Representations of locally compact groups – Fall 2013

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Contents

CHAPTER 1

Topological groups

1. Definitions and basic properties

DEFINITION 1.1. A *topological group* G is a group which is a topological space and has the property that the map $(x, y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous. (Equivalently, the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.) Here, $G \times G$ is given the product topology.

LEMMA 1.2. *Let G be a topological group. Then*

(1) *The map $g \mapsto g^{-1}$ is a homeomorphism of G onto itself.*
(2) *Fix $g_0 \in G$. The maps $g \mapsto g_0 g$, $g \mapsto g g_0$, and $g \mapsto g_0 g g_0^{-1}$ are homeomorphisms of G onto itself.*

EXAMPLE 1.3. Let F be a field that is also a topological group under addition and the nonzero elements $F^\times$ in F is a toplogical group under multiplication. (Some standard examples are $\mathbb{R}$, $\mathbb{C}$ and finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers). The set $M_n(F)$ of $n \times n$ matrices with entries in F is homoemorphic to F^{n^2}. Let $GL_n(F)$ (also denoted by $GL(n, F)$) be the group of invertible matrices in $M_n(F)$. Since $GL(n, F)$ is a subset of $M_n(F)$, we can make $GL_n(F)$ into a topological space using the the subspace topology. The determinant map $\det : M_n(F) \to F$ is continuous (it's a polynomial in the matrix entries). It is easy to see that the multiplication map $(A, B) \mapsto AB$ from $M_n(F) \times M_n(F) \to M_n(F)$ is continuous, so the restriction to $GL_n(F) \times GL_n(F)$ is also continuous. Given $A \in GL_n(F)$, let $\operatorname{adj} A$ be the adjoint of A. Using the fact that $A^{-1} = \operatorname{adj} A / \det A$, we can see that inversion is continuous. Hence $GL_n(F)$ is a topological group under matrix multiplication. If F is Hausdorff and locally compact, then $GL_n(F)$ is Hausdorff and locally compact. (Recall that a topological space X is locally compact if given a point $x \in X$ there exists a compact subset U of X such that U contains an open neighbourhood of x.)

More generally, if R is a commutative ring with identity, R is a topological space, and multiplication and addition are continuous maps from $R \times R$ to R, let $GL_n(R) = \{ A \in$

$M_n(R) \mid \det A \in R^\times \}$. Here, $R^\times$ is the group of units in R. Then $GL_n(R)$ is a topological group under matrix multiplication.

EXAMPLE 1.4. Let $B(\mathcal{H})$ be the set of bounded linear operators on a Hilbert space $\mathcal{H}$. (Recall that a linear operator T on $\mathcal{H}$ is bounded if there exists a positive constant C such that $\|T(v)\| \leq C\|v\|$ for all $v \in \mathcal{H}$.) If $T \in B(\mathcal{H})$ is bijective, then $T^{-1} \in B(\mathcal{H})$. The strong operator topology on $B(\mathcal{H})$ is the weakest locally convex topology such that the evaluation map sending T to $\|T(v)\|$ is continuous for each vector $v \in \mathcal{H}$. If $\mathcal{H}$ is infinite-dimensional, the group of bijective operators in B($\mathcal{H}$) is not a topological group with respect to the subspace topology induced by the strong operator topology on $B(\mathcal{H})$. Neither the multiplication map nor the inversion map is continuous.

If we take the subgroup $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$, this is a topological group with respect to the subspace topology. To see this, fix T_1, $T_2 \in U(\mathcal{H})$. Fix $\epsilon > 0$ and $v \in \mathcal{H}$. Then the set

$$\mathcal{V} := \{ S \in U(\mathcal{H}) \mid \|(S - T_1T_2(v)\| < \epsilon \}$$

is an open neighbourhood of T_1T_2 in the strong operator topology. Let

$$\mathcal{V}' = \{ (S_1, S_2) \in U(\mathcal{H}) \mid \|(S_1 - T_1)(T_2(v))\| < \epsilon/2, \text{ and } \|(S_2 - T_2)(v)\| < \epsilon/2 \}.$$

Then $\mathcal{V}'$ is an open neighbourhood of (T_1, T_2) in $U(\mathcal{H}) \times U(\mathcal{H})$. Let $(S_1, S_2) \in \mathcal{V}'$. Then

$$\begin{aligned}
\|(S_1S_2 - T_1T_2)(v)\| &= \|S_1(S_2 - T_2)(v) + (S_1 - T_1)(T_2(v))\| \\
&\leq \|S_1(S_2 - T_2)(v)\| + \|(S_1 - T_1)(T_2(v))\| \\
&= \|(S_2 - T_2)(v)\| + \|(S_1 - T_1)(T_2(v))\| < \epsilon/2 + \epsilon/2
\end{aligned}$$

To see that inversion is continuous, fix $T \in U(\mathcal{H})$. Then, given $v \in H$ and $\epsilon > 0$, let $\mathcal{V} = \{ S \in U(\mathcal{H}) \mid \|(S - T^{-1})(v)\| < \epsilon \}$. Set $w = T^{-1}(v)$. Show that the image of the open neighbourhood $\{ S' \in U(\mathcal{H}) \mid \|(S' - T)(w)\| < \epsilon \}$ of T under the inversion map is a subset of $\mathcal{V}$.

A subgroup H of a topological group G is a topological group in the subspace topology.

EXAMPLE 1.5. Let $(I, \leq)$ be a directed partially ordered set. Suppose that $\{G_i\}_{i\in I}$ is a family of finite groups. Assume that whenever $i \leq j \in I$, there exists a homomorphism $f_{ij} : G_j \to G_i$ such that

(1) f_{ii} is the identity map.
(2) $f_{ik} = f_{ij} \circ f_{jk}$ for $i \leq j \leq k \in I$.

Then $((G_i)_{i\in I}, (f_{ij})_{i\leq j\in I})$ is called an *inverse (or projective) system* of finite groups and homomorphisms. Let $\varprojlim G_i$ be the subset of the direct product $\prod_{i\in I} G_i$ consisting of elements $(g_i)_{i\in I}$ such that $g_i = f_{ij}(g_j)$ for all $i \leq j \in I$. It is easy to see that $\varprojlim G_i$ is a subgroup of $\prod_{i\in I} G_i$. We give each G_i the discrete topology and then we take the product topology on $\prod_{i\in I} G_i$. Then $\varprojlim G_i$ is closed in $\prod_{i\in I} G_i$. Furthermore, $\varprojlim G_i$ is compact and totally disconnected. The details are left as an exercise. Here is a simple concrete example: fix a prime p, take I to be the natural numbers, $G_i = \mathbb{Z}/p^i\mathbb{Z}$, and, for $i \leq j$, set $f_{ij}(n + p^j\mathbb{Z}) = n + p^i\mathbb{Z}$, $n \in \mathbb{Z}$. Then $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ is known as the group of p-adic integers. (In fact, $\mathbb{Z}_p$ is a ring.)

Lemma 1.6. *Any open subgroup of a topological group is closed.*

Proof. Let H be an open subgroup of a topological subgroup G. If $g \in G$, then gH is open. Hence $S := \bigcup_{g\in G\setminus H} gH$ is open. Since $H = G \setminus S$, it follows that H is closed. □

Lemma 1.7. *If H is a subgroup of a topological group G, the closure $\overline{H}$ is a subgroup of G. If H is normal in G, then $\overline{H}$ is normal in G.*

Proof. Let x, $y \in \overline{H}$. Let U be an open neighbourhood of xy. Let $S_U = \{\,(u, v) \in G\times G \mid uv \in U\,\}$. Then S_U is open and contains (x, y). Let U_1 and U_2 be open neighbourhoods of x and y, respectively, such that $U_1 \times U_2 \subset S_U$. Because x, $y \in \overline{H}$, $U_j \cap H$ is nonempty, $j = 1$, 2. Fix $u \in U_1 \cap H$ and $v \in U_2 \cap H$. Note that $uv \in U \cap H$. We have shown that $U \cap H$ is nonempty for any open neighbourhood U of xy. Hence $xy \in \overline{H}$.

Let V be an open neighbourhood of y^{-1}. Let $V' = \{\,v^{-1} \mid v \in V\,\}$. Since V' is open and contains y and $y \in \overline{H}$, the intersection $V' \cap H$ is nonempty. The inverse of any point in $V' \cap H$ belongs to $V \cap H$. Hence $V \cap H$ is nonempty. As before, this implies that $y^{-1} \in H$.

The assertion about normality is left as an exercise. □

Let H be a subgroup of a topological group G, and let $q : G \to G/H$ be the canonical mapping of G onto G/H. We define a topology $\mathbf{U}_{G/H}$ on G/H, called the *quotient topology*, by $\mathbf{U}_{G/H} = \{\,q(U) \mid U \in \mathbf{U}_G\,\}$. (Here, $\mathbf{U}_G$ is the topology on G). The canonical map q is open (by definition) and continuous.

Proposition 1.8. *Let H be a subgroup of a topological group G.*

1. *If H is compact, then q is a closed map.*
2. *G/H is a Hausdorff space if and only if H is closed.*

(3) *If G is locally compact, then G/H is locally compact. If, in addition, H is closed, then H is locally compact.*
(4) *If G is Hausdorff and H is a locally compact subgroup of G, then H is closed.*
(5) *If H is normal in G, then G/H is a topological group.*
(6) *H is open in G if and only if G/H is a discrete space. If G is compact, then H is open in G if and only if G/H is a finite discrete space.*

PROOF. (1): It suffices to show that if S is a closed subset in G, then the set SH is closed in G. In fact, if $S \subset G$ is closed and $T \subset G$ is compact, then ST is closed. The proof is left as an exercise.

(2): Suppose that G/H is Hausdorff. Then single points are closed in G/H. In particular $\{H\}$ is closed in G/H. By definition of the quotient topology, $q^{-1}(\{H\})$ is closed in G. Next, suppose that H is closed in G. If X is a topological space, then X is Hausdorff if and only if $\Delta_X := \{ (x, x) \mid x \in X \}$ is closed in $X \times X$. Define $f : G/H \times G/H \to (G \times G)/(H \times H)$ by $f(g_1H, g_2H) = (g_1, g_2)(H \times H)$. Then f is a homeomorphism. It suffices to show that $f(\Delta_{G/H})$ is closed. Equivalently, is suffices to show that the set $\{ (g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in H \}$ is closed. This set is the inverse image of H under the continuous map $\varphi : G \times G \to G$ defined by $\varphi(g_1, g_2) = g_1 g_2^{-1}$. Since H is closed, this set is also closed.

(3): Since q is continuous, the image under q of a compact subset of G in G/H is compact. Since q is open, this implies that if G is locally compact, then G/H is also locally compact. For the second statement, use the fact that a closed subset of a locally compact topological space is locally compact (in the subspace topology).

(4): Suppose that H is a locally compact subgroup of G. Let $U \subset H$ be a compact set such that U contains an open neighbourhood of 1 in H. Because compact subsets of Hausdorff spaces are closed, U is closed. Fix a closed set V in G such that $U = V \cap H$ and V contains an open neighbourhood of 1. Since U is compact in H, U is compact in G. Because G is Hausdorff, this implies that U is closed in G. Fix an open neighbourhood W of 1 in G such that $W \cdot W := \{ ww' \mid w, w' \in W \}$ is a subset of V.

Let $x \in \overline{H}$. We know that $Wx^{-1} \cap H$ is nonempty. Let $y \in Wx^{-1} \cap H$. Let W' be an open neighbourhood of yx in G. Because $y^{-1}W'$ and xW are both open neighbourhoods of x in G and $x \in \overline{H}$, the intersection $y^{-1}W' \cap xW \cap H$ is nonempty. Let $z \in y^{-1}W' \cap xW \cap H$. Note that $yz \in W \cdot W \subset V$, $yz \in H$ and $yz \in W'$. Hence $yz \in W' \cap (V \cap H) = W' \cap U$.

This shows that $W' \cap U$ is nonempty for every open neighbourhood W' of yx in G. Thus yx belongs to the closure $\overline{U}$. As observed above, U is closed. Hence $yx \in U \subset H$. Now we have that $x = y^{-1}yx \in H$.

(5): If g, $x \in G$, define $T_g(x) = gx$. Similarly, we define $T_{q(g)} : G/H \to G/H$, $g \in G$. Observe that $q \circ T_g = T_{q(g)} \circ p$ and $q(x)^{-1} = q(x)^{-1}$. Using the fact that q is an open map and T_g and the map $x \mapsto x^{-1}$ are continuous maps, we can show that $T_{q(g)}$ and the map $gH \mapsto g^{-1}H$ are continuous.

(6): Exercise. $\square$

Remark 1.9. In cases where certain G-invariant measures (known as Haar measures) exist, integration plays a key role in various aspects of representation theory. In order for such measures to exist, G must be locally compact. This is discussed in Section 4.

Exercise 1.10. Let G be a Hausdorff topological group.

a) Show that the centralizer of a nonempty subset of G is closed.
b) Show that the normalizer of a subgroup of G is closed.

Definition 1.11. The *identity component* G° of G is defined to be the connected component of G containing the identity element.

Lemma 1.12. *Let G be a topological group.*

(1) *G° is a closed normal subgroup of G.*
(2) *The quotient group G/G^0 is discrete if and only if G° is open in G.*

Exercise 1.13. Let G be a connected topological group. Show that a proper subgroup of G cannot contain an open neighbourhood of the identity.

Definition 1.14. If G and G' are topological groups a continuous homomorphism $f : G \to G'$ is called a *homomorphism of topological groups.* If f is also a homeomorphism, then f is an *isomorphism of topological groups.*

Lemma 1.15. *Let $f : G \to G'$ be a homomorphism of topological groups. Let H be the kernel of f.*

(1) *H is a closed normal subgroup of G.*
(2) *If G is locally compact, then H and G/H are locally compact.*
(3) *G/H and $f(G)$ are isomorphic (as topological groups) if and only if f is open when viewed as map from G to $f(G)$.*

2. Locally compact, totally disconnected groups

Recall that a topological space X is *totally disconnected* if given any two distinct elements x and y in X, there exist open sets U and V with $x \in U$, $y \in V$, $U \cap V = \emptyset$, and $X = U \cup V$. Equivalently, there is no connected subset of X with more than one element.

LEMMA 2.1. *A totally disconnected group is Hausdorff.*

The proof is left as an exercise. In fact, if a topological group is T_1 (that is, points are closed sets), then the group is Hausdorff.

DEFINITION 2.2. A topological group G is *profinite* if G is compact and totally disconnected.

LEMMA 2.3. *If G is profinite, the quotient maps combine into a topological isomorphism between G and the projective(aka inverse) limit $\varprojlim G/N$, where N runs over the set of open normal subgroups of G.*

LEMMA 2.4. . *(Chapter I, §1 of* [**S**]*) Let G be a topological group. The following are equivalent:*

1. *G is a projective limit of finite groups.*
2. *G is a compact, Hausdorff group in which the family of open normal subgroups forms a fundamental system of open neighbourhoods of the identity.*
3. *G is a compact, totally disconnected group.*

EXAMPLE 2.5. One example that has already been mentioned is the set $\mathbb{Z}_p$ of p-adic integers (where p is a fixed prime). Another example: Suppose that L is a Galois extension of a field K of infinite degree. Let F range over all extension fields of K such that F/K is a finite Galois extension. We can define a partial order on the set of such F using inclusion. If $F_2 \subset F_1$, we take the homomorphism $\mathrm{Gal}(F_1/K) \to \mathrm{Gal}(F_2/K)$ given by restriction to F_2. Then $\mathrm{Gal}(L/K) \simeq \varprojlim \mathrm{Gal}(F/K)$. The topology on $\mathrm{Gal}(L/K)$ is often referred to as the *Krull* topology.

DEFINITION 2.6. If G is a locally compact, totally disconnected group, we say that G is *locally profinite* (alternatively G is a *group of t.d. type*, or a *t.d. group*).

If G is a locally profinite group, then every open neighbourhood of the identity in G contains a compact open subgroup of G. A compact open subgroup of a locally compact, totally disconnected, group is a profinite group.

LEMMA 2.7. *A Hausdorff topological group is locally profinite if G has a countable neighbourhood basis at the identity consisting of compact open subgroups, and G/K is a countable set for every open subgroup K of G.*

Some locally profinite groups occur as matrix groups over p-adic fields. Let p be a prime. Let $x \in \mathbb{Q}^\times$. Then there exist unique integers m, n and r such that m and n are nonzero and relatively prime, p does not divide m or n, and $x = p^r m/n$. Set $|x|_p = p^{-r}$. This defines a function on $\mathbb{Q}^\times$, which we extend to a function from $\mathbb{Q}$ to the set of nonnegative real numbers by setting $|0|_p = 0$.

DEFINITION 2.8. The function $|\cdot|_p$ is called the *p-adic absolute value* on $\mathbb{Q}$.

The p-adic absolute value is a valuation on $\mathbb{Q}$ - that is, it has the properties

(1) $|x|_p = 0$ if and only if $x = 0$
(2) $|xy|_p = |x|_p|y|_p$
(3) $|x + y|_p \leq |x|_p + |y|_p$.

The usual absolute value on the real numbers is another example of a valuation on $\mathbb{Q}$. The p-adic abolute value satisfies the *ultrametric* inequality, that is, $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. Note that the ultrametric inequality implies property (iii) above. A valuation that satisfies the ultrametric inequality is called a *nonarchimedean* valuation.

Note that the set $\{\, |x|_p \mid x \in \mathbb{Q}^\times \,\}$ is a discrete subgroup of $\mathbb{R}^\times$. Hence we say that $|\cdot|_p$ is a *discrete* valuation. The usual absolute value on $\mathbb{Q}$ is an example of an *archimedean* valuation. Clearly it is not a discrete valuation. Two valuations on a field F are said to be *equivalent* is one is a positive power of the other.

THEOREM 2.9. *(Ostrowski; see Theorem 2.1 of* [**Cas**]*) A nontrivial valuation on $\mathbb{Q}$ is equivalent to the usual absolute value or to $|\cdot|_p$ for some prime p.*

DEFINITION 2.10. If F is a field and $|\cdot|$ is a valuation on F, the *topology on F induced by $|\cdot|$* has as a basis the sets of the form $U(x, \epsilon) = \{\, y \in F \mid |x - y| < \epsilon \,\}$, as x varies over F, and ϵ varies over all positive real numbers. A field F' with valuation $|\cdot|'$ is a *completion* of the field F with valuation $|\cdot|$ if $F \subset F'$, $|x|' = |x|$ for all $x \in F$, F' is complete with respect to $|\cdot|'$ (every Cauchy sequence with respect to $|\cdot|'$ has a limit in F') and F' is the closure of F with respect to $|\cdot|$.

