MAT267: 5th HW assignment. Due by 11:59pm on April 2.

1. Please do, but don't hand in question #1 on page 184 of Hirsch, Smale, & Devaney. For the phase portraits, feel free to use `http://www.bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html` although you should first plot the equilibrium points and the behaviour of the linearized systems at the equilibrium points (as long as the eigenvalues have nonzero real parts).

2. (10 pt) Please do (and hand in) question #3 on page 184 of Hirsch, Smale, & Devaney.

3. (15 pt) On pages 162-163 of Hirsch, Smale, & Devaney, there's an example of a nonlinear system $X' = F(X)$ that has an unstable fixed point and a stable periodic solution:

$$\begin{cases} x' = F_1(x,y) = \frac{x}{2} - y - \frac{x}{2}\left(x^2+y^2\right) \\ y' = F_2(x,y) = x + \frac{y}{2} - \frac{y}{2}\left(x^2+y^2\right) \end{cases}$$

The periodic solution is $(x(t), y(t)) = (\cos(t), \sin(t))$.

   (a) As described in the book, the system can be written in polar coordinates:

$$\begin{cases} r' &= f(r) = \frac{r}{2}(1-r^2) \\ \theta' &= 1 \end{cases}$$

   Solve the initial value problem with $r(0) = r_0$ and $\theta_0 = 0$. If $0 < r_0 < 1$, what does the solution do as $t \to \infty$? As $t \to -\infty$? If $1 < r_0$, what does the solution do as $t \to -\infty$? As t decreases from 0?

   (b) Linearize $r' = f(r)$ about its fixed points. Solve the linearized system

$$\begin{cases} u_1' &= f'(r_{ss}) u_1 \\ \theta' &= 1 \end{cases}$$

   for each fixed point where u_1 is the deviation from r_{ss}.

   (c) Find the variational equation along the periodic solution. That is, linearize about the periodic solution and find the linear system

$$U' = DF(X(t))\, U = A(t)\, U.$$

   Find the solution of the variational equation.

4. (15 pt) In section 8.2 of Hirsch, Smale, & Devaney, the authors consider the a nonlinear system $X' = F(X)$

$$\begin{cases} x' = F_1(X) = f(x,y) \\ y' = F_2(X) = g(x,y) \end{cases}$$

They assume that $F(\vec{0}) = \vec{0}$ (and hence $\vec{0}$ is a steady state solution) and that $DF(\vec{0})$ has two negative real eigenvalues $-\lambda < -\mu < 0$. They introduce a function $\mathcal{L}(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ and they show that if a point X is "close enough" to the fixed point then

$$h(x,y) := \nabla\mathcal{L}(X) \cdot F(X) < 0.$$

That is, $\mathcal{L}(X)$ is a Lyapunov function for the system $X' = F(X)$ for points that are "close enough" to the fixed point. (Note: they didn't use this language but this is what they were doing.)

(a) In their argument showing that $\mathcal{L}(X) \cdot F(X) < 0$, the authors had to address a term of the form $(\mu - \lambda)x^2$. This term is there because the authors used a Liapunov function that works great for the linear system

$$X' = \begin{pmatrix} -\mu & 0 \\ 0 & -\mu \end{pmatrix} X \quad \text{but not so great for} \quad X' = \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} X$$

Find a Lyapunov function $\widetilde{\mathcal{L}}(X)$ that avoids such a term, showing that the calculation proving $\widetilde{\mathcal{L}}(X) \cdot F(X) < 0$ for X "close enough to $\vec{0}$" goes through somewhat more cleanly.

(b) Actually, if you think carefully about what they authors proved with their Liapunov function, it follows that if X is close enough to $\vec{0}$ then

$$h(x, y) := \nabla\mathcal{L}(X) \cdot F(X) \leq -\frac{1}{2}\,\mu\,\mathcal{L}(X). \tag{1}$$

Prove that this means that if X_0 is close enough to $\vec{0}$ then the initial value problem $X' = F(X)$ with $X(0) = X_0$ will have solutions for all $t > 0$ and the solution converges to $\vec{0}$ as $t \to \infty$. (Hint: Compute $d/dt\ \mathcal{L}(X(t))$ and use this to find some function $\ell(t)$ such that $0 \leq \mathcal{L}(X(t)) \leq \ell(t)$ and $\ell(t) \to 0$ as $t \to \infty$.)

Is the 1/2 in (1) necessary or could it have been $-\alpha\,\mu\,\mathcal{L}(X)$ for some other value of α? If yes, what range of α would work?

