ny^{n-1}. □

Lemma 23. *(Uniform convergence of f_ϵ'' for $n > 2$) Let*

$$f_\epsilon(y) = \frac{y^{n+4}}{\epsilon y^n + y^4}.$$

Then for $n > 2$, f_ϵ'' converges uniformly on $[0, M]$ to $n(n-1)y^{n-2}$ as $\epsilon \to 0$.

Proof. In the following, we assume $\epsilon < 1$.

$$\begin{aligned} f_\epsilon''(h) &= \frac{n(n-1)y^{n+10}}{(\epsilon y^n + y^4)^3} + \frac{\epsilon(-20 + 15n - n^2)y^{2n+6}}{(\epsilon y^n + y^4)^3} + \frac{12y^2y^{3n+2}}{(\epsilon y^n + y^4)^3} \\ &= T_1 + T_2 + T_3. \end{aligned}$$

For $n < 3$, T_2 has a maximum determined by

$$0 = T_2'(y) = \frac{\epsilon(20 - 15n + n^2)y^{5+2n}((6-2n)y^4 + (\epsilon n - 6\epsilon)y^n)}{(\epsilon y^n + y^4)^4}$$

which occurs at $y = \left(\frac{\epsilon(6-n)}{6-2n}\right)^{\frac{1}{4-n}}$. Hence for $n < 3$

$$T_2(y) \leq C\epsilon^{\frac{n-2}{4-n}}.$$

For $n \geq 3$, $2n + 6 \geq 12$. Recalling that $0 \leq y \leq M$, we find $T_2(y) \leq C\epsilon y^{2n+6-12} \leq C\epsilon$.

This shows that for $n > 1$ T_2 converges uniformly to zero.

For $n < \frac{10}{3}$, T_3 has a maximum determined by

$$0 = T_3'(y) = 12\epsilon^2 \frac{(3n-10)y^{3n+5} + 2\epsilon y^{1+4n}}{(\epsilon y^n + e^4)^4}$$

which occurs at $y = \left(\frac{2\epsilon}{10-3n}\right)^{\frac{1}{4-n}}$. Hence for $n < \frac{10}{3}$

$$T_3(y) \leq C\epsilon^{\frac{n-2}{4-n}}.$$

For $n \geq \frac{10}{3}$, $3n+2 \geq 12$. Recalling that $0 \leq y \leq M$, we find $T_3(y) \leq C\epsilon y^{3n+2-12} \leq C\epsilon$.

This shows that for $n > 1$ T_3 converges uniformly to zero.

We now show that $T_1 - n(n-1)y^{n-2}$ converges unformly to zero.

$$\begin{aligned} T_1 - n(n-1)y^{n-2} &= n(n-1)y^{n-2}\left(-\frac{3\epsilon y^{n+8}}{(\epsilon y^n + y^4)^3} - \frac{\epsilon^3 y^{3n}}{(\epsilon y^n + y^4)^3} - \frac{3\epsilon^2 y^{2n+4}}{(\epsilon y^n + y^4)^3}\right) \\ &= n(n-1)y^{n-2}(T_{11} + T_{12} + T_{13}). \end{aligned}$$

Note that $T_{11} + T_{12} + T_{13}$ is bounded independently of y and ϵ. Hence

$$|T_1(y) - n(n-1)y^{n-2}| \leq C\epsilon^{\frac{n-2}{4}} \qquad \text{for } y \leq \epsilon^{\frac{1}{4}}.$$

Now assume $n < 4$ and $y > \epsilon^{\frac{1}{4}}$

$$T_{11}(y) \leq n(n-1)y^{n-2}\frac{3\epsilon y^{n+8}}{y^{12}}.$$

Recalling $y^{12} = y^{n+8}y^{4-n} > y^{n+8}\epsilon^{(4-n)/4}$,

$$\begin{aligned} &< n(n-1)y^{n-2}\epsilon\epsilon^{\frac{n-4}{4}} \\ &= n(n-1)y^{n-2}\epsilon^{\frac{n}{4}} \leq C_1\epsilon^{\frac{n}{4}}. \end{aligned}$$