The completion of F is the smallest field containing F such that F' is complete with respect to $|\cdot|'$.

The real numbers is the completion of $\mathbb{Q}$ with respect to the usual absolute value on $\mathbb{Q}$.

DEFINITION 2.11. The *p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

(We denote the extension of $|\cdot|_p$ to $\mathbb{Q}_p$ by $|\cdot|_p$ also).

In Example 1.5, we defined the p-adic integers $\mathbb{Z}_p$ as $\varprojlim \mathbb{Z}/p^i\mathbb{Z}$. Strictly speaking, before making the next definition, we should verify that $\varprojlim \mathbb{Z}/p^i\mathbb{Z}$ is isomorphic to the group $R = \{\, x \in \mathbb{Q}_p \mid |x|_p \leq 1 \,\}$. This can be done as follows. It is possible to show that a proper open subgroup of R has the form p^iR for some positive integer i and that R/p^iR is isomorphic to $\mathbb{Z}/p^i\mathbb{Z}$. Then, since R is profinite (it's compact and totally disconnected), $R \simeq \varprojlim R/p^iR \simeq \varprojlim \mathbb{Z}/p^i\mathbb{Z}$.

DEFINITION 2.12. The p-adic integers $\mathbb{Z}_p$ is the set $\{\, x \in \mathbb{Q}_p \mid |x|_p \leq 1 \,\}$.

Note that $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$ (this follows from the ultrametric inequality and the mulitiplicative property of $|\cdot|_p$), and $\mathbb{Z}_p$ contains $\mathbb{Z}$. The set $p\mathbb{Z}_p$ (the ideal of $\mathbb{Q}_p$ generated by the element p) is a maximal ideal of $\mathbb{Z}_p$ and $\mathbb{Z}_p/p\mathbb{Z}_p$ is therefore a field.

Let $x \in \mathbb{Q}^\times$. Write $x = p^r m/n$ with $r \in \mathbb{Z}$ and m and n nonzero integers such that m and n are relatively prime and not divisible by p. Because m and n are relatively prime and not divisible by p, the equation $nX \equiv m (\operatorname{mod} p)$ has a unique solution $a_r \in \{\, 1, \dots, p-1 \,\}$. That is, there is a unique integer $a_r \in \{1, \dots, p-1\,\}$ such that p divides $m - na_r$. Since $|n|_p = 1$, p divides $m - na_r$ is equivalent to $|(m/n) - a_r|_p < 1$, and also to $|x - a_r p^r|_p < |x|_p = p^{-r}$. Expressing $x - a_r p^r$ in the form $p^s m'/n'$ with $s > r$ and m' and n' relatively prime integers, we repeat the above argument to produce an integer $a_s \in \{1, \dots, p-1\}$ such that

$$|x - a_r p^r - a_s p^s|_p < |x - a_r p^r|_p = p^{-s}.$$

If $s > r+1$, set $a_{r+1} = a_{r+2} = \cdots = a_{s-1} = 0$, to get

$$|x - \sum_{n=r}^{s} a_n p^n|_p < p^{-s}.$$

Continuing in this manner, we see that there exists a sequence $\{\, a_n \mid n \geq r \,\}$ such that $a_n \in \{\, 0, 1, \dots, p-1 \,\}$ and, given any integer $M \geq r$,

$$|x - \sum_{n=r}^{M} a_n p^n|_p < p^{-M}.$$

It follows that $\sum_{n=r}^{\infty} a_n p^n$ converges in the p-adic topology to the rational number x.

On the other hand, it is quite easy to show that if $a_n \in \{0, 1, \ldots, p-1\}$ and r is an integer, then $\sum_{n=r}^{\infty} a_n p^n$ converges to an element of $\mathbb{Q}_p$ (though not necessarily to a rational number).

LEMMA 2.13. *A nonzero element x of $\mathbb{Q}_p$ is uniquely of the form $\sum_{n=r}^{\infty} a_n p^n$, with $a_n \in \{0, 1, 2, \ldots, p-1\}$, for some integer r with $a_r \neq 0$. Furthermore, $|x|_p = p^{-r}$. (Hence $x \in \mathbb{Z}_p$ if and only if $r \geq 0$).*

LEMMA 2.14. $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$.

PROOF. Let $a \in \mathbb{Z}_p$ be nonzero. According to Lemma 2.13, $a = \sum_{n=r}^{\infty} a_n p^n$ for some sequence $\{ a_n \mid n \geq r \}$, where $|a|_p = p^{-r} \leq 1$ implies that $r \geq 0$. If $r > 0$, then $a \in p\mathbb{Z}_p$. For convenience, set $a_0 = 0$ when $|a|_p < 1$. If $r = 0$, then $a_0 \in \{ 1, \ldots, p-1 \}$. If $a = 0$, set $a_0 = 0$. Define a map from $\mathbb{Z}_p$ to $\mathbb{Z}/p\mathbb{Z}$ by $a \mapsto a_0$. This is a surjective ring homomorphism whose kernel is equal to $p\mathbb{Z}_p$. □

DEFINITION 2.15. A *local field* F is a (nondiscrete) field F which is locally compact and complete with respect to a nontrivial valuation.

The fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$, p prime, are local fields. If $|\cdot|$ is a nontrivial nonarchimedean valuation on a field F, then $\mathfrak{p}_F := \{x \in F \mid |x| < 1\}$ is a maximal ideal in the ring $\mathfrak{o}_F := \{ x \in F \mid |x| \leq 1 \}$, so the quotient $\mathfrak{o}_F/\mathfrak{p}_F$ is a field, called the *residue class field* of F. The following lemma can be used to check that $\mathbb{Q}_p$ is a local field.

LEMMA 2.16. *([**Cas**], Corollary on p. 46) Let $|\cdot|$ be a nonarchimedean valuation on a field F. Then F is locally compact with respect to $|\cdot|$ if and only if*

(1) *F is complete (with respect to $|\cdot|$)*
(2) *$|\cdot|$ is discrete*
(3) *The residue class field of F is finite.*

For every integer N, $p^N\mathbb{Z}_p$ is a compact open (and closed) subgroup of $\mathbb{Q}_p$. It is not hard to see that $\{ p^N\mathbb{Z}_p \mid N \geq 0 \}$ forms a countable neighbourhood basis at the identity element 0. The group $\mathbb{Z}_p^\times$ of units in the ring $\mathbb{Z}_p$ is equal to $\{ a \in \mathbb{Q}_p \mid |a|_p = 1 \}$. It follows from Lemma 2.13 that $\mathbb{Q}_p^\times \simeq \langle p \rangle \times \mathbb{Z}_p^\times$. Hence $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete. It can be shown that any open subgroup of $\mathbb{Q}_p$ is of the form $p^N\mathbb{Z}_p$ for some integer N. Thus $\mathbb{Q}_p/K$ is discrete for every open subgroup K of $\mathbb{Q}_p$. Therefore $\mathbb{Q}_p$ is a locally profinite group. For more information on valuations, the p-adic numbers, and p-adic fields, see [**Cas**].

As mentioned in Example 1.3, because $\mathbb{Q}_p$ is locally compact, the topological group $GL_n(\mathbb{Q}_p)$ is also locally compact. In fact $GL_n(\mathbb{Q}_p)$ is a locally profinite group. If j is a positive integer, let K_j be the set of $g \in GL_n(\mathbb{Q}_p)$ such that every entry of $g-1$ belongs to $p^j\mathbb{Z}_p$. Then K_j is a compact open subgroup, and $\{ K_j \mid j \geq 1 \}$ forms a countable neighbourhood basis at the identity element 1.

Let K be an open subgroup of $GL_n(\mathbb{Q}_p)$. Then $K_j \subset K$ for some $j \geq 1$. Hence to prove that G/K is countable, it suffices to prove that G/K_j is countable for every j. For a discussion of the proof that G/K_j is countable, see [**M**]. At this point, we can apply Lemma 2.7 to conclude that $GL_n(\mathbb{Q}_p)$ is a locally profinite group.

Closed subgroups of locally profinite groups are locally profinite. Hence any closed subgroup of $GL_n(\mathbb{Q}_p)$ is locally profinite. Such groups are often called p-adic matrix groups. This gives a way to generate many examples of locally profinite groups.

3. Matrix groups

3.1. Lie groups. A *real Lie group* is a topological group G that is a finite-dimensional real smooth manifold with a group structure in which the multiplication and inversion maps from $G \times G$ to G and from G to G are smooth maps. Without referring to the differentiable manifolds, we may define a *matrix Lie group*, or a *closed Lie subgroup of* $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ to be a closed subgroup of the topological group $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$, respectively. (This latter definition is reasonable because $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ are real Lie groups, and it can be shown that a closed subgroup of a (real) Lie group is also a (real) Lie group). A *complex Lie group* is a complex-analytic manifold G that is also a group in which the multiplication and inversion maps are holomorphic functions.

A connected matrix Lie group is reductive if it is stable under conjugate transpose, and semisimple if it is reductive and has finite centre. The book of Hall [**Ha**] gives an introduction to matrix Lie groups, their structure, and their finite-dimensional representations. Some other introductory references on Lie groups and their representations are [**BD**], [**FH**], [**K1**] and [**K2**].

Example 3.1. Consider the hermitian form $z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n$ on $\mathbb{C}^n$. The $n \times n$ *complex unitary group* $U_n(\mathbb{C})$ is the group of complex $n \times n$ matrices that preserve this hermitian form. It is a closed subgroup of $GL_n(\mathbb{C})$, so it is a locally compact group. A matrix $g \in U_n(\mathbb{C})$ satisfies ${}^t\bar{g}g = I_n$ (the columns of the matrix form an orthonormal basis of $\mathbb{C}^n$ with respect to the standard inner form). The group $U_n(\mathbb{C})$ (sometimes written U_n) is a real Lie group. The group $SU_n(\mathbb{C}) := U_n(\mathbb{C}) \cap SL_n(\mathbb{C})$ is the $n \times n$ *complex special unitary group.*

Exercise 3.2. Verify that $U_n(\mathbb{C})$ is connected and compact.

Example 3.3. Consider the quadratic form $x_1^2 + x_2^2 + \cdots x_n^2$ on $\mathbb{R}^n$. The $n \times n$ *real orthogonal group* $O_n(\mathbb{R})$ is the group of matrices in $M_n(\mathbb{R})$ that preserve the form. It is a compact, disconnected Lie group. The *real special orthogonal group* $SO_n(\mathbb{R}) = SL_n(\mathbb{R}) \cap O_n(\mathbb{R})$ is the connected component of the identity in $O_n(\mathbb{R})$. The $n \times n$ *complex orthogonal group* $O_n(\mathbb{C})$ is the matrices in $M_n(\mathbb{C})$ that preserve the quadratic form $z_1^2 + z_2^2 + \cdots + z_n^2$ on $\mathbb{C}^n$.

Example 3.4. If F is a field, the $2n \times 2n$ symplectic group over F is defined by:

$$Sp_{2n}(F) = \{ g \in GL_{2n}(F) \mid {}^t g J g = J \}, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

If $F = \mathbb{R}$ or $\mathbb{C}$, then $Sp_{2n}(F)$ is a noncompact connected Lie group. There is another real Lie group, the so-called *compact symplectic group* $Sp(n)$, which may be defined as $U_{2n}(\mathbb{C}) \cap Sp_{2n}(\mathbb{C})$.

3.2. Linear algebraic groups. A linear algebraic group is an affine algebraic variety that is also a group, where the multiplication and inversion operations are given by regular functions on the variety. A linear algebraic group may be realized as a subgroup of some general linear group.

Many Lie groups are algebraic groups over the real or complex numbers. For example, a compact Lie group can be regarded as the group of points of a real linear algebraic group. Other examples of linear algebraic groups include various p-adic groups, that is, groups occuring as closed subgroups of $GL_n(F)$, where F is a p-adic field (a finite extension of $\mathbb{Q}_p$, p prime). If F is a finite field, the groups $GL_n(F)$, $SL_n(F)$, $Sp_n(F)$ are linear algebraic groups.

Orthogonal and unitary groups are the groups preserving nondegenerate quadratic and hermitian forms, respectively, on finite-dimensional vector spaces. If the vector space is a real vector space, there are several equivalence classes of quadratic forms, so there are various corresponding orthogonal groups. When the form is positive-definite, the associated orthogonal group is $O_n(\mathbb{R})$ (mentioned above). Over finite fields and p-adic fields, there can be more than one equivalence class of quadratic forms. Similary, different hermitian forms can give rise to nonisomorphic unitary groups. For example, when $F = \mathbb{Q}_p$, if n is even, there are two isomorphism classes of unitary groups and if n is odd, there is one such class. Almost all of these groups are noncompact.

If we have a topology on F, then $GL_n(F)$, $SL_n(F)$, $Sp_n(F)$, as well as many other matrix groups, are topological groups. When F is a p-adic field, these groups are locally profinite (with respect to the topology coming from F). Such groups have another topology, the Zariski topology (coming from the variety that is the algebraic group). In the representation theory, the topology arising from the topology on F plays an important role. Results in the structure theory of the groups depend on the underlying algebraic group structure, hence are related to the Zariski topology.

The adèles $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \prod' \mathbb{Q}_p$ of $\mathbb{Q}$ is a subring of the direct product $\mathbb{R} \times \prod_p \mathbb{Q}_p$. It consists of elements $(a_\infty, a_2, a_3, a_5, ...)$ such that $a_p \in \mathbb{Z}_p$ for all but finitely many primes. Consider the set of open neighbourhoods of zero of the form $U_\mathbb{R} \times \prod_p U_p$, where $U_\mathbb{R}$ is an open neighbourhood of zero in $\mathbb{R}$, U_p is an open neighbourhood of zero in $\mathbb{Q}_p$, and $U_p = \mathbb{Z}_p$ for all but finitely many primes p. This set forms a base of neighbourhoods of zero for a topology on $\mathbb{A}_\mathbb{Q}$. This topology is locally compact. More generally, if F is a number field (that is, a finite extension of $\mathbb{Q}$), we can define the ring of adèles of F similarly. Since $\mathbb{A}_F$ is a locally compact topological ring, $GL_n(\mathbb{A}_F)$ is a locally compact topological group. This is also the case if G is a linear algebraic group defined over F: $G(\mathbb{A}_F)$ is a locally compact group. Certain irreducible representations of $G(\mathbb{A}_F)$, known as automorphic representations, are basic objects in the theory of automorphic forms. Such representations can be expressed as restricted tensor products of infinite-dimensional representations of the locally compact groups $G(F_v)$, where F_v runs over the set of completions of F with respect to nontrivial valuations.

4. Haar measure on locally compact Hausdorff groups

Let G be a locally compact Hausdorff group. Integration on G and on various coset spaces plays a role in the construction of unitary representations, including discrete series representations, in the construction of induced representations, in relating representations of G to representations of the Banach space of integrable functions $L^1(G)$ on G, and in defining characters of infinite-dimensional representations.

Definition 4.1. If X is a topological space, a σ-ring in X is a nonempty family of subsets of X having the property that countable unions of elements in the family belong to the family, and if A and B belong to the family, then so does $\{ x \in A \mid x \notin B \}$. If X is a locally compact topological space, the *Borel ring* in X is the smallest σ-ring in X that contains the open sets. The elements of the Borel ring are called *Borel sets*. A function $f : X \to \mathbb{R}$ is *(Borel) measurable* if for every $t > 0$, the set $\{ x \in X \mid |f(x)| < t \}$ is a Borel set.

Definition 4.2. Let μ be a positive Borel measure on X.

(1) The measure μ is called *regular* if, for any Borel subset S of X,

$$\mu(S) = \inf\{ \mu(U) \mid U \supset S,\ U \text{ open} \} \quad \text{and} \quad \mu(S) = \sup\{ \mu(C) \mid C \subset S,\ C \text{ compact} \}.$$

(2) The measure μ is a called a *Radon measure* if
   (i) $\mu(C) < \infty$ for any compact set $C \subset X$,
   (ii) $\mu(S) = \inf\{ \mu(U) \mid U \supset S,\ U \text{ open} \}$ for any Borel subset S of X,
   (iii) $\mu(U) = \sup\{ \mu(C) \mid C \subset U,\ C \text{ compact} \}$ for any open subset U of X.

A σ-finite Radon measure is regular. (Recall that σ-finite means the space is a countable union of measurable sets, each having finite measure.)

Definition 4.3. A Radon measure μ on G is *left invariant* (respectively, *right invariant*) if $\mu(gS) = \mu(S)$ (respectively, $\mu(Sg) = \mu(S)$) for any $g \in G$ and Borel subset S of G.

Theorem 4.4. *(*[**Hal**], [**HR**], [**L**]*) There exists a nonzero left-invariant Radon measure μ_ℓ on G. It satisfies $\mu_\ell(U) > 0$ for any nonempty open subset U of G. If ν_ℓ is any nonzero left-invariant Radon measure on G, then there exists $c > 0$ such that $\nu_\ell = c\,\mu_\ell$.*

The measure μ_ℓ is called a *left Haar measure* on G. There is also a right Haar measure μ_r, unique up to positive constant multiples, on G. Right and left Haar measures do not usually coincide. See Proposition 4.11(3) for more information.

Exercise 4.5. Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^\times,\ y \in \mathbb{R} \right\}.$$

Show that $|x|^{-2} dx\, dy$ is a left Haar measure on G and $|x|^{-1} dx\, dy$ is a right Haar measure on G.

If X is a locally compact topological space, let $C_c(X)$ be the space of continuous complex-valued functions on X with compact support. Let $C_c^+(X)$ be the set of real-valued functions $f \in C_c(X)$ such that $f(x) \geq 0$ for all $x \in G$.

If $f \in C_c(G)$, then f is integrable with respect to μ_ℓ and we often write $\int_G f(g)\, d\mu_\ell(g)$ as $\int_G f(g)\, d_\ell g$. We use the same simplification for many kinds of functions on G (for example, non-negative measurable, Haar integrable, or vector valued-versions of integrability) for which $\int_G f(g)\, d_\ell g$ makes sense. The left invariance of μ_ℓ implies that

$$\int_G f(g_0 g)\, d_\ell g = \int_G f(g) d_\ell g, \qquad \forall\ g_0 \in G.$$

If $f \in C_c^+(G)$ and f is not identically zero, then $\int_G f(g)\, d_\ell g > 0$.

Existence of Radon measures can be expressed in terms of continuous linear functionals on the space $C_c(G)$. The following remark is useful when working with Haar measure.

