

Mat 1060 final exam, December 8, 2-5pm

Heat equation, Burger's equation, travelling waves. Consider the heat equation

$$u_t = \mu u_{xx}$$

and Burger's equation

$$\theta_t + \theta\theta_x = \mu\theta_{xx}$$

where $\mu > 0$.

a) Formally verify that if u is a solution of the heat equation then $\theta := -2\mu u_x/u$ is a solution of the viscous Burger's equation. This is the Cole-Hopf transform.

b) Can you find a reverse transformation? That is, given a solution θ of Burger's equation can you use it to construct a solution of the heat equation?

c) The inviscid Burger's equation ($\mu = 0$) can have solutions that shock in finite time. What would you guess about the viscous Burger's equation? (I.e. what wins? The nonlinearity or the viscosity?)

d) If $\mu > 0$ then Burgers equation has both diffusion and a nonlinearity and so maybe it has travelling wave solutions. Construct infinitely many travelling wave solutions. Are there any bounded ones?

Laplace equation, Poisson equation, gravity Consider a compactly supported mass distribution $\rho(\vec{x})$ where $\vec{x} \in \mathbb{R}^3$. The gravitational potential ϕ is the solution of

$$\Delta\phi = 4\pi G\rho.$$

a) Find an explicit solution for ϕ in terms of ρ .

b) The gravitational field is

$$\vec{F}(\vec{x}) = -\nabla\phi(\vec{x}).$$

This represents the gravitational force on a unit mass located at \vec{x} . Find an explicit expression for the gravitational field in terms of ρ .

c) Prove that the attraction $\vec{F}(\vec{\xi})$ exerted by Ω on a far away unit mass is approximately the same as if the total mass of Ω were concentrated at its centre of gravity \vec{x}_0 where

$$\vec{x}_0 \iiint_{\Omega} \mu(\vec{x}) d\vec{x} = \iiint_{\Omega} \vec{x}\mu(\vec{x}) d\vec{x}$$

d) Calculate the potential ϕ of a solid sphere Ω of radius a with centre at the origin and constant density μ . Specifically, find ϕ defined in *all* of \mathbb{R}^3 , not just outside the sphere. Your solution should decay at infinity and should be in $C^1(\mathbb{R}^3)$. Compute the gravitational field, \vec{F} , of your solution. Plot its magnitude as a function of the distance to the origin. Does your plot make sense to you?

Elastic waves, Fourier methods Consider the equations of linear elasticity

$$\begin{cases} u_{tt} = \mu\Delta u + (\mu + \lambda) [u_{xx} + v_{xy}] \\ v_{tt} = \mu\Delta v + (\mu + \lambda) [u_{xy} + v_{yy}] \end{cases}$$

where μ is the shear modulus and λ is Lamé's first parameter.

The unknowns, u and v , represent displacement from rest in the x and y directions. For example, if $u(x, y, t) > 0$ and $v(x, y, t) = 0$ then at the location (x, y) the material is being stretched in the x direction. (You saw this with the spring: if the rest length of the spring is L_0 we then found an ODE for the displacement from rest $L(t) - L_0$ where $L(t)$.)

a) Find the solution for initial data

$$\begin{cases} u(x, y, 0) = \sin(\xi_1 x) \\ u_t(x, y, 0) = 0 \\ v(x, y, 0) = 0 \\ v_t(x, y, 0) = 0 \end{cases}$$

b) Find the solution for initial data

$$\begin{cases} u(x, y, 0) = \sin(\xi_2 y) \\ u_t(x, y, 0) = 0 \\ v(x, y, 0) = 0 \\ v_t(x, y, 0) = 0 \end{cases}$$

c) Before deformation, each material point is located at (x, y) . After deformation, each material point is at $(U(x, y), V(x, y)) = (x + u(x, y), y + v(x, y))$. Find the Jacobian for this mapping. Thinking about silly putty or any other "reasonable" material, what should this Jacobian equal at every point in space?

d) Find the solution for initial data

$$\begin{cases} u(x, y, 0) = \sin(\xi_1 x + \xi_2 y) \\ u_t(x, y, 0) = 0 \\ v(x, y, 0) = -\sin(\xi_1 x + \xi_2 y) \\ v_t(x, y, 0) = 0 \end{cases}$$

What does your solution equal in the special case where $\xi_1 = \xi_2$?

e) Can you say something about elastic waves and the speed with which they propagate?

