Mat457Y/Mat1000Y Homework, due Friday February 14, 2003

1. **Fun with Function Spaces!** First, verify that if $f \in L^p(\mathbb{R}^n)$ then all dilations and amplifications of f are in $L^p(\mathbb{R}^n)$. That is,

$$f_{\sigma,\mu}(x) := \mu f(\sigma x) \in L^p(\mathbb{R}^n) \qquad \forall \mu, \sigma > 0.$$

In the following, you will use this observation to test inequalities. It won't prove inequalities for you, but it will allow you to quickly check if you're being lied to when you're told that something's true.

a) If someone told you that there exists $C < \infty$ so that

$$||fg||_{L^1} \leq C||f||_{L^p}||g||_{L^q}$$

but they didn't tell you that $1/p + 1/q = 1$, then you could guess that $1/q = 1 - 1/p$ as follows. First fix $p \in (1, \infty)$ and assume that you have $f \in L^p$ and $g \in L^q$. Now look at

$$||f_{\sigma\mu} g_{\sigma\mu}||_{L^1} \leq C||f_{\sigma\mu}||_{L^p}||g_{\sigma\mu}||_{L^q}$$

and vary σ and μ. Prove that if $1/q \neq 1 - 1/p$ then no C will work.

b) Let u be a function with compact support for which ∇u exists and is continuous. There is a C that depends only on p and n such that

$$||u||_{L^{p^*}} \leq C||\nabla u||_{L^p}$$

for all such u. What must p^* equal? *This is the Gagliardo-Nirenberg Inequality. It reflects the intuitively sensible fact that if the gradient of u is a certain size then u cannot get "too big" if u started out being zero somewhere.*

c) For $0 < \alpha < 1$, we define the Hölder seminorm

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

If $n < p < \infty$ then there exists $C < \infty$ depending only on p and n such that

$$[u]_\alpha \leq C||\nabla u||_{L^p}.$$

What must α equal? *This is Morrey's Inequality. It tells you what types of cusps a graph can have if it's the graph of a $W^{1,p}$ function.*

d) Recall the Fourier Transform $\mathcal{F}$ defined pointwise by:

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{ix\cdot\xi}\, dx.$$

We'd like to know whether $\mathcal{F}$ is a continuous linear operator. It's certainly linear. And if we have a normed topological vector space, we just want to know if it is a bounded linear operator:

$$||\mathcal{F}f||_Y \leq C||f||_X$$

for all $f \in X$. If $X = L^p$ with $1 < p < \infty$ and $Y = L^q$, for what values of q can we prove $\mathcal{F}$ is not continuous? *Please assume the integrals in question converge — i.e.* $f \in L^p$ *and* $\mathcal{F}f \in L^q$.

2. A topological vector space X is a *locally compact Hausdorff* topological vector space if it is Hausdorff and if for every $x \in X$ there is a compact set K so that $x \in K^\circ$. Prove that if X is a locally compact Hausdorff space and $A \subset X$ is a residual set then A is dense in X. Conclude that X is not a meagre set.

3. Kolmogorov & Fomin, page 238, #4. *Hint: you'll need to use the Open Mapping Theorem...*

4. Kolmogorov & Fomin, page 239, #8.

5. Consider $C[0,1]$, the space of continuous real-valued functions on $[0,1]$ with the L^∞ norm. We define the Volterra integral operator on this space:

$$Vx(t) = \int_0^t x(s)\, ds.$$

Prove that $V : C[0,1] \to C[0,1]$ is a bounded linear operator. Now prove that the spectrum of V is a single point: $\{0\}$.