Remark 4.6. The notion of (left) Haar measure is essentially equivalent to the notion of a linear functional Λ on the space $C_c(G)$ such that

(i) For each compact subset C of G, there exists a constant M_C such that for every $f \in C_c(G)$ with support inside C, $|\Lambda(f)| \leq M_C \sup_{g \in C} |f(g)|$. (If Λ is the linear functional corresponding to a particular left Haar measure μ_ℓ, then $M_C = \mu_\ell(C)$.)

(ii) $\Lambda(L_g f) = \Lambda(f)$ for all $g \in G$ and $f \in C_c(G)$, where $(L_g f)(x) = f(g^{-1}x)$, g, $x \in G$.

Definition 4.7. Let Let $L^1(G)$ be the Banach space obtained as the completion of $C_c(G)$ with respect to the norm $\|f\|_1 := \int_G |f(g)|\, d_\ell g$. (As usual, two functions in $L^1(G)$ are regarded as equivalent if and only if they differ on a set of measure zero.) Let $L^2(G)$ be the completion of $C_c(G)$ with respect to the norm $\|f\|_2 := \left(\int_G |f(g)|^2\, d_\ell g\right)^{1/2}$. Since the L^2 norm is associated to the inner product $(f_1, f_2) = \int_G f_1(g) \overline{f}_2(g)\, d_\ell g$, $L^2(G)$ is a Hilbert space.

Definition 4.8. A (locally compact Hausdorff) group G is called *unimodular* if each left Haar measure is also a right Haar measure.

If G is unimodular, we write $\int_G f(g)\, dg$ instead of $\int_G f(g)\, d_\ell g$. Clearly, G is unimodular if G is abelian.

Lemma 4.9. *Let μ_ℓ be a left Haar measure on G. If $\varphi : G \to G$ is an automorphism of G and a homeomorphism, then $\mu_\ell \circ \varphi$ is a left Haar measure on G. (Hence there exists a positive real number c_φ such that $\mu_\ell \circ \varphi = c_\varphi \mu_\ell$. Note that c_φ is independent of the choice of μ_ℓ.)*

If $g_0 \in G$, let $\operatorname{Int} g_0 : G \to G$ be defined by $\operatorname{Int} g_0(g) = g_0 g g_0^{-1}$, $g \in G$. Because $\operatorname{Int} g_0$ is an automorphism of G and a homeomorphism, the measure $S \mapsto \mu_\ell(\operatorname{Int} g_0(S)) = \mu_\ell(S g_0^{-1})$

(S measurable) is also a left Haar measure. By uniqueness of left Haar measure, there exists a constant $\Delta(g_0) > 0$ such that $\mu_\ell(Sg_0^{-1}) = \Delta(g_0)\mu_\ell(S)$. This can also be expressed as

$$\int_G (f \circ \mathrm{Int}(g_0))(g)\, d_\ell g = \int_G f(gg_0^{-1})\, d_\ell g = \Delta(g_0) \int_G f(g)\, d_\ell g, \qquad f \text{ integrable}$$

Definition 4.10. A *quasicharacter* of G is a continuous homomorphism from G to $\mathbb{C}^\times$.

Proposition 4.11. (1) *The function $\Delta : G \to \mathbb{R}_+^\times$ is a quasicharacter.*
(2) *The restriction of Δ to a compact subgroup of G is trivial.*
(3) *$\Delta(g)^{-1} d_\ell g$ is a right Haar measure on G.*

Proof. (1) The fact Δ is a homomorphism, follows from the fact that right multiplication is an action of G on itself. Suppose that $f \in C_c(G)$. Since f has compact support, f is uniformly continuous. Given $\epsilon > 0$, there is an open neighbourhood U of the identity 1 in G such that for $u \in U$ and $g \in G$, $|f(gu) - f(g)| < \epsilon$. Without loss of generality, we may assume that the closure $\overline{U}$ of U is compact. Let S be the support of f. For each $u \in U$, the function $g \mapsto f(gu) - f(g)$ is supported in the compact set $C := S\overline{U}^{-1} \cup S$. By continuity of integration (see Remark 4.6), there exists a constant M_C such that

$$\left|\int_G f(gu)\, d_\ell g - \int_G f(g)\, d_\ell g\right| < M_C \epsilon, \qquad u \in U.$$

This can be rewritten as

$$|\Delta(u)^{-1} - 1| \cdot \left|\int_G f(g)\, d_\ell g\right| < M_C \epsilon, \qquad u \in U.$$

We may choose f such that $\int_G f(g)\, d_\ell g$ is nonzero. Hence Δ is continuous at 1. The general case follows because Δ is a homomorphism.
(2): By continuity of Δ, the image of a compact subgroup of G under Δ is a compact subgroup of $\mathbb{R}_+^\times$. The trivial subgroup is the only compact subgroup of $\mathbb{R}_+^\times$.
(3): We know that

$$\Delta(g_0)^{-1} \int_G f(g)\, d_\ell g = \int_G f(gg_0)\, d_\ell g$$

for any Haar integrable function f on G (actually it's enough to check things for $f \in C_c(G)$). Replacing f by $f\Delta^{-1}$ and multiplying both sides by $\Delta(g_0)$, we obtain

$$\int_G f(g)\Delta(g)^{-1}\, d_\ell g = \int_G f(gg_0)\Delta(g)^{-1}\, d_\ell g.$$

This shows that $\Delta(g)^{-1}d_\ell g$ is right invariant. The rest of the proof is left as an exercise and involves checking continuity (see Remark 4.6) or verifying that $\Delta(g)^{-1}d_\ell g$ is a regular Borel measure using the fact that Δ is continuous and positive-valued. $\square$

Definition 4.12. The function Δ is called the *modular (quasi)character* or *modular function* of G.

Clearly, G is unimodular if and only if the modular quasicharacter is trivial. If G is unimodular, we simply refer to Haar measure on G. Reductive groups over local fields are unimodular. However, many of their closed subgroups, including the proper parabolic subgroups (see Exercise 4 below), are not unimodular.

Exercise 4.13.
1. Let dX denote Lebesgue measure on $M_n(\mathbb{R})$. This is a Haar measure on $M_n(\mathbb{R})$. Show that $|\det(g)|^{-n}dg$ is both a left and a right Haar measure on $GL_n(\mathbb{R})$. Hence $GL_n(\mathbb{R})$ is unimodular.
2. Fix a prime p. Let dt denote a Haar measure on $\mathbb{Q}_p$. For convenience, we normalize dt so that the measure of the compact subgroup $\mathbb{Z}_p$ is equal to one.
   - (i) Show that if n is a nonzero integer, the measure of $p^n\mathbb{Z}_p$ is equal to p^{-n}. (This can be done by induction. As a first step, note that the index of $p\mathbb{Z}_p$ in $\mathbb{Z}_p$ is p.)
   - (ii) Show that if $x \in \mathbb{Q}_p^\times$ is nonzero, then $\mu(x\mathbb{Z}_p)$ is equal to $|x|_p$.
   - (iii) Show that $|t|^{-1}dt$ is a Haar measure on $\mathbb{Q}_p^\times$. (*Hint*: If $c \in \mathbb{Q}_p^\times$, then $\varphi_c : x \mapsto cx$ is an automorphism of $\mathbb{Q}_p$ and a homeomorphism. Hence $\mu \circ \varphi_c$ is a Haar measure on $\mathbb{Q}_p$. Because $\mu(c\mathbb{Z}_p) = |c|_p\mu(\mathbb{Z}_p)$, it follows that $\mu \circ \varphi_c = |c|_p\mu$.)
3. Fix a prime p. Let dt denote a Haar measure on $\mathbb{Q}_p$. The product measure $dg = \prod_{i,j} dg_{ij}$ is a Haar measure on the additive group $M_n(\mathbb{Q}_p)$. Show that $|\det(g)|_p^{-n}dg$ is both a left and a right Haar measure on $GL_n(\mathbb{Q}_p)$.
4. Let n_1 and n_2 be positive integers such that $n_1+n_2 = n$. $P = P_{(n_1,n_2)}$ be the standard parabolic subgroup of $GL_n(F)$ ($F = \mathbb{R}$ or $F = \mathbb{Q}_p$) corresponding to the partition (n_1, n_2). That is, P is the matrices in $GL_n(F)$ of the form $g = \begin{pmatrix} g_1 & X \\ 0 & g_2 \end{pmatrix} \in P$, with $g_j \in GL_{n_j}(F)$, $j = 1$, 2, and $X \in M_{n_1\times n_2}(F)$. Let dg_j be Haar measure on $GL_{n_j}(F)$, and let dX be Haar measure on $M_{n_1\times n_2}(F) \simeq F^{n_1n_2}$. Show that $d_\ell g = |\det g_2|^{-n_2}dg_1\, dg_2\, dX$ and $d_r g = |\det g_2|^{-n_1}dg_1\, dg_2\, dX$ are left and right Haar measures on P (respectively). Here, $|\cdot|$ is the usual absolute value if $F = \mathbb{R}$

and $|\cdot| = |\cdot|_p$ if $F = \mathbb{Q}_p$. Hence the modular quasicharacter of P is equal to $\Delta(g) = |\det g_1|^{n_1} |\det g_2|^{-n_2}$.

(5) Show that the homeomorphism $g \mapsto g^{-1}$ turns μ_ℓ into a right Haar measure. Conclude that if G is unimodular, then $\int_G f(g)\, d_\ell g = \int_G f(g^{-1}) d_\ell g$ for all integrable functions f on G.

Proposition 4.14. *If G is compact, then G is unimodular and $\mu_\ell(G) < \infty$.*

Proof. It follows from Proposition 4.11(2) that G is unimodular. Haar measure on any locally compact group has the property that any compact subset has finite measure. Hence $\mu_\ell(G) < \infty$ whenever G is compact. □

If G is compact, *normalized* Haar meaure on G is the unique Haar measure μ on G such that $\mu(G) = 1$. When working with compact groups, we will usually work relative to normalized Haar measure.

Exercise 4.15. Prove that if G is such that the measure of G (with respect to left Haar measure) is finite, then G is compact. (*Hint*: Assume that G is noncompact. Let C be a compact set in G containing an open neighbourhood of 1. Let U be an open neighbourhood of 1 such that $g^{-1} \in U$ for all $g \in U$ and $gg' \in C$ for all $g,\ g' \in U$. Show that there exist elements $g_j \in G$, $j \in \mathbb{N}$, such that $g_i U \cap g_j U = \emptyset$ whenever $i \neq j$. Use this to show that the measure of G is infinite.)

5. Coset spaces and quasi-invariant measures

The most important method of constructing representations of locally compact groups is the procedure of inducing representations from subgroups. Let H be a closed (not necessarily normal) subgroup of a Hausdorff locally compact group G. If π is a representation of H, then the induced representation $\mathrm{Ind}_H^G \pi$ is a representation of G. When π is unitary, the representation $\mathrm{Ind}_H^G \pi$ should also be unitary. In order to achieve this, the measure-theoretic relations between G, H and G/H must be taken into account. In particular, the notion of quasi-invariant measure on the set of left cosets G/H (with the quotient topology) is key.

Let G/H be the set of left cosets of H in G, with the quotient topology. Fix a left Haar measure $\mu_{\ell,H}$ on H. For $f \in C_c(G)$, Define $f^H(g) = \int_H f(gh)\, d_{\ell,H}(h)$, for $g \in G$. By left-invariance of the Haar measure on H, $f^H(gh) = f^H(g)$ for all $h \in H$ and $g \in G$. Therefore, there is a unique function $f^\sharp$ on G/H such that $f^\sharp(gH) = f^H(g)$ for all $g \in G$.

Lemma 5.1. *If $f \in C_c(G)$, then $f^\sharp \in C_c(G/H)$.*

Proof. Fix $f \in C_c(G)$ and $g_0 \in G$. Choose an open neighbourhood U of 1. Since G is locally compact, we may assume that $\overline{U}$ is compact. Let $\epsilon > 0$. Because f is uniformly continuous and G is locally compact, there exists an open neighbourhood U' of 1 such that $U' \subset U$ and and $|f(ug) - f(g)| \leq \epsilon$ for all $u \in U'$ and $g \in G$.

Let $u \in U'$. Then $f(ug_0h) = 0$ and $f(g_0h) = 0$ whenever $h \in H \setminus (H \cap g_0^{-1}\overline{U}^{-1}\operatorname{supp} f)$. Since $H \cap g_0^{-1}\overline{U}^{-1}\operatorname{supp} f$ is a compact subset of H, continuity of Haar measure on H implies that there exists a nonnegative constant c (which we can take to be the measure of the set with respect to the chosen Haar measure) such that

$$|f^\sharp(ug_0H) - f^\sharp(uH)| = \left| \int_H (f(ug_0h) - f(g_0h))\, d_{\ell,H}h \right| \leq \epsilon\, c.$$

It follows that $f^\sharp$ is continuous.

Since $\operatorname{supp} f^\sharp \subset q(\operatorname{supp} f)$, the function $f^\sharp$ is compactly supported. □

Lemma 5.2. *Let $q : G \to G/H$ be the canonical map. If C is a compact subset of G/H, then there exists a compact subset C' of G such that $q(C') = C$.*

Proof. Let C be a compact subset of G/H. Since G is locally compact, there exists an open neighbourhood U of the identity in G such that $\overline{U}$ is compact. Since q is open, $q(U)$ is open in G/H, and so are the translates $g\cdot q(U)$ for $g \in G$. The sets $\{\, g\cdot q(U) \mid g \in G \,\}$ form an open cover of C. By compactness of C, there exists a finite subcover $\{\, g_i \cdot q(U) \mid 1 \leq i \leq n \,\}$. We know that $q^{-1}(C)$ is closed in G and, since G is Hausdorff, $g_1\overline{U} \cup \cdots \cup g_n\overline{U}$ is compact in G. The set $C' := q^{-1}(C) \cap (g_1\overline{U} \cup \cdots \cup g_n\overline{U})$ is compact and satisfies $q(C') = C$. □

Proposition 5.3. *The mapping $f \mapsto f^\sharp$ is onto $C_c(G/H)$. An element of $C_c^+(G/H)$ is of the form $f^\sharp$ for some $f \in C_c^+(G)$.*

Proof. Fix $F \in C_c(G/H)$. Let $C = \operatorname{supp} F$. Fix a compact subset C' of G such that $q(C') = C$. Choose $\varphi \in C_c(G)$ such that $\varphi(g) = 1$ for all $g \in C'$ and $0 \leq \varphi(g) \leq 1$ for all $g \in G$. (This uses Urysohn's Lemma – see Theorem 2.1 of [**K**].) Then $\varphi^\sharp > 0$ on C.

Define a function ψ on G/H by

$$\psi(gH) = \begin{cases} F(gH)/\varphi^\sharp(gH) & \text{if } gH \in C \\ 0 & \text{if } gH \notin C. \end{cases}$$

Then $\psi \in C_c(G/H)$. For $g \in G$, let $f(g) = \psi(gH)\varphi(g)$. Then $f \in C_c(G)$ and $f \geq 0$ if $F \geq 0$. If $g \in G$, then

$$f^\sharp(gH) = \int_G \psi(ghH)\varphi(gh)\, d_{\ell,H}h = \psi(gH)\varphi^\sharp(gH) = F(gH).$$

□

Given a bounded, continuous, complex-valued function f on G, let $\|f\|_\infty = \sup_{g\in G} |f(g)|$. ($\|f\|_\infty$ is called the *supremum norm* of f). If K is a compact subset of G that contains an open neighbourhood of 1, let $C_K(G) = \{\, f \in C_c(G) \mid \operatorname{supp} f \subset K \,\}$. We don't need the next result at the moment, but it may be used later.

PROPOSITION 5.4. *Let K be a compact subset of G that contains an open neighbourhood of 1. Then there exists a positive constant c_K such that $\|f^\sharp\|_\infty \leq c_K\|f\|_\infty$, for all $f \in C_K(G)$.*

PROOF. Fix $f \in C_c(G)$ with $\operatorname{supp} f \subset K$. Note that $f^\sharp(gH) = 0$ unless $g \in KH$. Assume that $g \in K$ and $h \in H$. If $gh \in \operatorname{supp} f$, then $h \in (K^{-1}K) \cap H$. But $(K^{-1}K) \cap H$ is compact, so has finite Haar measure. Let c_K be the H-Haar measure of $(K^{-1}K) \cap H$. Then, for $g \in K$,

$$|f^\sharp(gH)| \leq \int_H |f(gh)|\, d_{\ell,H}h \leq c_K\|f\|_\infty.$$

It follows that $\|f^\sharp\|_\infty \leq c_K\|f\|_\infty$. □

Define $(L_g\varphi)(g_0H) = \varphi(g^{-1}g_0H)$ for $\varphi \in C_c(G/H)$ and g, $g_0 \in G$. Note that $(L_gf)^\sharp = L_g(f^\sharp)$ for all $f \in C_c(G)$ and $g \in G$.

PROPOSITION 5.5. *Let Δ_G and Δ_H be the modular functions on G and H, respectively. If $f \in C_c(G)$, define*

$$\rho_f(g) = \int_H \Delta_G(h)\Delta_H(h)^{-1} f(gh)\, d_{\ell,H}(h), \qquad g \in G.$$

Then ρ_f is continuous and $\rho_f(gh) = \Delta_H(h)\Delta_G(h)^{-1}\rho_f(g)$ for $g \in G$ and $h \in H$. If $f \in C_c^+(G)$, then $\rho_f(g) \geq 0$ for all $g \in G$.

PROOF. Continuity of ρ_f follows from the fact that f has compact support. The transformation property follows from a change of variables. □

Functions of the form ρ_f are used to transfer integration between G and G/H.

PROPOSITION 5.6. *Let ρ be a locally integrable function on G which satisfies $\rho(gh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(g)$ for $g \in G$ and $h \in H$. If $f \in C_c(G)$ is such that $f^\sharp \equiv 0$, then*

$$\int_G f(g)\,\rho(g)\,d_\ell g = 0.$$

PROOF. The set $C := q(\operatorname{supp} f)$ is a compact subset of G/H. Choose a function in $C_c(G/H)$ that takes the value 1 on C. By Proposition 5.3, this function is of the form $\varphi^\sharp$ for some $\varphi \in C_c(G)$. In particular, $\varphi^H(g) = 1$ for all $g \in \operatorname{supp} f$. By assumption $f^H(g) = 0$ for all $g \in G$. So we have $\int_G \rho(g)\varphi(g)f^H(g)\,d_\ell g = 0$. That is

$$0 = \int_G \int_H \rho(g)\,\varphi(g)\,f(gh)\,d_{\ell,H}h\,d_\ell g.$$

Consider the function $(g,h) \mapsto \rho(g)\varphi(g)f(gh)$ on $G \times H$. If the value at (g,h) is nonzero, then $g \in \operatorname{supp}\varphi$ and $h \in g^{-1}\operatorname{supp} f$. This implies (g,h) lies in the compact set

$$\operatorname{supp}\varphi \times ((\operatorname{supp}\varphi)^{-1}\operatorname{supp} f) \cap H.$$

That is, the function is compactly supported. Since the function is integrable on $G \times H$ and vanishes outside a set of finite measure, we can use Fubini's Theorem.