Similarly,

$$T_{12}(y) \leq C_2\epsilon^{\frac{3n}{4}} \qquad \text{and} \qquad T_{13}(y) \leq C_3\epsilon^{\frac{2n}{4}}.$$

For $n < 4$ and $y > \epsilon^{1/4}$, this proves

$$|T_1(y) - n(n-1)y^{n-2}| \leq C\epsilon^{\frac{n}{4}}.$$

If $n \geq 4$,

$$\begin{aligned} |T_{11}| &\leq C\epsilon y^{(n+8)-12} \leq C\epsilon \\ |T_{12}| &\leq C\epsilon^3 y^{3n-12} \leq C\epsilon^3 \leq C\epsilon \\ |T_{13}| &\leq C\epsilon^2 y^{(2n+4)-12} \leq C\epsilon^2 \leq C\epsilon. \end{aligned}$$

Therefore, for $2 < n < 4$,

$$|f_\epsilon''(y) - n(n-1)y^{n-2}| \leq |T_1(y) - n(n-1)y^{n-2}| + |T_2(y)| + |T_3(y)| \leq C\epsilon^{\frac{n-2}{4}}.$$

For $4 \leq n$, $|f_\epsilon''(y) - n(n-1)y^{n-2}| \leq C\epsilon$ proving that for $n > 2$ $f_\epsilon''(y)$ converges uniformly on $[0, M]$ to $n(n-1)y^{n-2}$. $\square$

Lemma 24. *(Uniform convergence of $y^a f_n'(y)$) Let*

$$f_n(y) = \frac{y^{n+4}}{\epsilon y^n + y^4}.$$

Then for $1-n < a$, $y^a f_n'(y)$ converges uniformly on $[0, M]$ to ny^{a+n-1} as $\epsilon \to 0$.

Proof. Without loss of generality $a < min(4, 5-2n)$. In the following, we assume $\epsilon < 1$

$$f_n'(y) = \frac{ny^{n+7} + 4\epsilon y^{2n+3}}{(\epsilon y^n + y^4)^2}.$$

$$\text{Hence } \; y^a(f_n'(y) - ny^{n-1}) = \frac{\epsilon(4-2n)y^{2n+a+3} - \epsilon^2 y^{3n+a-1}}{(\epsilon y^n + y^4)^2}.$$

Fiy ϵ and $y \neq 0$. Take b with $y = \epsilon^b$. Therefore

$$\left|y^a(f_n'(y) - ny^{n-1})\right| = \left|\frac{(4-2n)\epsilon^{1+2nb+ab+3b} - n\epsilon^{3nb+ab-b+2}}{(\epsilon^{nb+1} + \epsilon^{4b})^2}\right|.$$

We start by noting that since $1-n < a$, we can take $\lambda > 0$ so that we also have $1-n < (1-\lambda)a$.

<u>case 1:</u> $b \geq \frac{1}{4-n}$

$$\begin{aligned} \left|y^a(f_1'(y) - ny^{n-1})\right| &\leq (4+3n)\frac{\epsilon^{3nb+ab-b+2}}{(\epsilon^{nb+1} + \epsilon^{4b})^2} \\ &\leq (4+3n)\frac{\epsilon^{3nb+ab-b+2}}{\epsilon^{2nb+2}} = (4+3n)\epsilon^{nb+ab-b} \\ &= (4+3n)\epsilon^{\frac{\lambda a}{4-n}}\epsilon^{nb+ab-b-\frac{\lambda a}{4-n}} \leq (4+3n)\epsilon^{\frac{\lambda a}{4-n}}. \end{aligned}$$

In the above, we used that $nb + ab - b - \frac{\lambda a}{4-n} \geq 0$ for $b \geq \frac{\lambda a}{n+a-1}\frac{1}{4-n}$, and that λ was chosen so that $\frac{\lambda a}{n+a-1} < 1$, hence $\epsilon^{nb+ab-b-\frac{\lambda a}{4-n}} \leq 1$ for all $b \geq \frac{1}{4-n}$.

