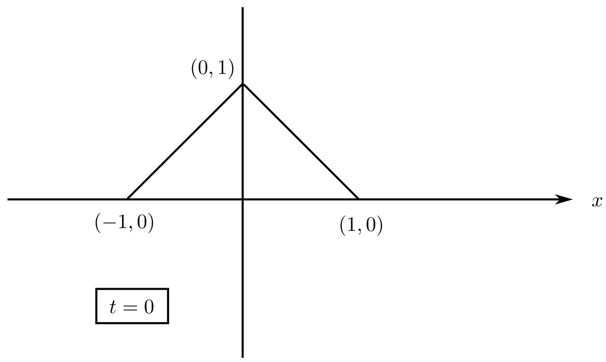
Mat 1060 midterm exam, October 22, 1:10-2pm

1. (5 points) **1-d wave equation on the line**

   Consider

$$\begin{cases} u_{tt} = 4\, u_{xx} & x \in \mathbb{R}\,, t > 0 \\ u(x,0) = f(x) & \\ u_t(x,0) = 0 & \end{cases}$$

   where the graph of the initial displacement f looks like



   Plot the solution at time $t = 2$. Label all "important" points. See above graph for what points are considered important for this purpose.

2. (10 points) **1-d wave equation on a bounded interval**

   Consider

$$\begin{cases} u_{tt} = 9\, u_{xx} & \text{for } x \in [0,1],\ t > 0 \\ u(0,t) = 2 & t > 0 \\ u(1,t) = 3 & t > 0 \\ u(x,0) = 1 & x \in [0,1] \\ u_t(x,0) = 0 & x \in [0,1] \end{cases}$$

   Find and plot the solution at time $t = 1/12$. Find and plot the solution at time $t = 1/3$.

3. (10 points)**What kind of equation am I?**

   Is

$$x^2\, u_{xx} - y^2\, u_{yy} = 0$$

   hyperbolic, parabolic, or elliptic? Reduce it to canonical form. Feel free to only consider the PDE on the domain $\{(x,y) \mid x \neq 0\ \&\ y \neq 0\}$.

4. (10 points) **Characteristics, characteristics...**

   Solve the initial value problem

$$\begin{cases} u_t + u\, u_x = e^{-x}\, v \\ v_t + c\, v_x = 0 \\ u(x,0) = x \\ v(x,0) = e^x \end{cases}$$

   where $c \neq 0$. *Hint: Try to solve the linear PDE first.*

5. (15 points) **Me, I don't like Monge Cones!**

   We used Monge Cones en route to finding a system of 5 ODEs whose solution will allow us to construct a solution of the fully nonlinear first-order PDE

$$F(x, y, u, u_x, u_y) = 0$$

   given acceptable initial data. There is another way to get those 5 ODEs; your challenge is to find it. Step 1: consider

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$$

   and differentiate with respect to x. This yields a quasilinear PDE (for what?). What are the three characteristic ODEs for this quasilinear PDE? Step 2: You're well on your way to finding the five characteristic ODEs that we use to construct a solution of the fully nonlinear PDE $F(x, y, u, u_x, u_y) = 0$. Can you finish?

6. (15 points) **A mysterious discovery while out for a walk**

   Your thesis supervisor would like you to solve Burger's equation with diffusion, $u_t + u\, u_x = 3\, u_{xx}$, on the interval $[0, 10]$ with periodic boundary conditions.

   The very next day, you trip over a black box on the sidewalk labelled "Burger's equation with diffusion — use me". If you give the box a diffusion constant D, initial data $v(\tilde{x}, 0) = v_0(\tilde{x})$ and a time τ_0, it gives you a function $v(\tilde{x}, \tau_0)$ which you can then plot. Wonderful! The only problem is... it's solving $v_\tau + v\, v_{\tilde{x}} = D\, v_{\tilde{x}\tilde{x}}$ with periodic boundary conditions on $[0, 2\pi]$ — it's solving it on the wrong domain!

   Explain how to use that black box to provide the solutions your supervisor seeks. If she asks you for a graph of a solution at time 1, how will you provide that?

   Consider the inviscid case ($D = 0$). Given that the problem on $[0, 2\pi]$ with initial data $\cos(x)$ blows up at time $T = 1$, when should the problem on $[0, 10]$ with initial data $\cos(\pi x/5)$ blow up?