Below we apply Fubini's theorem, replace g by gh^{-1}, apply Fubini's theorem again, make the change of variables $h \mapsto h^{-1}$, use properties of ρ, and the fact that $\phi^H(g) = 1$ for all $g \in \operatorname{supp} f$:

$$\begin{aligned}
0 &= \int_G \int_H \rho(g)\varphi(g)f(gh)d_{\ell,H}h\,d_\ell g = \int_H \int_G \rho(g)\varphi(g)f(gh)d_\ell g\,d_{\ell,H}h \\
&= \int_H \int_G \rho(gh^{-1})\varphi(gh^{-1})f(g)\Delta_G(h^{-1})d\mu_\ell g\,d_{\ell,H}h \\
&= \int_G \int_H \Delta_G(h)\Delta_H(h^{-1})\rho(gh)\varphi(gh)f(g)d_{\ell,H}h\,d_\ell g \\
&= \int_G \rho(g)f(g)\varphi^\sharp(gH)d_\ell g = \int_G \rho(g)f(g)\,d_\ell g
\end{aligned}$$

□

Given ρ a measurable function on G such that $\rho \geq 0$ and $\rho(gh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(g)$ for $h \in G$ and $g \in G$, we can define a linear functional λ_ρ on $C_c(G/H)$ by $\lambda_\rho(f^\sharp) = \int_G f(g)\rho(g)\,d_\ell g$, $f \in C_c(G/H)$. Note that we can apply Proposition 5.6 to see that λ_ρ is well-defined.

Proposition 5.7. *Let ρ be as above. Then there exists a regular Borel measure μ_ρ on G/H such that*

$$\int_{G/H} f^\sharp(\bar{g}) d\mu_\rho \bar{g} = \int_G f(g)\rho(g)\, d_\ell g, \qquad f \in C_c(G).$$

Proof. According to Proposition 5.3, given $\varphi \in C_c^+(G/H)$, there exists $f \in C_c^+(G)$ such that $f^\sharp = \varphi$. Note that $\lambda_\rho(\varphi) = \lambda_\rho(f^\sharp) = \int_G f(g)\rho(g) d_\ell g \geq 0$, since $f \in C_c^+(G)$. That is, λ_ρ is a positive linear functional. Applying the Riesz Representation Theorem, λ_ρ is given by a regular Borel measure μ_ρ on G/H. (The Riesz Representation Theorem holds for locally compact Hausdorff spaces. It says that each positive linear functional is given by integration against a positive measure.) □

Theorem 5.8. *There exists a nonzero positive G-invariant regular Borel measure on G/H if and only if $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. When it exists, such a measure $d\bar{g}$ is unique up to positive scalar multiples and can be normalized so that it satisfies*

$$\int_{G/H} \int_H f(gh)\, d_{\ell,H} h\, d\bar{g} = \int_G f(g)\, d_\ell g, \qquad f \in C_c(G).$$

Proof. Assume that $\Delta_G \,|\, H = \Delta_H$. We may apply Proposition 5.7 with $\rho(g) = 1$ for all $g \in G$. Let $d\bar{g} = d\mu_\rho \bar{g}$.

If $x \in G$, then

$$\begin{aligned}\int_{G/H} f^\sharp(x^{-1} \cdot \bar{g})\, d\bar{g} &= \int_{G/H} (L_x f)^\sharp(\bar{g})\, d\bar{g} = \int_G f(x^{-1}g)\, d_\ell g \\ &= \int_G f(g)\, d_\ell g = \int_{G/H} f^\sharp(\bar{g})\, d\bar{g}, \qquad f \in C_c(G).\end{aligned}$$

This shows that $d\bar{g}$ is G-invariant.

For the converse, assume that $d\bar{g}$ is a nonzero positive invariant measure on G/H. Set $\lambda(f) = \int_{G/H} f^\sharp(\bar{g})\, d\bar{g}$ for $f \in C_c(G)$. Then, if $x \in G$, we can show that $\lambda(L_x f) = \lambda(f)$ (similar to the above calculation). So λ is a nonzero positive left-invariant linear functional on $C_c(G)$. By uniqueness of the left Haar measure on G, there exists a positive c such that

$$\int_{G/H} f^\sharp(\bar{g})\, d\bar{g} = c \int_G f(g)\, d_\ell g, \qquad f \in C_c(G).$$

Choose $f \in C_c(G)$ such that $\int_G f(g)\, d_\ell g = 1$ Fix $h \in H$. Set $\varphi_h(g) = \Delta_H(h) f(gh)$, $g \in G$. Then $\varphi_h^\sharp = f^\sharp$ (details omitted). Therefore

$$\begin{aligned}
\Delta_H(h)\Delta_G(h)^{-1} - 1 &= \int_G (f(g)\Delta_H(h)\Delta_G(h^{-1}) - f(g))\, d_\ell g \\
&= \int_G (f(gh)\Delta_H(h) - f(g))\, d_\ell g \\
&= \int_G \varphi_h(g)\, d_\ell g - \int_G f(g)\, d_\ell g \\
&= c^{-1} \int_{G/H} (\varphi_h^\sharp(\bar{g}) - f^\sharp(\bar{g}))\, d\bar{g} = 0.
\end{aligned}$$

□

REMARK 5.9. Let H be a closed normal subgroup of G. Then $\Delta_H = \Delta_G \,|\, H$. Let $d\bar{g}$ be a left Haar measure on G/H. Then a left Haar measure on G/H is left G-invariant, so $f \mapsto \int_{G/H} f^\sharp(\bar{g})\, d\bar{g}$ defines a left Haar measure on G. Fix $h_0 \in H$. Replace f by the function $g \mapsto f(gh_0)$. Examining the value of the integral of this new function using the above realization of left Haar measure on G, we find that $\Delta_H(h) = \Delta_G(h)$.

If μ is a measure on G/H and $x \in G$, define a measure μ_x by $\mu_x(S) := \mu(x \cdot S)$ for any measurable subset S of G/H. If μ is a regular Borel measure on G/H, we say that μ is *quasi-invariant* if μ is nonzero and $\mu_x \sim \mu$ for all $x \in G$. Here $\sim$ denotes mutual absolute continuity of measures (essentially, μ and μ_x have the same sets of measure zero).

LEMMA 5.10. *There exists a continuous function ρ on G such that $\rho(g) > 0$ for all $g \in G$ and $\rho(gh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(g)$ for $h \in G$.*

Let ρ be as in Lemma 5.10 and let $\mu = \mu_\rho$. For $x \in G$, define μ_x by $x \cdot \mu(S) = \mu(xS)$ for any Borel set $S \subset G/H$. It is easy to see that $L_x\rho$ satisfies $(L_x\rho)(gh) = \Delta_H(h)\Delta_G(h)^{-1}(L_x\rho)(g)$ for all $g \in G$ and $h \in H$ and $\mu_{L_x\rho} = (\mu_\rho)_{x^{-1}}$. Because the function $y \mapsto \rho(xy)/\rho(y)$ is right H-invariant, we can view it as a function on G/H:

$$\sigma(x, q(y)) = \rho(xy)/\rho(y), \qquad x,\, y \in G$$

The function $(x, \bar{y}) \mapsto \sigma(x, \bar{y})$ is a continuous, strictly positive function on $G \times G/H$. We can easily see that

$$\sigma(x, \bar{y})\sigma(z, x \cdot \bar{y}) = \sigma(zx, \bar{y}), \qquad \bar{y} \in G/H,\, x,\, z \in G. \tag{5.1}$$

If $f \in C_c(G)$ and $x \in G$, then

$$\int_{G/H} f^\sharp(\bar{g})\, d\mu_x(\bar{g}) = \int_G f(y)\,(L_{x^{-1}}\rho)(y)\, d_\ell y = \int_G f(y)\,\sigma(x, q(y))\,\rho(y)\, dy$$
$$= \int_{G/H} f^\sharp(\bar{g})\,\sigma(x, \bar{y})\, d\mu(\bar{g}).$$

Applying Proposition 5.3, we have

$$\int_{G/H} \varphi(\bar{g})\, d\mu_x(\bar{g}) = \int_{G/H} \varphi(\bar{g})\,\sigma(x, \bar{y})\, d\mu(\bar{g}), \qquad \varphi \in C_c(G/H).$$

THEOREM 5.11. *Let ρ be as in Lemma 5.10. Then the measure μ_ρ (see Proposition 5.7) is quasi-invariant, the Radon-Nikodym derivative $\sigma(x, \bar{y}) = [d\mu_x/d\mu]\,(\bar{y})$ is continuous and satisfies equation (5.1). Furthermore,*

$$\int_G f(g)\rho(g)\, d_\ell g = \int_{G/H} \int_H f(gh) d_{\ell,H} h\, d\mu_\rho(gH) = \int_{G/H} f^\sharp(gH)\, d\mu_\rho(gH), \qquad f \in C_c(G).$$

Quasi-invariant measures on G/H are not unique up to scalar multiples, but any two of them are mutually absolutely continuous. For more details, see Section 1.3 of [**KT**].

EXAMPLE 5.12. (Exercise 1 (f)) from p.126 of [**R**]) Let $G = GL_2(\mathbb{R})$ and let H be the group of upper triangular matrices in G. Because G is unimodular and H is not unimodular, there is no G-invariant measure on G/H. The quotient G/H can be identified with the projective line. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad \neq -bc$, the coset gH is identified with the line generated by (a, c) in $\mathbb{R}^2$, or $x = a/c \in \mathbb{R} \cup \{\infty\}$. Show that Lebesgue measure on $\mathbb{R}$ gives a quasi-invariant measure on G/H.

CHAPTER 2

Continuous representations of locally compact groups

From now on, G will denote a locally compact Hausdorff group.

1. Definition of continuous representation

For V be a normed complex vector space, let $\mathrm{Aut}(V)$, $\mathrm{B}(V)$ and $\mathrm{GL}(V)$ be the set of invertible linear operators on V, the set of bounded linear operators on V, and the set of invertible elements in $\mathrm{B}(V)$ which have bounded inverses, respectively.

DEFINITION 1.1. If V is a vector space, (π, V) is an *abstract representation (of G)* if π is a homomorphism from G to $\mathrm{Aut}(V)$. If V is finite-dimensional, say $n = \dim(V)$, we say that π is finite-dimensional, or n-dimensional. When V is infinite-dimensional, we say that π is infinite-dimensional.

DEFINITION 1.2. If V is a complex normed vector space, (π, V) is a *continuous representation (of G)* if $\pi : G \to GL(V)$ is a homomorphism and the map $G \times V \to V$ is continuous.

EXAMPLE 1.3. Let π be a one-dimensional representation of $\mathbb{R}$. Then $\pi(0) = 1$, $\pi(t_1 + t_2) = \pi(t_1) + \pi(t_2)$ for all t_1 and $t_2 \in \mathbb{R}$, and $\pi(-t) = \pi(t)^{-1}$ for all $t \in \mathbb{R}$. Fix $t_0 \in \mathbb{R}$. Choose a function f on $\mathbb{R}$ such that f is infinitely differentiable, f is zero outside a neighbourhood of t_0, and there is a nonzero constant c such that $\int_{-\infty}^{\infty} f(t)\pi(t)\, dt = c \neq 0$. (Here, we have used that $\pi(t) \neq 0$ for all t.) Observe that

$$\int_{-\infty}^{\infty} \pi(t_1 + t_2) f(t_2)\, dt_2 = \pi(t_1) \int_{-\infty}^{\infty} \pi(t_2) f(t_2)\, dt_2 = \pi(t_1)\, c,$$

and

$$\pi(t_2) = c^{-1} \int_{-\infty}^{\infty} \pi(t_1 + t_2) f(t_2)\, dt_2 = c^{-1} \int_{-\infty}^{\infty} \pi(t_2) f(t_2 - t_1)\, dt_2.$$

The last integral is infinitely differentiable. Combining this with the property $\pi(t_1 + t_2) = \pi(t_1) + \pi(t_2)$, we have that π is a solution of the differential equation $\phi'(t) = z\,\phi(t)$ for some complex constant z. It follows that $\pi(t) = e^{zt}$. Note that π is unitary if and only if z is purely imaginary.

The proof of the next proposition uses the fact that the Banach-Steinhaus Theorem holds for normed barrelled spaces (or, more generally, locally convex barrelled spaces). Given a normed barrelled space V and a nonempty family $\mathcal{F}$ in $\mathrm{B}(V)$, pointwise boundedness of the family implies that the family is equicontinuous: If for each $v \in V$, $\sup\{\|T(v)\| \mid T \in \mathcal{F}\} < \infty$, then $\sup\{\|T\|_{op} \,|\, T \in \mathcal{F}\} < \infty$. Since G is locally compact, this can be applied for $\mathcal{F} = \{\, \pi(g) \mid g \in U \,\}$, where U is a compact neighbourhood of the identity.

PROPOSITION 1.4. *Let V be a normed vector space that is barrelled. Suppose that π is a homomorphism from G to* $\mathrm{Aut}(V)$*. Then (π, V) is a continuous representation of G if and only if*

(1) *The map $g \mapsto \pi(g)v$ is continuous for every $v \in V$.*
(2) *The map $v \mapsto \pi(g)v$ is continuous for every $g \in G$.*

REMARK 1.5. More generally, we can assume that V is a locally convex barrelled topological vector space. In a locally convex vector space, the topology is determined by a family of seminorms.

We can define various topologies on $\mathrm{B}(V)$. Let $V^* = \mathrm{B}(V, \mathbb{C})$ be the space of bounded (that is, continuous) linear functionals on V.

The *weak operator topology* on $\mathrm{B}(V)$ is the topology induced by the seminorms $T \mapsto |\lambda(T(v))|$, $v \in V$, $\lambda \in V^*$. If $\epsilon > 0$, $\lambda \in V^*$ and $v \in V$, the set $\{\, T \in \mathrm{B}(V) \mid |\lambda(T(v))| < \epsilon \,\}$ is an open neighbourhood of zero. (Of course, if V is a Hilbert space, then $\lambda \in V^*$ can be realized as $v \mapsto \langle v, w \rangle$ for some fixed $w \in V$.)

The *strong operator topology* on $\mathrm{B}(V)$ is the topology induced by the seminorms $T \mapsto \|T(v)\|$, $v \in V$. If $\epsilon > 0$ and $v \in V$, then $\{\, T \in \mathrm{B}(V) \mid \|T(v)\| < \epsilon \,\}$ is an open neighbourhood of zero. This topology has the property that each function $T \mapsto T(v)$ from $\mathrm{B}(V)$ to V is continuous for all $v \in V$.

In the *operator norm topology* (or *uniform topology*) on $\mathrm{B}(V)$, for $\epsilon > 0$, the set $\{\, T \in \mathrm{B}(V) \mid \|T\|_{op} < \epsilon \,\}$ is an open neighbourhood of zero.

If V is finite-dimensional, all three of the above topologies coincide. In the infinite-dimensional setting, the operator norm topology is stronger than the strong operator topology, which in turn is stronger than the weak operator topology.

Lemma 1.6. *Let (π, V) be a continuous representation of G. Then the function $g \mapsto \pi(g)$ from G to $\mathrm{GL}(V)$ is continuous when $\mathrm{GL}(V)$ is given the subspace topology arising from the strong operator topology on $\mathrm{B}(V)$.*

Proof. Fix $g_0 \in G$. Given $\epsilon > 0$ and $v_0 \in V$, the set $U_{g_0,v_0,\epsilon} := \{\, T \in \mathrm{GL}(V) \mid \|(T - \pi(g_0))(v_0)\| < \epsilon \,\}$ is an open neighbourhood of $\pi(g_0)$ in $\mathrm{GL}(V)$. By assumption, the function $g \mapsto \pi(g)v_0$ from G to V is continuous. Therefore, there exists an open neighbourhood U of g_0 in G such that if $g \in U$, then $\|\pi(g)v_0 - \pi(g_0)v_0\| < \epsilon$. That is, for $g \in U$, $\pi(g)$ belongs to the set $U_{g_0,v_0,\epsilon}$. Thus the function $g \mapsto \pi(g)$ is continuous at g_0. □

Note that when V is infinite-dimensional, $\mathrm{GL}(V)$ is not a topological group with respect to the strong operator topology. However, when V is finite-dimensional, the continuous function $g \mapsto \pi(g)$ is also a continuous homomorphism. function. In the infinite-dimensional setting, $\mathrm{GL}(V)$ is a topological group in the uniform topology. But very few continuous representations (π, V) have the property that the function $g \mapsto \pi(g)$ is continuous when $\mathrm{GL}(V)$ has the norm topology. In particular, as shown in [**Sg**], the trivial representation is the only uniformly continuous unitary representation of a complex semisimple Lie group, and all topologically irreducible uniformly continuous representations of real Lie groups are finite-dimensional. See [**Ka**] for information about characterizations of uniformly continuous unitary representations of connected locally compact groups.

Exercise 1.7. Let $G = \mathbb{R}/\mathbb{Z}$ (or, equivalently, $G = SO_2(\mathbb{R})$).

a) Let $V = C(G)$ be the space of continuous complex-valued functions on G, equipped with the supremum norm: $\|f\|_\infty = \sup_{g\in G} |f(g)|$. Then V is a Banach space. The right regular representation ρ of G on V is given by $(\rho(t)f)(x) = f(x+t)$, $f \in V$, x, $t \in G$. Show that the homomorphism $t \mapsto \rho(t)$ from G to $\mathrm{GL}(V)$ is not continuous when $\mathrm{GL}(V)$ is given the operator norm topology. (*Hint*: Viewing G as the interval $[0, 1]$ with its endpoints identified, suppose that $0 < t < 1/2$. Take $f_t \in V$ such that f_t is supported in $[0, t]$, $0 \le f_t(x) \le 1$, and $f_t(t/2) = 1$. Note that $\|f_t\|_\infty = 1$. Show that $\|\rho(t)f_t - f_t\|_\infty = 1$. This implies that $\|\rho(t) - I\|_{op} \ge 1$, where I is the identity operator on V.)
b) Let $W = L^2(G)$. Let ρ be the right regular representation of G on W. Show that the homomomorphism $t \mapsto \rho(t)$ from G to the group of unitary operators $U(W)$ on W is not continuous when $U(W)$ is given the operator norm topology.

