Homework #3, due October 17 at the beginning of class.

1. Please do problem 3 in Section 2.3.

2. Please do problems 3, 4, and 6 in Section 3.1.

   For the problems from the book — there are hints in the back of the book. Please don't spend more than fifteen minutes on each problem before looking at the hints!

3. Consider the wave equation on the line with initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \forall t > 0, x \in \mathbb{R} \\ u(x,0) = g(x) & \forall x \in \mathbb{R} \\ u_t(x,0) = h(x) & \forall x \in \mathbb{R} \end{cases}$$

   Assume the initial data satisfies $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$.

   (a) Can you show that the solution depends continuously on the initial data? That is, if u_1 solves the IVP with g_1 and h_1 and u_2 solves the IVP with g_2 and h_2 then can you relate the norm of their difference at time t, $u_1(\cdot,t) - u_2(\cdot,t)$, to the difference of the norms of the initial data? Use the $C^2(\mathbb{R})$ norm:

$$\|w\|_{C^2(\mathbb{R})} := \max_{x\in\mathbb{R}} |w(x)| + \max_{x\in\mathbb{R}} |w'(x)| + \max_{x\in\mathbb{R}} |w''(x)|$$

   (b) If you fix the time t and take the initial data closer and closer to one another, does $u_1(\cdot,t) - u_2(\cdot,t) \to 0$?

   (c) What if you ask about *all* times?

4. **Transmitted and Reflected Waves in a Composite Bar**

   The wave equation can describe displacements from rest in a solid material. If $u(x,t)$ represents the displacement in the x direction in a uniform material that's axially loaded and if one assumes a linear relationship between stress and strain (a reasonable assumption for small displacements) then the displacement satisfies

$$u_{tt} = \frac{E}{\rho}\, u_{xx}$$

   where E is the Young's modulus and ρ is the density of the material.

   Consider an infinitely long composite bar. For $x < 0$ the density is ρ_1 and the Young's modulus is E_1. For $x > 0$ the density is ρ_2 and the Young's modulus is E_2. The material changes type at $x = 0$.

   Please search on "wave reflection at a medium boundary" at youtube. The top video should come from "bockphysics". Watch the video and observe the incident (incoming), reflected (bounced back) and transmitted (got through) waves.

(a) Verify that

$$u(x,t) = Ae^{i\omega(t \pm \frac{x}{c_1})}$$

solves the wave equation in material 1 if $c_1 = \sqrt{E_1/\rho_1}$. Using this observation, what is a corresponding *real-valued* solution of the wave equation?

(b) You want to understand how a rightward moving incident wave produces a leftward moving reflected wave and a rightward moving transmitted wave at the interface at $x = 0$. Let

$$u_1(x,t) = A_I e^{i\omega(t - \frac{x}{c_1})} + A_R e^{i\omega(t + \frac{x}{c_1})}$$

and

$$u_2(x,t) = A_T e^{i\omega(t - \frac{x}{c_2})}.$$

"A_I" is the amplitude of the *incident* wave, "A_R" is the amplitude of the *reflected* wave, and "A_T" the amplitude of the *transmitted* wave. Check that u_1 solves the wave equation in the material on the left and u_2 solves the wave equation in the material on the right. Make sure that you understand why the incident wave in u_1 is rightward moving and the reflected wave is leftward moving. Similarly, make sure you understand why the transmitted wave (u_2) is rightward moving.

(c) The displacement should be continuous at the interface between the materials:

$$u_1(0,t) = u_2(0,t).$$

Similarly, the stresses should be continuous at the interface:

$$E_1 \frac{\partial u_1}{\partial x}(0,t) = E_2 \frac{\partial u_2}{\partial x}(0,t).$$

Use these to find two equations involving A_I, A_R, and A_T. Solve the equations to find A_T and A_R in terms of A_I, c_1, ρ_1, c_2, and ρ_2.

(d) In terms of reflected and transmitted waves, the key quantities are the "mechanical impedances" $\rho_1 c_1$ and $\rho_2 c_2$. For what types of materials will there be no reflected wave? If the materials have been chosen to have no reflected wave, how is the magnitude of the transmitted wave (A_T) related to the magnitude of the incident wave (A_I)? Is it possible to create materials for which there is *no* transmitted wave?

If you're curious, google "impedance matching"!

5. Consider the wave equation $u_{tt} - c^2 \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}$. You would like to find out what dimensions n allow for distortionless radially symmetric wave propagation.

We say that distortionless radially symmetric wave propagation is possible if there are functions $\alpha(r) > 0$ and $\beta(r) > 0$ with $\alpha(1) = 1$ and $\beta(0) = 0$ such that

$$u(\vec{x},t) := \alpha(r)\, \phi(t - \beta(r))$$

solves the wave equation for every "reasonable" ϕ. Let's say $\phi(0) = \phi'(0) = 0$ and $\phi \in C^2(\mathbb{R})$. Here, $r := |\vec{x}|$.

(a) Find $\alpha(r)$ and $\beta(r)$ so that u is a solution of the wave equation for every reasonable ϕ. (Hint: you'll succeed only in some dimensions.) Separately, convince yourself that if you'd made the ansatz

$$u(\vec{x}, t) := \alpha(r)\,\phi(\beta(r) - t) \tag{1}$$

this would have led to solutions too. And convince yourself that the assumption $\alpha(1) = 1$ was a red-herring — what if you'd assumed $\alpha(1) = 2$?

(b) Assume the form (1) and choose $\phi(r)$ so that initially $u(x, 0)$ is compactly supported on $1 \leq |x| \leq 2$ (for example). Plot $u(\cdot, t)$ at some subsequent times t. (Note that these plots are pretty simple because you can just plot things as a function of r.)

(c) Consider a point $\vec{x}_0$ so that $|\vec{x}_0| = R > 2$. At what time would $u(\vec{x}_0, t)$ become nonzero? At what time would it become zero again for all perpetuity? Define the cumulative energy at $\vec{x}_0$ to be

$$\mathcal{E}(\vec{x}_0) = \frac{1}{2}\int_{t_0}^{t_1} u_t(\vec{x}_0, t)^2 + c^2|\nabla u(\vec{x}_0, t)|^2 \, dt$$

where t_0 and t_1 are these two times. How does the cumulative energy depend on R? (For this one, if you like just choose the initial $u(\vec{x}, 0)$ to be simple enough that you can do whatever integrals you need.)

(d) Are you happy that you live in the dimension you live in? Why?