Heat equation, parabolic equations, maximum principles Let u be the solution of

$$\begin{cases} Lu := u_t - u_{xx} = 0 & \forall x \in \mathbb{R}, t \in [0, T] \\ u(x, 0) = f(x) \end{cases}$$

where f is continuous. Fix $h > 0$ and $k > 0$. Let Σ denote the lattice of points (x_n, t_m) where $x_n := nh$, $t_m := mk$, $n \in \mathbb{Z}$, and $m \in \mathbb{N}$. Let v be the solution on the lattice of

$$\begin{cases} \Lambda v := \frac{v(x_n, t_m+k) - v(x_n, t_m)}{k} - \frac{v(x_n+h, t_m) - 2v(x_n, t_m) + v(x_n-h, t_m)}{h^2} = 0 \\ v(x_n, 0) = f(x_n). \end{cases} \quad (1)$$

a) Introducing $\lambda := k/h^2$, use (1) to solve for $v(x_n, t_{m+1})$ in terms of $v(x_{n-1}, t_m)$, $v(x_n, t_m)$, and $v(x_{n+1}, t_m)$. Find the total heat at “time level” $m + 1$ and the total heat at the m th time level. Does your answer make sense?

b) Using your expression for $v(x_n, t_{m+1})$, find a relation between the l^∞ norm of $v(\cdot, t_{m+1})$ and the l^∞ norm of $v(\cdot, t_m)$. What does your expression become if h and k are chosen so that $\lambda \leq 1/2$? Using this, what result can you prove for the discretized heat equation if you assume $\lambda \leq 1/2$?

c) Show that for $\lambda = k/h^2 = 1/2$

$$v(x_n, t_m) = 2^{-m} \sum_{j=0}^m \binom{m}{j} f(x_{n-m+2j}) \quad (2)$$

and hence, by the binomial formula,

$$\sup_{\sigma} |v| \leq \sup_{\mathbb{R}} |f|.$$

d) Equation (2) gives the value of v at a point (x_n, t_m) on the lattice as a combination of values of f evaluated at some lattice points. Call these lattice points the “domain of dependence” for $v(x_n, t_m)$. How does the domain of dependence depend on h and k ? Assume you take h and k to zero while preserving $\lambda = 1/2$. What happens to the domain of dependence?

e) Let $0 < \lambda \leq 1/2$ and fix $(x_n, t_m) \in \Sigma$. Show that there is a finite C so that

$$|u(x_n, t_m) - v(x_n, t_m)| \leq Ch^2.$$

Hint: expand λu about (x_n, t_m) using Taylor’s theorem. Does C depend on the point (x_n, t_m) ? Show that if $\lambda = 1/6$ then there is a finite \tilde{C} so that

$$|u(x_n, t_m) - v(x_n, t_m)| \leq \tilde{C}h^4.$$

f) How can you use the results of part e) to prove a maximum principle for u ? Consider the heat equation with spatially dependent diffusivity

$$\begin{cases} u_t = (D(x)u_x)_x \\ u(x, 0) = f(x) \end{cases}.$$

Discretize this to create a new operator Λ on the lattice Σ . (Hint: $u_x(x_n, t_m) \approx u(x_{n+1}, t_m) - u(x_n, t_m)$.) Let v be the solution of the discretized equation on the lattice Σ . Solve for $v(x_n, t_{m+1})$ in terms of $v(x_{n+1}, t_m)$, $v(x_n, t_m)$, and $v(x_{n-1}, t_m)$. Assume that $0 < D(x) \leq D < \infty$ for all x . Can you prove something that will lead to a maximum principle for v in terms of its initial data?