Proposition 1.8. *Let V be a Hilbert space and let $U(V)$ be the unitary subgroup of* $\mathrm{GL}(V)$*. If $\pi : G \to U(V)$ is an abstract homomorphism and for each $v \in V$, the map $g \mapsto \langle v, \pi(g)v\rangle$ is continuous at the identity, then (π, V) is a unitary representation of G.*

It follows from the above proposition that if $g \mapsto \pi(g)v$ is continuous in the weak topology, then it is continuous in the strong topology. The converse follows from the Cauchy-Schwarz inequality: Given u, $v \in V$,

$$|\langle \pi(g)u, v\rangle - \langle u, v\rangle| \leq \|\pi(g)u - u\|\|v\|.$$

2. Representations of compact groups are unitary

Proposition 2.1. *Let (π, V) be a continuous representation of a compact group in a Hilbert space V. Then there exists a G-invariant positive-definite Hermitian form on V and defines the same topology on V.*

Proof. (The general idea). Define $\varphi(v, w) = \int_G \langle \pi(g)v, \pi(g)w\rangle \, dg$, v, $w \in G$. Show that φ is G-invariant, positive, hermitian. Show that the norm $v \mapsto \varphi(v, v)^{1/2}$ is equivalent to $\|\cdot\|$ are equivalent norms – this involves applying the Banach-Steinhaus Theorem to see that $g \mapsto \|\pi(g)\|_{op}$ is bounded above. □

3. Irreducibility, subrepresentations, etc.

If (π, V) is a representation of G and W is a subspace of V, we say that V is G-invariant if $\pi(g)W \subset W$ for all $g \in G$.

Definition 3.1. An abstract representation (π, V) of a group G is *algebraically irreducible* if there does not exist a nonzero proper G-invariant subspace of V.

Definition 3.2. A continuous representation (π, V) is *topologically irreducible* if there does not exist a nonzero proper closed G-invariant subspace of V.

Recall that every subspace of a finite-dimensional normed space is closed. Hence for continuous finite-dimensional representations, the notions of algebraic irreducibility and topological irreducibility coincide.

It is easy to show that if V is an normed vector space, $A \in B(V)$, and W is a subspace of V such that $A(W) \subset W$, then $A(\overline{W}) \subset \overline{W}$.

Lemma 3.3. *If (π, V) is a continuous representation of G and W is a G-invariant subspace of V, then $\overline{W}$ is a G-invariant subspace.*

DEFINITION 3.4. Let (π, V) be a continuous representation of G. If W is a closed subspace of V, we can define a continuous representation (π_W, W) of G, by taking $\pi_W(g)$ to be the restriction of $\pi(g)$ to W, for $g \in G$. In this case, (π_W, W) is called a *subrepresentation* of (π, V). If (π_W, W) is a subrepresentation of G, we can define a continuous representation $(\pi_{V/W}, V/W)$ of G on the quotient space V/W (which is also a normed space, since W is closed in V) by $\pi_{V/W}(g)(v + W) = \pi(g)v + W$, $g \in G$, $v \in V$. The representation $(\pi_{V/W}, V/W)$ is called a *quotient (representation)* of (π, V). A representation of G that occurs as $(\pi_{W_2/W_1}, W_2/W_1)$ where $W_1 \subset W_2$ are closed G-invariant subspaces of V is called a *subquotient* of (π, V).

DEFINITION 3.5. A representation (π, V) is *pre-unitary* if V is an inner product space and $\langle \pi(g)v_1, \pi(g)v_1 \rangle = \langle v_1, v_2 \rangle$ for all $g \in G$ and v_1, $v_2 \in V$. If V is a Hilbert space and π is continuous and pre-unitary, we say that π is *unitary.*

DEFINITION 3.6. If (π_i, V_i), $1 \leq i \leq n$ are continuous representations of G, we may define a continuous representation $(\oplus_i \pi, \oplus V_i)$ where $(\oplus \pi_i)(g)(v_1, \ldots v_n) = (\pi_1(g)v_1, \ldots, \pi_n(g)v_n)$. Here, we use the norm

$$\|(v_1, \ldots, v_n)\| = \left(\sum_{i=1}^{n} \|v_i\|_i^2 \right)^{1/2}$$

on $\oplus_i V_i$, where $\| \cdot \|_i$ is the norm on V_i, $1 \leq i \leq n$. The representation $(\oplus \pi_i, \oplus_i V_i)$ is called the *direct sum* of the representations (π_i, V_i), $1 \leq i \leq n$. We may also define direct sums of infinite collections of representations. For example, suppose that (π_i, V_i), $i \in I$ (I some index set) is a collection of continuous unitary representations of G. Let $\langle \cdot, \cdot \rangle_i$ be the inner product on V_i. Let

$$\oplus_{i \in I} V_i := \{ \, (v_i)_{i \in I} \in \prod_{i \in I} V_i \mid \sum_{i \in I} \|v_i\|_i^2 < \infty \, \}.$$

Define $\langle (v_i), (w_i) \rangle = \sum_{i \in I} \langle v_i, w_i \rangle_i$. This defines an inner product on V that makes $\oplus_i V_i$ into a Hilbert space. Set $(\oplus \pi_i)(g)(v_i) = (\pi_i(g)v_i)$, $g \in G$, $(v_i) \in V$. Then the representation $(\oplus \pi_i, \oplus V_i)$ is unitary. Representations of the form $(\oplus \pi_i, \oplus V_i)$ are said to be *completely reducible.*

If W is a closed G-invariant subspace of a Hilbert space V, then W is a Hilbert space. Hence a subrepresentation of a unitary representation is a unitary representation.

Lemma 3.7. *Let (π, V) be a continuous unitary representation of G. A closed subspace W of V is G-invariant if and only if the orthogonal complement $W^\perp$ of W is G-invariant. In this case (π, V) is the direct sum of the subrepresentations (π_W, W) and $(\pi_{W^\perp}, W^\perp)$.*

Corollary 3.8. *A finite-dimensional unitary representation of G is completely reducible.*

Recall that we have shown (see Proposition 2.1) that a representation of a compact group in a Hilbert space is unitary.

Corollary 3.9. *A finite-dimensional continuous representation of a compact group is completely reducible.*

Definition 3.10. If (π, V) is a continuous representation of G, let $V^* = \mathrm{B}(V, \mathbb{C})$ be the space of continuous linear functionals on V. Given $\lambda \in V^*$ and $v \in V$, the continuous function $g \mapsto \lambda(\pi(g)v)$ is called a *matrix coefficient* of π. If V is a Hilbert space, a matrix coefficient of π has the form $g \mapsto \langle \pi(g)v, w\rangle$, where v and w are fixed vectors in V.

Exercise 3.11. Let (π, V) be a unitary representation of G. Prove that π is topologically irreducible if and only if the the matrix coefficient $g \mapsto \langle\, \pi(g)v, w\,\rangle$ is not identically zero for all nonzero vectors v and w in V.

Example 3.12. Consider the two-dimensional representation $(\pi, \mathbb{C}^2)$ of $G = \mathbb{R}$ defined by $\pi(x)(z_1, z_2) = (z_1 + xz_2, z_2)$, $x \in G$, z_1, $z_2 \in \mathbb{R}$. The subspace $W := \mathrm{Span}\{(1, 0)\}$ is the unique one-dimensional invariant subspace of $\mathbb{C}^2$. In particular, there is no G-invariant complementary subspace, so π is not completely reducible.

4. The regular representation on $L^2(G/H)$

Let H be a closed subgroup of G. Let μ be a quasi-invariant measure on G/H. This measure has the property that

$$\int_{G/H} f(\bar{g}) d\mu(x \cdot \bar{g}) = \int_{G/H} \sigma(x, \bar{g})\, f(\bar{g})\, d\bar{g}, x \in G,\ f \in C_c(G/H),$$

where σ is a nonnegative continuous function on $G \times G/H$ that satisfies

$$\sigma(y, \bar{g})\sigma(x, y \cdot \bar{g}) = \sigma(xy, \bar{g}), \qquad x,\, y \in G,\ \bar{g} \in G/H,$$

and $\sigma(x, yH) = \rho(xy)/\rho(y)$, x, $y \in G$, where ρ is a strictly positive continuous function as in Lemma 5.10.

If $\Delta_G \,|\, H = \Delta_H$, we will take μ to be a G-invariant measure on G/H. (In this case, $\sigma(x, \bar{g}) = 1$ for all $x \in G$ and $\bar{g} \in G/H$.)

For $x \in G$, $\bar{g} \in G/H$ and $f \in L^2(G/H, \mu)$, set

$$(\pi(x)f)(\bar{g}) = \sqrt{\sigma(x^{-1}, \bar{g})}\, f(x^{-1} \cdot \bar{g}).$$

It is a simple matter to check that $\|\pi(x)f\| = \|f\|$ and $\pi(xy)f = \pi(x)\pi(y)f$ for $f \in L^2(G/H, \mu)$ and $x, y \in G$. We refer to π as the *left regular representation* of G on $L^2(G/H, \mu)$.

Proposition 4.1. *Let π be as above. Then π is a continuous unitary representation of G.*

More will be added later. For the moment, we only mention that given $\epsilon > 0$ and $f \in L^2(G/H, \mu)$, there exists $\varphi \in C_c(G/H)$ such that $\|f - \varphi\|_2 < \epsilon/3$. Using the fact that each $\pi(x)$ is an isometry, we can show that

$$\|\pi(x)f - f\|_2 < \|\pi(x)\varphi - \varphi\|_2 + 2\epsilon/3, \qquad x \in G.$$

Consider the map $x \mapsto L_x\varphi$ from G to $C_c(G/H)$ given by $(L_x\varphi)(\bar{g}) = \varphi(x^{-1} \cdot \bar{g})$, $x \in G$, $\bar{g} \in G/H$. If $C_c(G/H)$ is given the topology associated to the sup norm, this map is continuous (details to be added). Combining this fact with continuity of σ, we can show that there exists an open neighbourhood U of 1 in G such that $\|\pi(x)\varphi - \varphi\|_2 < \epsilon/3$ for all $x \in U$.

In the special case where $H = \{1\}$, we have the left regular representation of G on $L^2(G)$, where we use a left Haar measure on G. For x and y in G and $f \in L^2(G)$, set $(\rho(x)f)(y) = \Delta_G(x)^{1/2} f(yx)$. Then ρ is a continuous unitary representation of G, called the *right regular representation* of G.

Exercise 4.2. Let $X = \mathrm{GL}_2(\mathbb{R})/O_2(\mathbb{R})$, where

$$O_2(\mathbb{R}) = \{\, g \in GL_2(\mathbb{R}) \mid (gv, gw) = (v, w)\ \forall\, v, w \in \mathbb{R}^2 \,\},$$

where $(\cdot, \cdot)$ is the standard inner product. Consider the space $L^\infty(X)$ of essentially bounded measurable functions on X, equipped with the essential supremum norm. Set $(\pi(x)f)(\bar{g}) = f(x^{-1} \cdot \bar{g})$, $x \in G$, $\bar{g} \in X$, $f \in L^\infty(X)$. Show that π is not a continuous representation of $\mathrm{GL}_2(\mathbb{R})$.

5. Intertwining operators and Schur's Lemma

Definition 5.1. Let (π_1, V_1) and (π_2, V_2) be continuous representations of G.

(1) An *intertwining operator from* π_1 *to* π_2 is a linear transformation $A \in \mathrm{B}(V_1, V_2)$ such that $A \circ \pi_1(g) = \pi_2(g) \circ A$ for all $g \in G$. The notation $\mathrm{Hom}_G(\pi_1, \pi_2)$ (or $\mathrm{Hom}_G(V_1, V_2)$) will be used for the set of intertwining operators from π_1 to π_2.
(2) We say that π_1 and π_2 are *equivalent* if $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains a homoemorphism. In this case, we write $\pi \simeq \pi_2$.
(3) If π_1 and π_2 are unitary representations, we say that π_1 and π_2 are *unitarily equivalent* if $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains a homeomorphism that is an isometry.

It is easy to see that equivalence of representations is an equivalence relation. For the rest of this section, assume that (π_1, V_1) and (π_2, V_2) are continuous representations of G.

Proposition 5.2. *Suppose that* $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$. *Let* W_1 *be the kernel of* A *and let* W_2 *be the closure of the range of* A. *Then*
(1) W_1 *and* W_2 *are* G*-invariant.*
(2) *If* π_1 *and* π_2 *are finite-dimensional and irreducible and* A *is nonzero, then* A *is an isomorphism.*
(3) *If* π_1 *and* π_2 *are unitary, let* $(\pi_1^\perp, W_1^\perp)$ *be the restriction of* π_1 *to* $W_1^\perp$ *and let* (π_2', W_2) *be the restriction of* π_2 *to* W_2. *Then* $\pi_1^\perp$ *and* π_2' *are unitarily equivalent.*

More will be added later. For the third part, we use the polar decomposition of A: $A = A_1 A_2$, where A_1 is a partial isometry of V_1 onto W_2 with kernel $W_1^\perp$ and $A_2 = (A^* A)^{1/2}$. We can show that $A_1 \in \mathrm{Hom}_G(\pi_1^\perp, \pi_2')$. Since the restriction of A_1 to $W_1^\perp$ is an isometry of $W_1^\perp$ onto W_2, $\pi_1^\perp$ and π_2' are unitarily equivalent.

Corollary 5.3. *Assume that* π_1 *and* π_2 *are unitary and there exists* $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$ *such that* A *is one-to-one and the range of* A *is dense in* V_2, *then* π_1 *and* π_2 *are unitarily equivalent.*

Corollary 5.4. *Assume that* π_1 *and* π_2 *are unitary and topologically irreducible. If* $\mathrm{Hom}_G(\pi_1, \pi_2)$ *is nonzero, then* π_1 *and* π_2 *are unitarily equivalent.*

Corollary 5.5. *Assume that* π_1 *and* π_2 *are unitary. Then* $\mathrm{Hom}_G(\pi_1, \pi_2)$ *is nonzero if and only if some subrepresentation of* π_1 *is (unitarily) equivalent to a subrepresentation of* π_2.

Example 5.6. Let $G = \mathbb{C}^\times \ltimes \mathbb{C}$ and $V = \mathbb{C}^2$ (written as column vectors). Let (π, V) be the representation of G such that $\pi(t, x) = \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{C}^\times$, $x \in \mathbb{C}$. The subspace

$W = \mathrm{Span}(e_1)$ is a unique one-dimensional G-invariant suspace of V. Let (π', V) be the representation of G such that $\pi'(t,x) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{C}^\times$, $x \in \mathbb{C}$. Then $\mathrm{Hom}_G(\pi, \pi')$ and $\mathrm{Hom}_G(\pi', \pi)$ are both one-dimensional. A nonzero element of $\mathrm{Hom}_G(\pi', \pi)$ intertwines a subrepresentation of π' and the unique subrepresentation of π. On the other hand, a nonzero element of $\mathrm{Hom}_G(\pi, \pi')$ does not interwine a subrepresentation of π and a subrepresentation of π'. Instead, it interwines the unique nonzero quotient of π and a subrepresentation of π'.

Exercise 5.7. Let (π, V) be a unitary representation of G. Suppose that $\{\, W_i \mid i \in I \,\}$ is a family of closed G-invariant subspaces of V such that

(1) $W_i \perp W_j$ if $i \neq j$
(2) $\cup_{i\in I} W_i$ is total in V (that is, the set of finite linear combinations of vectors from $\cup_{i \in I} W_i$ is dense in V)

For $g \in G$, let $\pi_i(g)$ be the restriction of $\pi(g)$ to W_i. Prove that (π, V) is equivalent to $(\oplus \pi_i, \oplus W_i)$.

Proposition 5.8. *(Schur's Lemma) Let V be a Hilbert space and let $\mathcal{F}$ be a collection of operators in $B(V)$ such that $T \in \mathcal{F}$ if and only if $T^* \in \mathcal{F}$. Assume that $\mathcal{F}$ is topologically irreducible. (That is, $\{0\}$ and V are the only closed subspaces of V which are invariant under all of the operators in $\mathcal{F}$.) Then $\{\, S \in \mathrm{B}(V) \mid ST = TS \,\forall\, T \in \mathcal{F}. \,\}$ consists of scalar multiples of the identity operator.*

Further comments will be added here. It is easy to see that if $S \in \mathrm{B}(V)$ commutes with all operators in $\mathcal{F}$, then $S + S^*$ and $i(S - S^*)$ (which are both self-adjoint) also commute with all operators in $\mathcal{F}$. The spectral theorem for self-adjoint operators can be applied to show that $S + S^*$ and $i(S - S^*)$ are both scalar multiples of the identity operator.

Corollary 5.9. *(Schur's Lemma for unitary representations) Let (π, V) be a unitary representation of G. Then π is topologically irreducible if and only if $\mathrm{Hom}_G(\pi, \pi)$ consists of scalar multiples of the identity.*

Lemma 5.10. *If (π, V) is a (nonunitary) finite-dimensional irreducible representation of G, then $\mathrm{Hom}_G(\pi, \pi)$ consists of scalar multiples of the identity operator.*

Proof. Let A be a nonzero element of $\mathrm{Hom}_G(\pi, \pi)$. Because V is finite-dimensional, A has an eigenvalue, say λ. Note that $A - \lambda I \in \mathrm{Hom}_G(\pi, \pi)$. Since the kernel of $A - \lambda I$ is nonzero and G-invariant and π is irreducible, the kernel must equal V. That is, $A = \lambda I$. $\square$

The next example shows that the converse of the lemma does not necessaily hold in the nonunitary setting.

EXAMPLE 5.11. Consider the representation (π, V) from Example 5.6. We can see that, although π is not irreducible, $\mathrm{Hom}_G(\pi, \pi)$ consists of scalar multiples of the identity operator on V.

COROLLARY 5.12. *Assume that π_1 and π_2 are unitary and topologically irreducible (or finite-dimensional, not necessarily unitary, and irreducible). Then*

$$\dim \mathrm{Hom}_G(\pi_1, \pi_2) = \begin{cases} 1 & \textit{if } \pi_1 \simeq \pi_2 \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Suppose that $\mathrm{Hom}_G(\pi_1, \pi_2) \neq 0$. Let $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$ be nonzero. Because A is nonzero, the kernel of A is a closed G-invariant subspace of V_1, and π_1 is topologically irreducible, we have that A is one-to-one. In the unitary case, note that $A^* \in \mathrm{Hom}_G(\pi_2, \pi_1)$. It follows that $AA^* \in \mathrm{Hom}_G(\pi_2, \pi_2)$. Since π_2 is topologically irreducible, AA^* must be a scalar multiple of the identity operator on V_2. This can be used to see that A is onto. In the finite-dimensional case, the range of A is a nonzero closed G-invariant subspace of V_2, so, by irreducibility of π_2, A is onto. We have shown that nonzero elements of $\mathrm{Hom}_G(\pi_1, \pi_2)$ are invertible. It follows that π_1 and π_2 are equivalent if and only if $\mathrm{Hom}_G(\pi_1, \pi_2)$ is nonzero.

