Mat1060, Homework #5, due November 19 at the beginning of class.

1. Please do problems 4, 6, 8, 10 and 11 in Section 4.2.

2. Find u which is harmonic ($\Delta u = 0$) in the unit disc in $\mathbb{R}^2$ centred at $(0,0)$ with boundary data

$$u(1,\theta) = \begin{cases} 0 & 0 < \theta < 2\pi/3 \\ 1 & 2\pi/3 < \theta < 4\pi/3 \\ -1 & 4\pi/3 < \theta < 2\pi \end{cases}$$

   Plot $u(r,\theta)$ as a function of θ for some r values of your choice. (If you used Fourier methods to solve the problem then plot the partial sum $u_N(r,\theta)$ for N chosen large.)

3. (a) Consider the Laplace problem on the half-space

$$\begin{cases} \Delta u(x,y) = 0 & x \in \mathbb{R},\ y > 0 \\ u(x,0) = \begin{cases} k_1 & x \leq 0 \\ k_2 & x > 0 \end{cases} \end{cases} \tag{1}$$

   where k_1 and k_2 are constants. You can solve this problem by first solving

$$\begin{cases} \Delta v(x,y) = 0 & x \in \mathbb{R},\ y > 0 \\ v(x,0) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \end{cases}$$

   and then constructing u from v. Show how to construct u from v.

   (b) Find a solution v which is bounded.(Note: you can do this using the integral formula. Or you can use symmetry arguments to guess that in polar coordinates u would be a function of angle only...)

   (c) Plot $v(x,y)$ as a function of x for some y values of your choice. Describe what you see. As $(x,y) \to (0,0)$ what does $v(x,y)$ do?

   (d) Now consider the Laplace problem on the half-space

$$\begin{cases} \Delta u(x,y) = 0 & x \in \mathbb{R},\ y > 0 \\ u(x,0) = \begin{cases} 0 & x \leq a \\ f & x \in (a,b) \\ 0 & x \geq b \end{cases} \end{cases} \tag{2}$$

   where f is a constant. Find the bounded solution u. Hint: use the solution of (1).

(e) Consider the Laplace problem on the half-space

$$\begin{cases} \Delta u(x,y) = 0 & x \in \mathbb{R},\ y > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases} \tag{3}$$

Fix a mesh-size ϵ. Use it to define a grid on the x-axis by $x_n = n\epsilon$ for $n \in \mathbb{Z}$. Take $a = x_n$, $b = x_{n+1}$ and $f = g(x_n)$. What is the corresponding solution of part (2)? Use this to create a a bounded solution u_ϵ of (3) with piecewise constant initial data g_ϵ. (Assume g is bounded.) Think about how u_ϵ approximates u (the solution of (3)). (Assume whatever you want about g here.)