<u>case 2:</u> $b < \frac{1}{4-n}$

$$\begin{aligned} \left|y^a(f_n'(y) - ny^{n-1}\right| &\leq (4+3n)\frac{\epsilon^{2nb+3b+ab+1}}{(\epsilon^{nb+1} + \epsilon^{4b})^2} \\ &\leq (4+3n)\frac{\epsilon^{2nb+3b+ab+1}}{\epsilon^{8b}} = (4+3n)\epsilon^{2nb+ab-5b+1} \\ &\leq (4+3n)\epsilon^{\frac{\lambda a}{4-n}}\epsilon^{2nb+ab+1-5b-\frac{\lambda a}{4-n}} \\ &< (4+3n)\epsilon^{\frac{\lambda a}{4-n}}. \end{aligned}$$

In the above we used that $2nb + ab + 1 - 5b - \frac{\lambda a}{4-n} \geq 0$ for $b \leq \frac{\lambda a - 4 + n}{2n+a-5}\frac{1}{4-n}$, and that λ was chosen so that $\frac{\lambda a-4+n}{2n+a-5} \geq 1$, hence $\epsilon^{2nb+ab+1-5b-\frac{\lambda a}{4-n}} \leq 1$ for all $b \leq \frac{1}{4-n}$. $\square$

Lemma 25. *(Uniform convergence of $y^a f_2''(y)$) Let*

$$f_2(y) = \frac{y^6}{\epsilon y^2 + y^4}$$

Then for $0 < a < 4$, $y^a f_2''$ converges uniformly to $2y^a$ as $\epsilon \to 0$.

Proof. Again, the upper bound $a < 4$ is simply to provide a shorter proof. In the following, we assume $\epsilon < 1$.

$$f_2''(y) = \frac{2y^2(6\epsilon^2 + 3\epsilon y^2 + y^4)}{(\epsilon + y^2)^3}$$

$$\text{Hence} \quad y^a f_2''(y) - 2y^a = \frac{2\epsilon^2 y^a(-\epsilon + 3y^2)}{(\epsilon + y^2)^3}.$$

Fix ϵ and $y \neq 0$. Take b with $y = \epsilon^b$. Therefore

$$|y^a f_2''(y) - 2y^a| = \left| \frac{-2\epsilon^{ab+3} + 6\epsilon^{ab+2b+2}}{(\epsilon + \epsilon^{2b})^3} \right|.$$

case 1: $b \geq \frac{1}{2}$

$$\begin{aligned} |y^a f_2''(y) - 2y^a| &\leq 8\frac{\epsilon^{ab+3}}{(\epsilon + \epsilon^{2b})^3} \\ &\leq 8\frac{\epsilon^{ab+3}}{\epsilon^3} = \epsilon^{ab} \\ &= 8\epsilon^{\frac{a}{2}}\epsilon^{ab-\frac{a}{2}} \leq 8\epsilon^{\frac{a}{2}}. \end{aligned}$$

In the above, we used that $ab - \frac{a}{2} \geq 0$ for $b \geq \frac{1}{2}$, hence $e^{ab-\frac{a}{2}} \leq 1$.

case 2: $b < \frac{1}{2}$

$$\begin{aligned} |y^a f_2''(y) - 2y^a| &\leq 8\frac{\epsilon^{ab+2b+2}}{(\epsilon + \epsilon^{2b})^3} \\ &\leq 8\frac{\epsilon^{ab+2b+2}}{\epsilon^{6b}} = 8\epsilon^{ab-4b+2} \\ &= 8\epsilon^{\frac{a}{2}}\epsilon^{ab-4b-\frac{a}{2}+2} \leq 8\epsilon^{\frac{a}{2}}. \end{aligned}$$

In the above, we used that $ab - 4b - \frac{a}{2} + 2 > 0$ for $b < \frac{1}{2}$, hence $e^{ab-4b-\frac{a}{2}+2} < 1$.