It remains to show that the dimension of $\mathrm{Hom}_G(\pi_1, \pi_2)$ is one whenever $\pi_1 \simeq \pi_2$. Let A_1, $A_2 \in \mathrm{Hom}_G(\pi_1, \pi_2)$ be such that A_2 is nonzero. Then we know that A_2 is invertible. Since $A_2^{-1} A_1 \in \mathrm{Hom}_G(\pi_1, \pi_1)$, by Schur's Lemma, $A_2^{-1} A_1$ is a scalar multiple of the identity. That is, $A_1 = c\, A_2$ for some complex number c. Hence $\dim \mathrm{Hom}_G(\pi_1, \pi_2) = 1$. □

EXAMPLE 5.13. The left regular representation of G on $L^2(G)$ is not topologically irreducible. When G is compact, it is easy to see that the trivial representation of G occurs as a subrepresentation. However, if G is noncompact, it can be shown that the left regular representation has no finite-dimensional subrepresentation. Instead, show that the operators of the right regular representation intertwine the left regular representation. Apart from the identity element, they are not scalar multiples of the identity operator. Apply Schur's Lemma.

CHAPTER 3

Representations of compact groups: the general theory

Throughout this chapter, unless we explicitly say otherwise, G is assumed to be a compact group. Recall (Proposition 2.1) that a continuous representation of G in a Hilbert space is unitary (with respect to some inner product on the space that defines a norm which is equivalent to the Hilbert space norm).

The "Peter-Weyl Theorem" is a collection of results about continuous representations of G, which may be viewed as analogues of results from the representation theory of finite groups.

1. The Peter-Weyl Theorem

Recall that G is unimodular and has finite measure with respect to any Haar measure on G. We fix a Haar measure on G such that the measure of G is equal to one. Let $C(G)$ be the space of continuous complex-valued functions on G. With the supremum norm $\|f\|_\infty = \sup_{g\in G}|f(g)|$, $C(G)$ is a Banach space.

Using continuity of integration and the fact that the measure of G is equal to one, we can see that the L^2 norm of a continuous function is bounded above by the supremum norm the function:

Lemma 1.1. *If $f \in C(G)$, then $\|f\|_2 \leq \|f\|_\infty$.*

According to Proposition 4.1, the left regular representation of G on $L^2(G)$ is continuous. Unitarity follows from G-invariance of Haar measure. These properties will be used below.

Definition 1.2. If $f_1,\ f_2 \in C(G)$, we define the convolution $f_1 * f_2$ of f_1 with f_2 by: $(f_1 * f_2)(x) = \int_G f_1(y) f_2(y^{-1}x)\, dy$, $x \in G$.

We can define convolution more generally, for f_1 and $f_2 \in L^2(G)$. Let π denote the left regular representation of G on $L^2(G)$. Let $\check{f}_2$ be the function defined by $\check{f}_2(y) = f_2(y^{-1})$, $y \in G$. Note that $(f_1 * f_2)(x) = \int_G f_1(y)(\pi(x)\check{f}_2)(y)\, dy = \langle \pi(x)\check{f}_2, \bar{f}_1\rangle$. Therefore $f_1 * f_2$ is the matrix coefficient of π attached to the vectors $\check{f}_2$ and $\bar{f}_1$. By continuity of π, $f_1 * f_2 \in C(G)$.

The next lemma follows from the Cauchy-Schwartz inequality and the fact that π is unitary.

LEMMA 1.3. *If $f_1,\ f_2 \in L^2(G)$, then $\|(f_1 * f_2)\|_\infty \leq \|f_1\|_2\|f_2\|_2$.*

LEMMA 1.4. *If $f_2 \in L^2(G)$, define a linear transformation $K_{f_2} : L^2(G) \to C(G)$ by $K_{f_2}(f_1) = f_1 * f_2$. This is a compact transformation.*

PROOF. A compact transformation is one that maps bounded sets to relatively compact sets (sets having compact closure). Let $\mathcal{F} = \{K_{f_2}(f_1) \mid f_1 \in L^2(G),\ \|f_1\|_2 \leq 1\}$. To see that K_{f_2} is compact, it suffices to prove that $\mathcal{F}$ is a relatively compact subset of $C(G)$.

According to Ascoli's Theorem (see p.7 of [**R**] for the general statement), $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is equicontinuous and each of the sets $\mathcal{F}(x) = \{\, K_{f_2}(f_1)(x) \mid f_1 \in L^2(G),\ \|f_1\|_2 \leq 1\,\}$ is relatively compact in $\mathbb{C}$.

To see that $\mathcal{F}(x)$ is relatively compact, apply Lemma 1.3 to conclude that $\|K_{f_2}(f_1)(x)| \leq \|f_2\|_2$ whenever $\|f_1\|_2 \leq 1$. This shows that $\mathcal{F}(x)$ is bounded. A bounded subset of $\mathbb{C}$ is relatively compact.

To verify that $\mathcal{F}$ is equicontinuous, note that if $x_1,\ x_2 \in G$ and $\|f_1\| \leq 1$, then

$$\begin{aligned}|K_{f_2}(f_1)(x_1) - K_{f_2}(f_1)(x_2)| &= |\langle \pi(x_1)\check{f}_2 - \langle \pi(x_1)\check{f}_2, \bar{f}_1\rangle| \\ &\leq \|\pi(x_1)\check{f}_2 - \pi(x_2)\check{f}_2\|_2\|f_1\|_2 \leq \|\pi(x_1)\check{f}_2 - \pi(x_2)\check{f}_2\|_2.\end{aligned}$$

Let $\epsilon > 0$. By continuity of π, there exists an open neighbourhood U of x_1 such that whenever $x_2 \in U$, we have $\|\pi(x_1)\check{f}_2 - \pi(x_2)\check{f}_2\|_2 < \epsilon$. That is, $|K_{f_2}(f_1)(x_1) - K_{f_2}(f_1)(x_2)| < \varepsilon$ for all f_1 such that $\|f_1\| \leq 1$. This says that $\mathcal{F}$ is equicontinuous. Now we may apply Ascoli's Theorem to conclude that K_{f_2} is compact. $\square$

COROLLARY 1.5. *K_{f_2} is a compact operator on $L^2(G)$.*

PROOF. A subset S of a complete normed space X is relatively compact if and only if S is totally bounded. By definition, S is totally bounded if for each $\epsilon > 0$, there exist finitely many $s_1, \ldots, s_n \in S$ such that

$$S \subset \cup_{i=1}^n \{\, x \in X \mid \|x - s_i\| < \epsilon\,\}.$$

Let $\mathcal{F}$ be as in Lemma 1.4. By Lemma 1.4, $\mathcal{F}$ is a totally bounded subset of $C(G)$. Using Lemma 1.1, we can show that it follows that $\mathcal{F}$ is a totally bounded subset of $L^2(G)$. Hence $K_{f_2}(f_1)$ is a compact operator on $L^2(G)$. $\square$

THEOREM 1.6. *(Peter-Weyl Theorem 1) If $g_0 \in G$ and $g_0 \neq 1$, then there exists an irreducible finite-dimensional representation π of G such that $\pi(g_0)$ is not equal to the identity operator.*

PROOF. Let V be an open neighbourhood of the identity in G such that $g_0 \notin V^2$ and $V = V^{-1}$. Fix a function $\psi \in C(G)$ such that $\psi(1) > 0$, $\psi(g^{-1}) = \psi(g) \geq 0$ for all $g \in G$, and the support of ψ is contained in V. Let $K_\psi(f) = \psi * f$, $f \in L^2(G)$. We can check that

$$K_\psi^*(f)(g) = \int_G f(x)\overline{\psi(g^{-1}x)}\, dx,$$

which, by properties of ψ, is equal to $K_\psi(f)(g)$. So K_ψ is a compact self-adjoint operator on $L^2(G)$. Applying the Hilbert-Schmidt Spectral Theorem, we have that $L^2(G)$ is the Hilbert space direct sum

$$L^2(G) = \bigoplus_{\lambda \in \sigma(K_\psi)} V_{\lambda_i},$$

where $\sigma(K_\psi)$ is the spectrum of K_ψ, and for $\lambda_i > 0$, $V_i = \mathrm{Ker}(K_\psi - \lambda_i I)$ is finite-dimensional. Applying K_ψ to ψ, we have $K_\psi(\psi) = \sum_{\lambda_i \in \sigma(K_\psi), \lambda_i > 0} \lambda_i \psi_i$, where $\psi_i \in C(G) \cap V_i$. Using properties of ψ, we can verify that $K_\psi(\psi)(1)$ is nonzero and $K_\psi(\psi)(g_0) = 0$. Hence there exists i such that $\pi(g_0)\psi_i \neq \psi_i$.

Next, observe that $(K_\psi f)(g) = \langle \pi(g)\psi, \bar{f} \rangle$, $f \in L^2(G)$. (Here, π is the left regular representation of G.) This can be used to see that $K_\psi \in \mathrm{Hom}_G(\pi, \pi)$. This implies $K_\psi - \lambda_i I \in \mathrm{Hom}_G(\pi, \pi)$. We conclude that V_i is G-invariant. The restriction π_i of π to the finite-dimensional G-invariant subspace V_i satisfies $\pi_i(g_0) \neq I_{V_i}$. We know that (π_i, V_i) is completely reducible. Hence exists at least one irreducible subrepresentation of π_i under which g_0 does not act as the identity on the space of the subrepresentation. □

COROLLARY 1.7. *A compact group G is abelian if and only if all of its irreducible finite-dimensional continuous representations are one-dimensional.*

PROOF. Let G' be the derived group of G. (That is, G' is the smallest closed subgroup of G containing all commutators – elements of the form $xyx^{-1}y^{-1}$, x, $y \in G$.) Note that any continuous one-dimensional representation of G is trivial on G'. Therefore, if all irreducible finite-dimensional representations of G are one-dimensional, we can apply Theorem 1.6 to conclude that $G' = \{1\}$, that is, G is abelian.

The converse follows from Schur's Lemma. □

Before stating the next theorem, we need a few basic facts about matrix coefficients of finite-dimensional continuous representations of G.

The following definition is valid for noncompact groups and the representations don't have to be continuous.

DEFINITION 1.8. Let (π_1, V_1) and (π_2, V_2) be finite-dimensional representations of groups G_1 and G_2. Let $V = V_1 \otimes V_2$. We may define a representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$ by setting

$$(\pi_1(g_1) \otimes \pi_2(g_2))(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2, \qquad v_i \in V_i,\ g_i \in V_i,$$

and then extending by linearity. This is called the *external tensor product* of π_1 and π_2. When $G = G_1 = G_2$, we can restrict the external direct product representation to the subgroup $\{\,(g, g) \mid g \in G\,\}$ of $G \times G$. This representation of G is also written as $\pi_1 \otimes \pi_2$: $(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2$. It is called the *internal tensor product* of π_1 and π_2.

LEMMA 1.9. *The product of a matrix coefficient of π_1 and a matrix coefficient of π_2 is a matrix coefficient of $\pi_1 \otimes \pi_2$.*

DEFINITION 1.10. Let (π, V) be a representation of G and let V^* be the space of continuous linear functionals on V. The *contragredient* of π, is the representation $\pi^\vee$ of G in V^* defined by $(\pi^\vee(g)\lambda)(v) = \lambda(\pi(g^{-1}v))$, $g \in G$, $v \in V$.

LEMMA 1.11. *Let (π, V) be a finite-dimensional continuous representation of G. Then*

(1) *π is irreducible if and only if $\pi^\vee$ is irreducible.*
(2) *A matrix coefficient of $\pi^\vee$ is the complex conjugate of a matrix coefficient of π.*

Note that the last part of the next lemma is an immediate consequence of Theorem 1.6.

LEMMA 1.12. *Let $\mathcal{A}(G)$ be the span of the set of matrix coefficients of irreducible representations of G. Then*

(1) *The characteristic function of G belongs to $\mathcal{A}(G)$.*
(2) *The product of two functions in $\mathcal{A}(G)$ belongs to $\mathcal{A}(G)$.*
(3) *The complex conjugate of a function in $\mathcal{A}(G)$ belongs to $\mathcal{A}(G)$.*
(4) *If $g_1 \neq g_2 \in G$, there exists $f \in \mathcal{A}(G)$ such that $f(g_1) \neq f(g_2)$.*

THEOREM 1.13. *(Peter-Weyl Theorem 2) $\mathcal{A}(G)$ is dense in $C(G)$.*

PROOF. Apply Lemma 1.12 and the (complex version of the) Stone-Weierstrass Theorem. $\square$

COROLLARY 1.14. *If U is a neighbourhood of 1 in G, there exists a finite-dimensional representation of G whose kernel is contained in U.*

2. Faithful representations

DEFINITION 2.1. A representation (π, V) of a group G is *faithful* if $\pi(g) \neq I$ for all $g \in G$ such that $g \neq 1$.

PROPOSITION 2.2. *Let G be a compact group. The following are equivalent*

(1) *There exists a neighbourhood U of 1 in G that does not contain a nontrivial subgroup of G.*
(2) *There exists a faithful finite-dimensional representation of G.*
(3) *G is a real Lie group.*

PROOF. To be added... □

Comments about faithful and almost-faithful representations of Lie groups... to be added...

3. Finite-dimensional representations of locally profinite groups

Locally profinite groups (for example $SL_2(\mathbb{Q}_p)$) are not necessarily compact. However, due to the fact that they have so many compact open subgroups, the representation theory of locally profinite groups is heavily influenced by properties of representations of compact groups.

PROPOSITION 3.1. *Let (π, V) be a continuous finite-dimensional representation of a locally profinite group. G. Then*

(1) *The kernel of π is an open (normal) subgroup of G.*
(2) *For $v \in V$, let $\mathrm{Stab}_G(v) = \{\, g \in G \mid \pi(g)v = v \,\}$. Then $\mathrm{Stab}_G(v)$ is an open subgroup of G.*
(3) *If G is profinite, then $G/\mathrm{Ker}\,\pi$ is finite. (Hence $\pi(G)$ is finite.)*

REMARK 3.2. A representation (π, V) of a locally profinite group G having the property that $\mathrm{Stab}_G(v)$ is open for all vectors $v \in V$ is called a *smooth* representation.

The next lemma is false for noncompact locally profinite groups. Later in the course, we will see that the trivial representation of $SL_2(\mathbb{Q}_p)$ is the only irreducible finite-dimensional continuous (complex) representation of $SL_2(\mathbb{Q}_p)$.

Corollary 3.3. *If G is a profinite group and K is an open compact normal subgroup of G, there exists a finite-dimensional continuous representation π of G such that* $\operatorname{Ker}\pi = K$.

4. Finite-dimensional subrepresentations of the regular representation

Several of the results here are valid for finite-dimensional representations of abstract groups.

Lemma 4.1. *Let $\mathbb{C}^G$ be the space of complex-valued functions on G. Let ρ be the right regular representation of G on $\mathbb{C}^G$. Suppose that V is a finite-dimensional G-invariant subspace of $\mathbb{C}^G$. If $f \in V$, there exists $A \in \operatorname{End}(V)$ such that $f(g) = \operatorname{tr}(A\rho(g))$, $g \in G$.*

Definition 4.2. If (π, V) is a finite-dimensional representation of G, define a representation $(\ell_\pi, \operatorname{End}(V))$ by $\ell_\pi(g)(A) = \pi(g) \circ A$, $A \in \operatorname{End}(V)$. Define $T_\pi : \operatorname{End}(V) \to \mathbb{C}^G$ by $T_\pi(A)(g) = \operatorname{tr}(\ell_\pi(g)(A)) = \operatorname{tr}(\pi(g) \circ A)$.

Lemma 4.3. $T_\pi \in \operatorname{Hom}_G(\ell_\pi, \rho)$.

If $\lambda \in V^*$ and $v \in V$, then we define $A_{\lambda\times v} \in \operatorname{End}(V)$ by $A_{\lambda\times v}(w) = \lambda(w)v$, $w \in V$. The map $\lambda \otimes v \to A_{\lambda\otimes v}$ extends to a linear isomorphism between $V^* \otimes V$ and $\operatorname{End}(V)$.

The next lemma shows that the image of T_π consists of linear combinations of matrix coefficients of π.

Lemma 4.4. *If $\lambda \in V^*$ and $v \in V$, then $T_\pi(A_{\lambda\otimes v})(g) = \lambda(\pi(g)v)$, $g \in G$.*

Lemma 4.5. *Every finite-dimensional subrepresentation of $(\rho, \mathbb{C}^G)$ that is equivalent to (π, V) occurs as a subrepresentation of $(\rho, T_\pi(\operatorname{End}(V))$.*

Proof. Let (ρ, W) be a finite-dimensional subrepresentation of $(\rho, \mathbb{C}^G)$ that is equivalent to (π, V). Then there exists a linear isomorphism $C : V \to W$ such that $C\pi(g)C^{-1} = \rho(g) \,|\, W$ for all $g \in G$. Let $f \in W$. By Lemma 4.1, there exists $A \in \operatorname{End}(V)$ such that $f(g) = \operatorname{tr}(A\rho(g))$ for all $g \in G$. To complete the proof, verify that $T_\pi(CAC^{-1})(g) = f(g)$ for $g \in G$. It follows that $W \subset T_\pi(\operatorname{End}(V))$. □

Theorem 4.6. *(Burnside's Theorem) If (π, V) is an irreducible finite-dimensional representation of a group G, then* $\operatorname{End}(V) = \operatorname{span}\{\, \pi(g) \mid g \in G \,\}$.

As the next example shows, it is easy to see that when π is not irreducible, $\operatorname{span}\{\, \pi(g) \mid g \in G \,\}$ is a proper subspace of $\operatorname{End}(V)$.

EXAMPLE 4.7. Let $G = \mathbb{R}^\times \ltimes \mathbb{R}$. Define a representation π of G by $\pi(t,x) = \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}^\times$, $x \in \mathbb{R}$. Clearly, $\operatorname{span}\{\, \pi(t,x) \mid t \in \mathbb{R}^\times,\ x \in \mathbb{R}\,\}$ consists of the 2×2 matrices whose $(2,1)$ entry is equal to zero.

PROOF. More details to be added at a later time.

Step 1 Given $v \in V$, define $\alpha_v : \mathrm{End}(V) \to V$ by $\alpha_v(A) = A(v)$. Show that $\alpha_v \in \mathrm{Hom}_G(\ell_\pi, \pi)$.

