Homework #1, due September 26 at the beginning of class.

Section 1.1: Please do problems 3, 4b, and 6b.

Section 1.2: Please do problems 3, 8, and 9.

In the following, you are asked to construct "solutions" to the Eikonal equation in various domains. What you will construct will be functions that satisfy the PDE at almost all points in the domain — the function may be discontinuous or continuous but not differentiable on a set of measure zero. And so your functions will not be classical solutions.

We did this for the inviscid Burger's equation with piecewise constant initial data: using the method of characteristics there were many ways to construct functions that would satisfy the PDE at all points away from a curve. If that curve was, in turn, correctly chosen then the constructed function would be a weak solution. We're not going to get into the question of defining weak solutions of the Eikonal equation — you're simply being asked to construct some functions that satisfy the Eikonal equation almost everywhere.

Section 1.3: The Eikonal Equation on the interval Consider the eikonal equation $|u_x| = 1$ on the interval $(0,1)$ with boundary data $u(0) = u(1) = 0$.

For a Cauchy problem in two dimensions, we specify data on some non-characteristic curve $\Gamma(s)$ and then use the characteristics to "propagate" the data off of the curve and into a neighbourhood of the curve. In this question, there is no curve $\Gamma(s)$; there are simply two points: $x = 0$ and $x = 1$ with data on them. You're trying to propagate that data "into" the interval $(0,1)$.

a) What are the characteristic ODE? What is the initial data? (NOTE: at each point on the boundary you have two possible sets of initial data.)

b) Take the initial data $p(0) = 1$ at the left-hand boundary point and $p(0) = -1$ at the right-hand boundary point. Explicitly solve the characteristic ODE. Now, given $X \in (0,1)$, we would like to define $u(X)$ by giving it the value carried to it by a characteristic emenating from the boundary. The problems is: you can reach X using characteristics from more than one boundary point. So let's make some choices. . . What solution $u(X)$ do you construct if you assign X to the characteristic starting at the nearest boundary point? What solution do you construct if you assign X to the characteristic starting at $x_0 = 0$ if $X < \alpha$ and to the characteristic starting at $x_0 = 1$ if $X \geq \alpha$? (Assume $\alpha \in (0,1)$ is fixed.) Graph your solutions. Is one solution "better" than the others? What does it correspond to geometrically?

c) What types of solutions would you have found if you had chosen the other options for $p(0)$ on the boundary?

d) In b), you constructed infinitely many solutions. All but one of them were discontinuous. Without using the method of characteristics, construct infinitely many continuous solutions.

Section 1.3: The Eikonal Equation on the strip Consider the eikonal equation $|\nabla u| = 1$ on the strip $U = 0 < x_1 < 1 \in \mathbb{R}^2$ with boundary data $u(0, x_2) = u(1, x_2) = 0$ for all $x_2 \in \mathbb{R}$. Repeat a), b), and d) above. *Again, this isn't a Cauchy problem. Instead, you're propagating data off of the boundary of the domain into the domain: off of $x_2 = 0, 1$ and into $x_2 \in (0, 1)$.*

Section 1.3: The Eikonal Equation on the disk Consider the eikonal equation $|\nabla u| = 1$ on the disk $U = \{x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$ with boundary data $u(x_1, x_2) = 0$ for all $(x_1, x_2) \in \partial U$.

a) What are the characteristic ODE? What is the initial data? (NOTE: at each point on the boundary you have four possible sets of initial data.)

b) Take the initial data $(p_1(0), p_2(0))$ to point "inwards". Explicitly solve the characteristic ODE. Now, given $X \in U$, note that you cannot uniquely determine a point on the boundary whose characteristic reaches X. So let's make some choices... What solution $u(X)$ do you construct if you use the characteristic starting at the nearest boundary point? What solution do you construct if you use the characteristic starting at points in the "southern hemisphere"? (That is starting at a point (x_1, x_2) where $x_2 \leq 0$.) Describe your solutions. Is one solution "better" than the others? What does it correspond to geometrically? The second solution, where in $\overline{U}$ is it discontinuous? *Again, this isn't a Cauchy problem. Instead, you're propagating data off of the boundary of the domain into the domain.*

c) Above, you constructed infinitely many solutions. All but one of them were discontinuous. Without using the method of characteristics, construct infinitely many continuous solutions.

Section 1.3: The Eikonal equation on other domains

a) What would you expect to get as the "preferred" solution of the Eikonal equation on an ellipse? Where would it fail to be differentiable?

b) What would you expect to get as the "preferred" solution of the Eikonal equation on a rectangle? Where would it fail to be differentiable?