This proves that for all y, $|y^a f_2''(y) - 2y^a| \leq C\epsilon^{\frac{a}{2}}$, implying uniform convergence. □

References

[1] F. Bernis and A. Friedman. Higher order nonlinear degenerate parabolic equations. *J. Diff. Equations*, 83:179–206, 1990.

[2] F. Bernis, L. A. Peletier, and S. M. Williams. Source type solutions of a fourth order nonlinear degenerate parabolic equation. *Nonlinear Analysis, Theory, Methods and Applications*, 18(3), 1992.

[3] Francisco Bernis. Nonlinear parabolic equations arising in semiconductor and viscous droplet models. In N. G. Lloyd, W.-M. Ni, and L. A. Peletier, editors, *nonlinear diffusion equations and their equilibrium states, 3*, pages 77–88. Springer-Verlag, Basel, 1992.
[4] A. Bertozzi. Self-similar singularities in fourth order degenerate diffusion equations. in preparation.
[5] A. L. Bertozzi and M. Pugh. in preparation, 1994.
[6] Andrea L. Bertozzi, Michael P. Brenner, Todd F. Dupont, and Leo P. Kadanoff. Singularities and similarities in interface flow. In L. Sirovich, editor, *Centennial Edition, Applied Math. Series*. Springer–Verlag, 1993.
[7] Stephanella Boatto, Leo Kadanoff, and Piero Olla. Travelling wave solutions to thin film equations. *Phys. Rev. E*, 48:4423, 1993.
[8] Michael Brenner and Andrea Bertozzi. Spreading of droplets on a solid surface. *Physical Review Letters*, 71(4):593–596, July 1993.
[9] Peter Constantin, Todd F. Dupont, Raymond E. Goldstein, Leo P. Kadanoff, Michael J. Shelley, and Su-Min Zhou. Droplet breakup in a model of the Hele-Shaw cell. *Physical Review E*, 47(6):4169–4181, June 1993.
[10] Todd F. Dupont, Raymond E. Goldstein, Leo P. Kadanoff, and Su-Min Zhou. Finite–time singularity formation in Hele Shaw systems. *Physical Review E*, 47(6):4182–4196, June 1993.
[11] E. B. Dussan V. The moving contact line: the slip boundary condition. *J. Fluid Mech.*, 77:665–684, 1986.
[12] E. B. Dussan V and S. Davis. On the motion of a fluid–fluid interface along a solid surface. *J. Fluid Mech.*, 65:71–95, 1974.
[13] Peter Ehrhard and Stephen H. Davis. Non-isothermal spreading of liquid drops on horizontal plates. *J. Fluid. Mech.*, 229:365–388, 1991.
[14] H. P. Greenspan. On the motion of a small viscous droplet that wets a surface. *J. Fluid Mech.*, 84:125–143, 1978.
[15] Patrick J. Haley and Michael J. Miksis. The effect of the contact line on droplet spreading. *J. Fluid Mech.*, 223:57–81, 1991.
[16] L. M. Hocking. Rival contact-angle models and the spreading of drops. *Journal of Fluid Mechanics*, 239:671–681, 1992.
[17] Chun Huh and L. E. Scriven. Hydrodynamic model of steady movement of a solid /liquid/fluid contact line. *J. Colloid Interface Sci.*, 35:85–101, 1971.
[18] J. L. Lions. *Quelque méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.
[19] P. Neogi and C. A. Miller. *J. Colloid Interface Sci.*, 92:338, 1984.
[20] L. A. Peletier. The porous media equation. In H. Amman et. al., editor, *Applications of nonlinear analysis in the physical sciences*, pages 229–241. Pitman, 1981.
[21] V. M. Starov. Spreading of droplets of nonvolatile liquids over a flat solid. *J. Colloid Interface Sci. USSR*, 45, 1983.
[22] S. Zhou. *Interface Dynamics: Bubble Growth and Droplet Breakup in the Hele–Shaw Cell*. PhD thesis, University of Chicago, 1992.

EMAIL: bertozzi@@math.uchicago.edu, pugh@@aloss.cims.nyu.edu

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

COURANT INSTITUTE, NEW YORK UNIVERSITY, NEW YORK, NY 10012