Step 2 Show that if (ℓ_π, W_0) is a (nonzero) irreducible subrepresentations of $(\ell_\pi, \mathrm{End}(V))$, then for each nonzero $A \in W_0$, $\mathrm{tr}(\pi(g)A) \neq 0$ for some $g \in G$.

Step 3 Let $W' = \{\, A \in \mathrm{End}(V) \mid \mathrm{tr}(AB) = 0\ \forall\ B \in \operatorname{span}\{\, \pi(g) \mid g \in G\,\}\,\}$. Prove that W' is a G-invariant subspace of $\mathrm{End}(V)$.

Let W' be as above. Suppose that W' is nonzero. Since W' is G-invariant, there exists a (nonzero) irreducible subrepresentation (ℓ_π, W_0) of (ℓ_π, W'). Let $A \in W_0$ be nonzero. According to the result of step 2, the function $g \mapsto \mathrm{tr}(\pi(g)A)$ is nonzero. This contradicts the fact that A belongs to W'. Hence $W' = \{0\}$. To complete the proof, use nondegeneracy of the bilinear form $(A,B) \mapsto \mathrm{tr}(AB)$. □

LEMMA 4.8. *The external tensor product $\pi^\vee \otimes \pi$ is equivalent to the representation $(\sigma, \mathrm{End}(V))$ of $G \times G$ defined by $\sigma(g_1, g_2)(A) = \pi(g_1)A\pi(g_2)^{-1}$, $g_1,\ g_2 \in G$, $A \in \mathrm{End}(V)$.*

PROOF. The isomorphism between $V^* \otimes V$ and $\mathrm{End}(V)$ defined above belongs to $\mathrm{Hom}_G(\pi^\vee \otimes \pi, \sigma)$. □

The next two results are corollaries of Burnside's Theorem.

COROLLARY 4.9. *If (π, V) is a finite-dimensional irreducible unitary representation of G, then the representation σ of $G \times G$ defined in Lemma 4.8 is irreducible.*

PROOF. Let $T \in \mathrm{Hom}_{G \times G}(\sigma, \sigma)$. Then, for $g \in G$ and $A \in \mathrm{End}(V)$,

$$T(\pi(g)A) = T(\sigma(g,1)(A)) = \sigma(g,1)(T(A)) = \pi(g)T(A)$$

Taking $A = I$, we have $T(\pi(g)) = \pi(g)T(I)$ for $g \in G$.

Let $A \in \mathrm{End}(V)$. By Burnside's Theorem, since π is irreducible, there exist finitely many elements $x_i \in G$ and complex numbers c_i such that $A = \sum_i c_i \pi(x_i)$. Since $T(\pi(x_i)) = \pi(x_i)T(I)$ for each i, we have $T(A) = A\,T(I)$.

Similarly, $T(\pi(g)) = T(I)\pi(g)$ for all $g \in G$. It follows that $\pi(g)T(I) = \pi(g)T(I)$ for all $g \in G$. That is, $T(I) \in \mathrm{Hom}_G(\pi, \pi)$. By irreducibility of π, there exists $\lambda \in \mathbb{C}$ such that $T(I) = \lambda I$. Therefore $T(A) = A\, T(I) = A\, \lambda I = \lambda\, A$, for $A \in \mathrm{End}(V)$.

This shows that $\mathrm{Hom}_{G\times G}(\sigma, \sigma)$ consists of scalar multiples of the identity operator on $\mathrm{End}(V)$. Since σ is unitary and irreducible, we may apply Schur's Lemma to conclude that σ is irreducible. $\square$

REMARK 4.10. The corollary still holds when π is nonunitary, but the proof has to be adjusted, as the last part of the argument cannot be applied in the nonunitary setting.

COROLLARY 4.11. *Let π be an irreducible finite-dimensional representation of G. Then T_π (see Definition 4.2) is one-to-one.*

PROOF. Let $A \in \mathrm{End}(V)$ be such that $T_\pi(A) = 0$. By definition of T_π, $\mathrm{tr}(\pi(g)A) = 0$ for every $g \in G$. According to Burnside's Theorem, $\mathrm{End}(V) = \mathrm{span}\{\, \pi(x) \mid x \in G\,\}$. It follows that $\mathrm{tr}(BA) = 0$ for all $B \in \mathrm{End}(V)$. Nondegeneracy of trace forces $A = 0$. $\square$

DEFINITION 4.12. The *biregular* representation of $G\times G$ on $\mathbb{C}^G$ is defined by $(\omega(g_1, g_2)f)(x) = f(g_1^{-1} x g_2)$. (Of course, when G is a compact group, we may restrict the biregular representation to a continuous representation of G on $C(G)$ or on $L^2(G)$.)

LEMMA 4.13. *Let σ be as in Lemma 4.8 and T_π be as in Definition 4.2. Then $T_\pi \in \mathrm{Hom}_{G\times G}(\sigma, \omega)$. The composition of the isomorphism between $V^* \otimes V$ and $\mathrm{End}(V)$ with T_π belongs to $\mathrm{Hom}_{G\times G}(\pi^\vee \otimes \pi, \omega)$.*

COROLLARY 4.14. *Let π be a finite-dimensional irreducible unitary representation of G. Then the restriction of the biregular representation of $G \times G$ to the subspace of $\mathbb{C}^G$ spanned by the matrix coefficients of π is equivalent to $\pi^\vee \otimes \pi$.*

COROLLARY 4.15. *Let π be a finite-dimensional irreducible unitary representation of G. Then $(\rho, T_\pi(\mathrm{End}(V))$ is equivalent to the direct sum of $\dim V$ copies of (π, V).*

PROOF. Restrict the biregular representation to the subgroup $\{(1, g) \mid g \in G\}$ to obtain the right regular representation ρ. If we examine the intertwining isomorphism between $\pi^\vee \otimes \pi$ and $(\omega, T_\pi(\mathrm{End}(V))$ as it restricts to this subgroup of $G \times G$, we find that it becomes an equivalence between the representation τ of G on $V^* \times V$ defined by $\tau(g)(\lambda \otimes v) = \lambda \otimes \pi(g)v$, $\lambda \in V^*$, $v \in V$. It is easy to see that τ is isomorphic to the direct sum of $\dim V$ copies of (π, V). $\square$

REMARK 4.16. Once we have established the orthogonality relations for matrix coefficients of representations of compact groups, we will see that when G is compact, the direct sum in the above corollary is an orthogonal direct sum.

Now we assume that G is compact and (π, V) is an irreducible finite-dimensional unitary representation of G. Then $T_\pi(\mathrm{End}(V)) \subset C(G) \subset L^2(G)$.

DEFINITION 4.17. Let $L^2(G, \pi)$ be the algebraic direct sum of the irreducible subrepresentations of $(\rho, L^2(G))$ which are equivalent to π.

LEMMA 4.18. *$L^2(G, \pi) = T_\pi(\mathrm{End}(V))$ and $\dim L^2(G, \pi) = (\dim V)^2$ and $L^2(G, \pi)$ is the subspace of $C(G)$ spanned by the matrix coefficients of π.*

PROOF. According to Lemma 4.5, $L^2(G, \pi) \subset T_\pi(\mathrm{End}(V))$. By Corollary 4.15, $(\rho, T_\pi(\mathrm{End}(V))$ is equivalent to the direct sum of $\dim V$ copies of π. In particular, each irreducible subrepresentation of $(\rho, T_\pi(\mathrm{End}(V))$ is equivalent to π. The final statement of the lemma follows from Lemma 4.4 and continuity of π. $\square$

DEFINITION 4.19. The *unitary dual* of G is the set $\widehat{G}$ of equivalence classes of irreducible unitary finite-dimensional representations of G.

REMARK 4.20. Later we will show that all irreducible unitary representations of G are finite-dimensional.

Our goal is to prove that $L^2(G)$ is the Hilbert space direct sum of the subrepresentations $L^2(G, \pi)$ as π ranges over $\widehat{G}$. One part of this involves showing that if σ and π are inequivalent finite-dimensional irreducible representations of G, then $L^2(G, \pi)$ and $L^2(G, \sigma)$ are orthogonal in $L^2(G)$.

5. Orthogonality relations for matrix coefficients

In this section, G is a compact group. More will be added to this section soon.

LEMMA 5.1. *Let (π, V_π) and (σ, V_σ) be continuous representations of G. Let $A \in B(V_\pi, V_\sigma)$. Define $A^\natural = \int_G \sigma(g) A \pi(g)^{-1}\, dg$. Then $A^\natural \in \mathrm{Hom}_G(\pi, \sigma)$.*

PROPOSITION 5.2. *Let (π, V_π) be a finite-dimensional irreducible unitary representation of G. Then $A^\natural = (\mathrm{tr}\, A / \dim V_\pi) \cdot I$.*

PROOF. By Lemma 5.1, $A^\natural \in \mathrm{Hom}_G(\pi, \pi)$. By Schur's Lemma, $A^\natural$ is a scalar multiple of I. To complete the proof, evaluate the trace of $A^\natural$. $\square$

Remark 5.3. Let (π, V_π) and (σ, V_σ) be equivalent finite-dimensional irreducible representations of G. It is easy to show that the set of matrix coefficients of π coincides with the set of matrix coefficients of σ.

Theorem 5.4. *(Schur orthogonality relations) Let (π, V_π) and (σ, V_σ) be finite-dimensional irreducible unitary representations of G. Let $\langle \cdot, \cdot \rangle_\pi$ $\langle \cdot, \cdot \rangle_\sigma$ be G-invariant inner products on V_π and V_σ, respectively. For v, $v' \in V_\pi$ and u, $u' \in V_\sigma$,*

$$\int_G \langle \pi(g)v, v' \rangle_\pi \overline{\langle \sigma(g)u, u' \rangle_\sigma} dg = \begin{cases} 0 & \text{if } \pi \not\simeq \sigma \\ \langle v, u \rangle_\pi \overline{\langle v', u' \rangle_\pi} & \text{if } \pi = \sigma. \end{cases}$$

Proof. Define $A \in B(V_\sigma, V_\pi)$ by $A(w) = \langle w, u \rangle_\sigma v$. Show that

$$\langle A^\natural(u'), v' \rangle_\pi = \int_G \langle \pi(g)v, v' \rangle_\pi \overline{\langle \sigma(g)u, u' \rangle_\sigma} dg.$$

According to Proposition 5.1, $A^\natural \in \operatorname{Hom}_G(\sigma, \pi)$. When $\pi \not\simeq \sigma$, we know that $A^\natural = 0$.

Suppose that $\sigma = \pi$. Show that $\operatorname{tr} A = \langle v, u \rangle_\pi$. Then apply Proposition 5.2. □

6. Decomposition of the regular representation

If $\pi \in \widehat{G}$, let $L^2(G, \pi)$ be as in Definition 4.17.

Theorem 6.1. *(Peter-Weyl Theorem 3) Let G be a compact group. Then $L^2(G)$ is the Hilbert space direct sum of the finite-dimensional subrepresentations $L^2(G, \pi)$ as π ranges over $\widehat{G}$.*

Proof. Let π and σ be inequivalent finite-dimensional irreducible unitary representations of G. We know (see Lemma 4.18) that $L^2(G, \pi)$ (resp. $L^2(G, \sigma)$) is the space spanned by the matrix coefficients of π (resp. σ). By the Schur orthogonality relations, we have that $L^2(G, \pi)$ and $L^2(G, \sigma)$ are orthogonal in $L^2(G)$.

Take the algebraic direct sum $\oplus_{\pi \in \widehat{G}} L^2(G, \pi)$. In view of the orthogonality of the spaces $L^2(G, \pi)$, it suffices to show that $\oplus_{\pi \in \widehat{G}} L^2(G, \pi)$ is dense in $L^2(G)$ (see Exercise 5.7), Chapter 2). Since $L^2(G, \pi)$ is the span of the matrix coefficients of π, we have that $\oplus_{\pi \in \widehat{G}} L^2(G, \pi)$ is the space $\mathbb{A}(G)$ consisting of the span of all matrix coefficients of finite-dimensional irreducible representations of G. By the Peter-Weyl Theorem 2, $\mathbb{A}(G)$ is dense in $C(G)$. It follows that it is dense in $L^2(G)$. □

Exercise 6.2. Let G be a compact Lie group. Let π be a faithful finite-dimensional representation of G. Consider the set of irreducible subrepresentations of representations of the form $\sigma_1 \otimes \cdots \otimes \sigma_n$, where σ_i is equivalent to an irreducible subrepresentation of π or to the contragedient of the subrepresentation. Prove that this set contains all irreducible finite-dimensional representations of G (up to equivalence).

EXERCISE 6.3. Suppose that H is a closed subgroup of a compact abelian group G. Let $\chi : H \to \mathbb{C}^\times$ be a character of H (that is, a unitary one-dimensional continuous representation). Show that χ extends to a character of G. (*Hint*: Consider the space $V = \{ f \in L^2(G) \mid f(hg) = \chi(h)f(g) \}$. To show that V is nonzero, note that if $\varphi \in C(G)$, then $f(g) = \int_G \varphi(hg)\chi(h^{-1})\, dh$ defines an element of V. Show that there exists φ such that $f \neq 0$.)

7. Decomposition of unitary representations of compact groups

PROPOSITION 7.1. *Let (π, V) be a unitary representation of a compact group G. Then π has a finite-dimensional irreducible subrepresentation.*

PROOF. Let $\langle\ ,\ \rangle$ be a G-invariant inner product on V. Let $w \in V$. Define $T_w : V \to C(G)$ by $T_w(v) = \langle \pi(g)v,\, w \rangle$. Show that

$$\|T_w(v)\|_\infty = \sup_{g \in G} |T_w(v)(g)| \leq \|v\|\|w\|, \qquad v \in V.$$

It follows that $T_w \in \mathrm{B}(V, C(G)) \subset \mathrm{B}(V, L^2(G))$. In particular, $T_w : V \to L^2(G)$ is a continuous linear transformation. Note that

$$T_w(\pi(x)v)(g) = \langle \pi(g)\pi(x)v,\, w \rangle = T_w(v)(gx) = (\rho(x)T_w(v))(g), \qquad x,\, g \in G,\ v \in V.$$

Hence $T_w \in \mathrm{Hom}_G(\pi, \rho)$. If $w \neq 0$, then $T_w(w)(1) = \|w\|^2 \neq 0$. Hence $T_w(w)$ is nonzero. Since $T_w(w) \in C(G) \in L^2(G)$, there exists an irreducible finite-dimensional representation σ of G such that $T_w(w) \in L^2(G, \sigma)$. Let P_σ be the orthogonal projection of V onto $L^2(G, \sigma)$. Then $P_\sigma \in \mathrm{Hom}_G((\rho, L^2(G)), (\rho, L^2(G, \sigma))$.

The composition $P_\sigma \circ T_w$ belongs to $\mathrm{Hom}_G(\pi, (\rho, L^2(G, \sigma))$. By an earlier result concerning interwining of unitary representations, there exists a subrepresentation of π that is equivalent to a subrepresentation of $(\rho, L^2(G, \sigma))$. Since $L^2(G, \sigma)$ is finite-dimensional, this shows that (π, V) has a finite-dimensional subrepresentation. $\square$

COROLLARY 7.2. *A topologically irreducible unitary representation of a compact group is finite-dimensional.*

THEOREM 7.3. *(Peter-Weyl Theorem 3) Let (π, V) be a unitary representation of a compact group G. Then V is a Hilbert space direct sum of finite-dimensional irreducible representations of G.*

PROOF. Without loss of generality V is nonzero. By Proposition 7.1, (π, V) has at least one irreducible (finite-dimensional) subrepresentation. Let Σ be the collection of sets of pairwise orthogonal finite-dimensional irreducible subrepresentations of V. (That is, if $S \in \Sigma$ and (σ_1, V_1), (σ_2, V_2) belong to S, then σ_1 and σ_2 are irreducible subrepresentations of π and $V_1 \perp V_2$.) By Zorn's Lemma, Σ has a maximal element, say S.

To prove the theorem, it suffices to show that finite linear combinations of vectors from subspaces in the set S are dense in V (see Exercise 5.7, Chapter 2). Let V' be the orthogonal complement of the span of vectors from subspaces in S. Then V' is G-invariant and closed. Assume that V' is nonzero. By Proposition 7.1, V' contains a finite-dimensional irreducible subrepresentation, say (σ, W). We can append W to S, contradicting maximality of S. It follows that $V' = \{0\}$. $\square$

DEFINITION 7.4. Let (π, V) be a unitary representation of G. We say that a vector $v \in V$ is *G-finite* if v belongs to a finite-dimensional subrepresentation of (π, V). (The set of G-finite vectors in V is a subspace of V.)

COROLLARY 7.5. *Let (π, V) be a unitary representation of G. The subspace of G-finite vectors in V is dense in V.*

In a later section, we will see how to use characters and integration to construct orthogonal projections onto finite-dimensional subrepresentations of unitary representations of compact groups.

8. Characters of finite-dimensional representations

DEFINITION 8.1. Let (π, V) be a finite-dimensional representation of a group G. The *character* of π is the function $\chi_\pi : G \to \mathbb{C}$ defined by $\chi_\pi(g) = \operatorname{tr} \pi(g)$, $g \in G$.

DEFINITION 8.2. A function $f : G \to \mathbb{C}$ is a *class function* if $f(gxg^{-1}) = f(x)$ for all g and $x \in G$.

Clearly the character of a finite-dimensional representation is a class function.

PROPOSITION 8.3. *Let (π, V) and (σ, W) be finite-dimensional representations of a group G. Then*

(1) *If* $\pi \simeq \sigma$, *then* $\chi_\pi = \chi_\sigma$.
(2) $\chi_{\pi\oplus\sigma} = \chi_\pi + \chi_\sigma$.
(3) $\chi_{\pi\otimes\sigma} = \chi_\pi \chi_\sigma$.
(4) $\chi_{\pi^\vee}(g) = \chi_\pi(g^{-1})$, $g \in G$.
(5) *If* π *is unitary, then* $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$, $g \in G$.
(6) $\chi_\pi(1) = \dim V$.
(7) χ_π *is a sum of matrix coefficients of* π.
(8) *If* π *is continuous, then* χ_π *is a continuous class function.*

EXAMPLE 8.4. Let $G = \mathbb{R}^\times \ltimes \mathbb{R}$. Define $\pi(t,x) = \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix}$ and $\sigma(t,x) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}^\times$, $x \in \mathbb{R}$. Then $\chi_\pi(t,x) = \chi_\sigma(t,x) = t+1$, $t \in \mathbb{R}^\times$, $x \in \mathbb{R}$. The characters of π and σ coincide, but π and σ are not equivalent.

