

Mat1060, Homework #6, due Thursday November 28 at the beginning of class.

1. Please do problems 1, 3, 4, and 6 in Section 4.4.
2. Please do problems 1, 2, and 5 in Section 5.1.
3. We know that if  $u \in C^2(\Omega)$  and  $\Delta u = 0$  and  $\xi \in \Omega$  and  $B_r(\xi) \subset \Omega$  then  $u(\xi)$  equals its average over the sphere  $\partial B_r(\xi)$ :

$$u(\xi) = \frac{1}{\omega_n r^{n-1}} \int_{|x-\xi|=r} u(x) dS_x. \quad (1)$$

This then implies that  $u(\xi)$  equals its average over the ball  $B_r(\xi)$ :

$$u(\xi) = \frac{n}{\omega_n r^n} \int_{|x-\xi|\leq r} u(x) dS_x. \quad (2)$$

Note that the above holds for *every* ball contained in  $\Omega$ , not just for sufficiently small balls.

- (a) Prove that if  $u \in C^2(\Omega)$  and (1) holds for all sufficiently small balls then  $u$  is harmonic and therefore (1) and (2) hold for all balls contained in  $\Omega$ .
  - (b) Prove that if  $u \in C(\Omega)$  and (1) holds for all sufficiently small balls then, in fact,  $u \in C^\infty(\Omega)$ . Conclude that  $u$  is harmonic and therefore (1) and (2) hold for all balls contained in  $\Omega$ . *Hint: try smoothing  $u$  by convolving it with a radially symmetric function  $\phi_\epsilon$  such that  $\phi_\epsilon$  has integral 1. If this doesn't help, google "mollifier". Note: you didn't really need that  $u$  be continuous in  $\Omega$ ; being locally integrable would have sufficed.*
4. We know that if  $v \in C^2(\Omega)$  and  $\Delta v \geq 0$  and  $\xi \in \Omega$  and  $B_r(\xi) \subset \Omega$  then  $v(\xi)$  is less than or equal to its average over the sphere  $\partial B_r(\xi)$ :

$$v(\xi) \leq \frac{1}{\omega_n r^{n-1}} \int_{|x-\xi|=r} v(x) dS_x. \quad (3)$$

This then implies that  $v(\xi)$  is less than or equal to its average over the ball  $B_r(\xi)$ :

$$v(\xi) \leq \frac{n}{\omega_n r^n} \int_{|x-\xi|\leq r} v(x) dS_x. \quad (4)$$

Note that the above holds for *every* ball contained in  $\Omega$ , not just for sufficiently small balls.

- (a) Prove that if  $v \in C(\Omega)$  and (3) holds for all sufficiently small balls then the strong maximum principle holds:

$$\max_{x \in \overline{\Omega_0}} v(x) \leq \max_{x \in \partial\Omega_0} v(x)$$

for every open, bounded set  $\Omega_0$  such that  $\overline{\Omega_0} \subset \Omega$ . We need  $\Omega_0$  to be bounded because otherwise we could take  $\Omega = \mathbb{R}^2 - B_1(0)$ ,  $\Omega_0 = \mathbb{R}^2 - B_2(0) \subset \Omega$ , and  $v(x) = \ln(|x|)$ .  $v$  is harmonic but it's not true that  $\sup_{\Omega_0} v = \infty \leq \sup_{\partial\Omega_0} v = \ln(2)$ .

- (b) Prove that if  $v \in C(\Omega)$  and (3) holds for all sufficiently small balls then “the comparison with harmonic functions property” holds: if  $\Omega_0$  is an open, bounded set such that  $\overline{\Omega_0} \subset \Omega$  and  $u \in C^2(\Omega_0) \cap C(\overline{\Omega_0})$  is harmonic then

$$v(x) \leq u(x) \quad \forall x \in \partial\Omega_0 \quad \implies \quad v(x) \leq u(x) \quad \forall x \in \Omega_0$$

Again, we need  $\Omega_0$  to be bounded. Otherwise you could take  $\Omega$  and  $\Omega_0$  as above and take  $v(x) = \ln(|x|)$  and  $u(x) = \ln(2)$ . Both  $u$  and  $v$  are harmonic functions and  $v \leq u$  on  $\partial\Omega_0$  but it's not true that  $v \leq u$  on  $\Omega_0$ .

- (c) Prove that if  $v \in C(\Omega)$  and “the comparison with harmonic functions property” holds then (3) holds for all balls contained in  $\Omega$ . Note: you didn't really need that  $v$  be continuous in  $\Omega$ ; being upper semicontinuous would suffice. But you'd have to work with a monotonically decreasing sequence of continuous functions  $\{v_n\}$  that converges to  $v$  and then take the limit to get the result for  $v$ .

Lesson learned: we could define “ $v$  is subharmonic” by asking that (3) holds for all balls or by asking that it holds for all sufficiently small balls or by asking that “the comparison with harmonic functions property” holds.