(c) Show that for the Liapunov function you found in part (a), you can find a positive number ω so that

$$\nabla\widetilde{\mathcal{L}}(X) \cdot F(X) \leq -\omega\,\widetilde{\mathcal{L}}(X).$$

How does ω depend on λ and μ? Prove that this means that if X_0 is close enough to $\vec{0}$ then the initial value problem $X' = F(X)$ with $X(0) = X_0$ will have a solution for all $t > 0$ and the solution converges to $\vec{0}$ as $t \to \infty$.

(d) Let's go into $\mathbb{R}^n$. Imagine you have a nonlinear system $X' = F(X)$ so that $\vec{0}$ is a steady state and $DF(\vec{0})$ is diagonalizable and all of its eigenvalues are negative:

$$DF(\vec{0}) \sim \text{diag}(-\lambda_1, -\lambda_2, \ldots, -\lambda_n)$$

where $-\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_n < 0$. What Liapunov function, $\mathcal{L}(X)$, would you create in order to prove that all solutions that start close to $\vec{0}$ will exist for all $t > 0$ and will converge to $\vec{0}$ as $t \to \infty$? In proving this, you'll end up proving that for X sufficiently close to $\vec{0}$ you have

$$\nabla\mathcal{L}(X) \cdot F(X) \leq -\omega\,\mathcal{L}(X).$$

How does ω depend on λ_1, λ_2, $\ldots\lambda_n$? Does this dependence make sense to you?

5. (10 pt) For the case where $X' = F(X)$ in $\mathbb{R}^2$ and $F(\vec{0}) = \vec{0}$ and $DF(\vec{0})$ has complex conjugate eigenvalues with negative real parts ($\alpha \pm i\,\beta$ where $\alpha < 0$), the authors simply write, "It is straightforward to check that the same result holds..." (See towards the bottom of page 167.) Make this rigourous by choosing a Liapunov function $\mathcal{L}$ and doing the necessary calculations to find $\omega > 0$ so that

$$\nabla\mathcal{L}(X) \cdot F(X) \leq -\omega\,\mathcal{L}(X).$$

for X_0 "close enough" to $\vec{0}$. How does ω depend on α and β? Prove that this means that if X_0 is close enough to $\vec{0}$ then the initial value problem $X' = F(X)$ with $X(0) = X_0$ will have a solution for all $t > 0$ and the solution converges to $\vec{0}$ as $t \to \infty$.

6. (15 pt) For the case $X' = F(X)$ in $\mathbb{R}^2$ and $F(\vec{0}) = \vec{0}$ and $DF(\vec{0})$ has repeated negative eigenvalues and isn't diagonalizable, the authors point you to pages 70-71 to find a change of coordinates so that the new system $Y' = G(Y)$ has $G(\vec{0}) = \vec{0}$ and $DG(\vec{0})$ yielding the linearized system

$$DG(\vec{0}) = \begin{pmatrix} -\lambda & \epsilon \\ 0 & -\lambda \end{pmatrix}$$

where ϵ can be taken as small as you desire by choosing the change of coordinates accordingly. They then write that it follows that "the vector field points inside circles of sufficiently small radius."

Make all of this rigourous by understanding the change of coordinates, choosing a Liapunov function $\mathcal{L}$, computing $\nabla\mathcal{L}(Y) \cdot G(Y)$ and then showing $\nabla\mathcal{L}(Y) \cdot G(Y) \leq -\omega\,\mathcal{L}(Y)$ for some $\omega > 0$ if Y is "sufficiently close" to $\vec{0}$ and then proving that X_0 is close enough to $\vec{0}$ then the initial value problem $X' = F(X)$ with $X(0) = X_0$ will have a solution for all $t > 0$ and the solution converges to $\vec{0}$ as $t \to \infty$.

7. Not to hand in, just to think about it if you're curious

If

$$X' = \begin{pmatrix} -\lambda & 0 \\ 0 & \mu \end{pmatrix} X = AX$$

where $\lambda, \mu > 0$ then $\mathcal{L}(X) = \mu x^2 - \lambda y^2$ has $\nabla\mathcal{L}(X) \cdot AX = -2\,\mu\,\lambda\,(x^2 + y^2) \leq 0$. Draw level sets of $\mathcal{L}(X)$ and think about what this $\nabla\mathcal{L}(X) \cdot AX \leq 0$ means about solutions of $X' = AX$. Can you find $\omega > 0$ so that $\nabla\mathcal{L}(X) \cdot AX < -\omega\,\mathcal{L}(X)$? If you could show $\mathcal{L}(X(t)) \to 0$ as $t \to \infty$, what would this tell you about the solution? Must $X(t)$ converge at $t \to \infty$?