THEOREM 8.5. *Let* (π, V_π) *and* (σ, V_σ) *be continuous finite-dimensional unitary represe-nations of a compact group* G. *Then*
(1) $\int_G \chi_\pi(g)\, dg = \dim\{\, v \in V_\pi \mid \pi(g)v = v \ \forall\, g \in G \,\}$.
(2) $\langle\, \chi_\pi,\, \chi_\sigma \,\rangle_{L^2(G)} = \dim \mathrm{Hom}_G(\sigma, \pi)$.
(3) *If* χ *and* σ *are irreducible, then*

$$\langle\, \chi_\pi,\, \chi_\sigma \,\rangle_{L^2(G)} = \begin{cases} 1, & \textit{if } V_\sigma \simeq V_\pi, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. Let $W = \{\, v \in V_\pi \mid \pi(g)v = v \ \forall\, g \in G \,\}$. Clearly, W is a subrepresentation of π. Let $P(v) = \int_G \pi(g)\, v\, dg$, $v \in V_\pi$. By invariance of Haar measure on G, $pi(g)P(v) = P(v)$ for all $v \in V_\pi$. Hence $P(V) \subset W$. It follows from the definition of P that $P(v) = v$ for all $v \in W$. Hence $W = P(V)$. Moreover, $P^2(v) = P(P(v)) = P(v)$, since $P(v) \in W$. So P is a projection of V onto W. (Indeed, we can see that P is the orthogonal projection of V onto W.) Hence the trace of P is equal to $\dim W$. Observe that the trace of P is also equal to $\int_G \chi_\pi(g)\, dg$.

Define a representation τ of G on $\mathrm{Hom}_{\mathbb{C}}(V_\sigma, V_\pi)$ by $\tau(g)A = \pi(g)A\sigma(g^{-1})$. Note that

$$\mathrm{Hom}_G(\sigma, \pi) = \{\, A \in \mathrm{Hom}_{\mathbb{C}}(V_\sigma, V_\pi) \mid \tau(g)A = A \,\}$$

Given $\lambda \in V_\sigma^*$ and $v \in V_\pi$, define $A_{\lambda\otimes v} \in \mathrm{Hom}_{\mathbb{C}}(V_\sigma, V_\pi)$ by $A_{\lambda\otimes v}(w) = \lambda(w)v$, $w \in V_\sigma$. Note that

$$A_{\sigma^\vee(g)\lambda\otimes\pi(g)v} = \pi(g)A_{\lambda\otimes v}\sigma(g^{-1}), \qquad g \in G.$$

Extending this to an equivalence between $\sigma^* \otimes \pi$ and τ, we have

$$\chi_\tau = \chi_{\sigma^*\otimes\pi} = \chi_{\sigma^\vee}\chi_\pi = \chi_\pi\overline{\chi_\sigma}.$$

Applying part (1) to the representation $(\tau, \mathrm{Hom}_{\mathbb{C}}(V_\sigma, V_\pi))$, we see that (2) holds.

Part (3) is a consequence of (2). It can also be obtained as a consequence of the Schur orthogonality relations for matrix coefficients(Theorem 5.4), since each character is a sum of matrix coefficients. □

PROPOSITION 8.6. *Let (π, V) be a finite-dimensional unitary representation of a compact group G. Then π is determined up to equivalence by χ_π.*

PROOF. We know that π is completely reducible. Let $\{\pi_j\}$ be the set of irreducible constituents of π. Then there exist positive integers n_j such that $\chi_\pi = \sum_j n_j\chi_{\pi_j}$. By Theorem 8.5, we have $n_j = \langle \chi_{\pi_j}, \chi_\pi\rangle_{L^2(G)}$. Consequently, if (σ, W) is a finite-dimensional unitary representation of G such that $\chi_\sigma = \chi_\pi$, we can see that, up to equivalence, the representations π_j are the irreducible constituents of σ, and the multiplicity of π_j in σ is equal to $\langle \chi_{\pi_j}, \chi_\sigma\rangle_{L^2(G)} = n_j$. Hence $\sigma \simeq \pi$. □

COROLLARY 8.7. *Let (π, V) be a finite-dimensional unitary representation of a compact group G. Then $\langle \chi_\pi, \chi_\pi\rangle_{L^2(G)} = 1$ if and only if π is irreducible.*

THEOREM 8.8. *The span of the characters of the irreducible unitary representations of a compact group G is dense in the space of continuous class functions on G.*

PROOF. Let $\varphi \in C(G)$ be a class function. Fix $\epsilon > 0$. By the Peter-Weyl Theorem 2, there exists $f \in \mathcal{A}(G)$ such that $\|\varphi - f\|_\infty < \epsilon$. For $x \in G$, let $\psi(x) = \int_G f(gxg^{-1})\,dg$. Then

$$\|\varphi - \psi\|_\infty = \sup_{g\in G} |\varphi(g) - \psi(g)| \le \sup_{g\in G} \int |\varphi(xgx^{-1}) - f(xgx^{-1})|\,dg \le \|\varphi - f\|_\infty < \epsilon.$$

To complete the proof, it suffices to show that ψ is a linear combination of characters of irreducible representations of G. Since $f \in \mathcal{A}(G)$, there exist finitely many irreducible unitary representations (π_j, V_j) of G, together with $v_j \in V_j$, $\lambda_j \in V_j^*$, such that $f(g) = \sum_j \lambda_j(\pi_j(g)v_j)$, $g \in G$. Observe that, thinking of $\pi_j(x)$ as an element of $\mathrm{End}(V_j)$, we have $\pi_j(x)^\natural = \int_G \pi_j(gxg^{-1})\,dg \in \mathrm{Hom}_G(\pi_j, \pi_j)$ (see Lemma 5.1). Since π_j is irreducible, by Proposition 5.2, $\pi_j(x)^\natural = -(\mathrm{tr}\, pi_j(x)/\dim V_j)I_{V_j}$. Therefore

$$\psi(x) = \sum_j \int_G \lambda_j(\pi_j(gxg^{-1})v_j)\,dg = \sum_j \lambda_j(\pi_j(x)^\natural v_j) = (\chi_{\pi_j(x)}/\dim V_j)\lambda_j(v_j).$$

That is, ψ is a linear combination of the characters χ_{π_j}. □

9. Convolution, characters and orthogonal projections

Let $f \in C(G)$ and let (π, V) be a unitary representation of G, with $\langle\ ,\ \rangle$ a G-invariant inner product on V. Define $\pi(f) : V \to V$ by $\pi(f)v = \int_G f(g)\pi(g)v\, dg$, $v \in V$. Recall that if f_1, $f_2 \in G$, the convolution $f_1 * f_2$ belongs to $C(G)$ and is defined by $(f_1 * f_2)(x) = \int_G f_1(g) f_2(g^{-1}x)\, dg$, $x \in G$. For $f \in C(G)$, define $\widetilde{f} \in C(G)$ by $\widetilde{f}(g) = \overline{f(g^{-1})}$, $g \in G$.

Lemma 9.1. *Let (π, V) be a unitary representation of G.*

(1) *If $f \in C(G)$, then $\pi(f) \in \mathrm{B}(V)$.*
(2) *If f_1, $f_2 \in C(G)$, then $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$.*
(3) *If v, $w \in V$ and $f \in C(G)$, then $\langle \pi(f)v, w \rangle = \langle v, \pi(\widetilde{f})w \rangle$.*
(4) *If $f \in C(G)$ and $g \in G$, then $\pi(g)\pi(f) = \pi(L_g f)$. (Here, $(L_g f)(x) = f(g^{-1}x)$, $x \in G$.)*

Exercise 9.2. Prove Lemma 9.1.

Exercise 9.3. If $x \in G$ and $f \in C(G)$, define $\tau(x)f$ by $(\tau(x)f)(g) = f(x^{-1}gx)$, $g \in G$. Show that $\pi(x)\pi(f) = \pi(\tau(x)f)\pi(x)$.

Lemma 9.4. *Let (π, V) be a unitary representation of G. Let $f \in C(G)$.*

(1) *If f is a class function, then $\pi(f) \in \mathrm{Hom}_G(\pi, \pi)$.*
(2) *If $f = \widetilde{f}$ and $f * f = f$, then $\pi(f)$ is an orthogonal projection.*

Proof. The first statement follows from Exercise 9.3 and Lemma 9.1(1).

For the second statement, $f * f = f$ implies $\pi(f)$ is a projection. Then, from $\widetilde{f} = f$ and Lemma 9.1(3), we have that $\pi(f)$ is self-adjoint. Self-adjoint projections are orthogonal projections. □

Corollary 9.5. *If $f \in C(G)$ is a class function such that $f * f = f$ and $\widetilde{f} = f$, then $\pi(f)$ is an orthogonal projection of V onto a subrepresentation of (π, V). Furthermore, if f_1 and f_2 are two such functions and $f_1 * f_2 = 0$, then $\pi(f_1)V$ and $\pi(f_2)V$ are orthogonal.*

Proposition 9.6. *Let (π, V) and (σ, W) be irreducible unitary representations of G. Then*

$$\chi_\pi * \chi_\sigma = \begin{cases} (\dim V)^{-1}\chi_\pi, & \text{if } \pi \simeq \sigma \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\langle\ ,\ \rangle$ be a G-invariant inner product on W. Let $\{v_1,\dots,v_n\}$ be an orthonormal basis of W. For g and $x \in G$,

$$\chi_\sigma(g^{-1}x) = \operatorname{tr}(\sigma(g^{-1})\sigma(x)) = \sum_{i,j=1}^{n} \langle v_j, \sigma(g^{-1})v_i\rangle\langle \sigma(x)v_i, v_j\rangle = \sum_{i,j=1}^{n} \overline{\langle \sigma(g)v_i, v_j\rangle}\langle \sigma(x)v_i, v_j\rangle$$

Here we have used the fact that the matrix of $\sigma(g)$ with respect to the orthonormal basis is a unitary matrix.

$$(\chi_\pi * \chi_\sigma)(x) = \int_G \chi_\pi(g)\operatorname{tr}(\sigma(g^{-1})\sigma(x))\,dg = \sum_{i,j=1}^{m}\left(\int_G \chi_\pi(g)\overline{\langle\sigma(g)v_i, v_j\rangle}\,dg\right)\langle\sigma(x)v_i, v_j\rangle.$$

Because χ_π is a sum of matrix coefficients of π, if π is not equivalent to σ, by Theorem 5.4, the $L^2(G)$ inner product of χ_π and the matrix coefficient $g \mapsto by\langle\sigma(g)v_i, v_j\rangle$ is equal to zero.

Suppose that $\pi \simeq \sigma$. Then $\chi_\pi = \chi_\sigma$, so we may assume that $\pi = \sigma$ and replace $\sigma(g)$ and $\sigma(x)$ by $\pi(g)$ and $\pi(x)$ above. We have $\chi_\pi(g) = \sum_{\ell=1}^{n}\langle\pi(g)v_\ell, v_\ell\rangle$. By the Schur Orthogonality Relations (Theorem 5.4),

$$\int_G \chi_\pi(g)\overline{\langle\pi(g)v_i, v_j\rangle}\,dg = \sum_{\ell=1}^{n}\int_G \langle\pi(g)v_\ell, v_\ell\rangle\overline{\langle\pi(g)v_i, v_j\rangle} = \sum_{\ell=1}^{n}\langle v_\ell, v_i\rangle\overline{\langle v_\ell, v_j\rangle}/m,$$

which equals 1 if $i = j$ and 0 if $i \neq j$. It follows that

$$(\chi_\pi * \chi_\pi)(x) = \sum_{j=1}^{n} 1/n(\langle\sigma(x)v_j, v_j\rangle) = (\dim V)^{-1}\chi_\pi(x).$$

$\square$

Exercise 9.7. Let (π, V) and (σ, W) be irreducible unitary representations of G.

(1) Suppose that f_π and f_σ are matrix coefficients of π and σ, respectively and $\pi \not\simeq \sigma$. Prove that $f_\pi * f_\sigma = 0$.
(2) Let $\langle\ ,\ \rangle$ be a G-invariant inner product on the space of π. For vectors u, $v \in V$, set $f_{u,v}(g) = \langle\pi(g)u, v\rangle$, $g \in G$. Let u, v, u' and v' be vectors in V. Prove that $f_{u,v} * f_{u',v'} = (\langle u, v'\rangle/\dim V)f_{u',v}$.

Definition 9.8. Let (π, V) and (σ, W) be unitary representations of G. Assume that σ is irreducible. (Then σ is finite-dimensional.) Let V^σ be the largest subrepresentation of (π, V) having the property that every irreducible subrepresentation of (π, V^σ) is equivalent to σ. The space V^σ is called the σ-*isotypic* (or σ-*isotypical*) subspace of π.

According to the Peter-Weyl Theorem 3 (Theorem 7.3), (π, V) is equivalent to the Hilbert space direct sum $(\oplus\pi \mid V^\sigma, \oplus V^\sigma)$ (described in Exercise 5.7 of Chapter 2), where σ ranges over the irreducible unitary representations of G such that V^σ is nonzero.

Proposition 9.9. *Let (π, V) be a unitary representation of G. Let (σ, W) be an irreducible unitary representation of G such that V^σ is nonzero. Let $e_\sigma = \dim(W)\overline{\chi_\sigma}$. Then $\pi(e_\sigma)$ is the orthogonal projection of V onto V^σ. That is, $v \mapsto (\dim W)\int_G \overline{\chi_\sigma(g)}\pi(g)v\,dg$ is the orthogonal projection of V onto V^σ.*

Proof. By Proposition 8.3, e_σ is a class function and $\widetilde{e_\sigma} = e_\sigma$. According to Proposition 9.6, we have $e_\sigma * e_\sigma = e_\sigma$. Hence (see Corollary 9.5), $\pi(e_\sigma) \in \mathrm{Hom}_G(\pi, \pi)$ and is an orthogonal projection.

We need to verify that $\pi(e_\sigma)V = V^\sigma$. Let (π', V') be an irreducible subrepresentation of (π, V) that is equivalent to σ. Let $\{ v_1, \ldots, v_n \}$ be an orthonormal basis for V'. Then

$$\begin{aligned}\pi(e_\sigma)v_i &= \int_G (\dim W)\overline{\chi_\sigma(g)}\pi'(g)v_i\,dg = (\dim W)\sum_{j=1}^n \left(\int_G \overline{\chi_\sigma(g)}\langle \pi'(g)v_i, v_j\rangle\,dg\right) v_j \\ &= (\dim W)\sum_{j=1}^n \left(\int_G \overline{\chi_{\pi'}(g)}\langle \pi'(g)v_i, v_j\rangle\,dg\right) v_j \\ &= (\dim W)\sum_{j,\ell=1}^n \left(\int_G \overline{\langle \pi'(g)v_\ell, v_\ell\rangle}\langle \pi'(g)v_i, v_j\rangle\,dg\right) v_j \\ &= (\dim W)\sum_{j,\ell=1}^n \left(\langle v_i, v_\ell\rangle\overline{\langle v_j, v_\ell\rangle}/\dim W\right) v_j = v_i.\end{aligned}$$

(Above, we have used Theorem 5.4). Because $\pi(e_\sigma)$ is an orthogonal projection, $\pi(e_\sigma)v_i = v_i$ is equivalent to $v_i \in \pi(e_\sigma)V$. Hence $(\pi', V') \subset \pi(e_\sigma)V$. That is, $V^\sigma \subset \pi(e_\sigma)V$.

To complete the proof, it suffices to show that any irreducible subrepresentation of $(\pi, \pi(e_\sigma)V)$ is equivalent to (σ, W). Equivalently, it's enough to show that an irreducible subrepresentation (π', V') of (π, V) that is not equivalent to σ belongs to the kernel of $\pi(e_\sigma)$. As above, let $\{ v_1, \ldots, v_n \}$ be an orthonormal basis for V'. Then

$$\pi(e_\sigma)v_i = \int_G (\dim W)\overline{\chi_\sigma(g)}\pi'(g)v_i\,dg = (\dim W)\sum_{j=1}^n \left(\int_G \overline{\chi_\sigma(g)}\langle \pi'(g)v_i, v_j\rangle\,dg\right) v_j.$$

Since χ_σ is a sum of matrix coefficients of σ and $\sigma \not\simeq \pi'$, it follows from Theorem 5.4 that $\int_G \overline{\chi_\sigma(g)}\langle \pi'(g)v_i, v_j\rangle = 0$ for all i and j. That is, $\pi(e_\sigma)v_i = 0$ for all i. □

REMARK 9.10. Let f be a matrix coefficient of σ. We can show that $f * e_\sigma = e_\sigma * f = f$ (see Exercise 9.7). It follows that $\pi(f) = \pi(e_\sigma * f) = \pi(e_\sigma)\pi(f)$. So $\pi(f)V \subset V^\sigma$

EXAMPLE 9.11. Consider the right regular representation $(\rho, L^2(G))$. Let (σ, W) be an irreducible unitary representation of G. For $f \in L^2(G)$ and $x \in G$,

$$\begin{aligned}(f * \overline{e_\sigma})(x) &= (\dim W)^{-1} \int_G f(g)\, \chi_\sigma(g^{-1}x)\, dg = (\dim W)^{-1} \int_G f(xg)\,]\chi_\sigma(g^{-1})\, dg \\ &= (\dim W)^{-1} \int_G \overline{\chi_\sigma(g)}\, (\rho(g)f)(x)\, dg = \int_G e_\sigma(g)(\rho(g)f)(x)\, dg = (\rho(e_\sigma)f)(x)\end{aligned}$$

That is, $f * \overline{e_\sigma} = \rho(e_\sigma)f$. So in this example, the orthogonal projection onto the σ-isotypic subspace is given by convolution with $(\dim W)^{-1}\chi_\sigma$.

10. Induced representations of compact groups

DEFINITION 10.1. Let W be a topological vector space and let $C(G, W)$ be the space of continuous functions from G to W. If K is a compact subset of G and U is an open subset of W, define $\mathcal{V}(K, U) = \{\, f \in C(G, W) \mid f(K) \subset U\,\}$. The *compact-open* topology on $C(G, W)$ is the topology for which the sets $\mathcal{V}(K, U)$ form a subbasis.

LEMMA 10.2. *$C(G, W)$ is a topological vector space and the map $G \times C(G, W) \to C(G, W)$ given by $(g, f) \mapsto L_g f$ is continuous.*

DEFINITION 10.3. Let (σ, W) be a continuous representation of a closed subgroup H of G. Let $i_H^G(W) = \{\, f \in C(G, W) \mid f(gh) = \sigma(h)^{-1}f(g)\ \forall\, g \in G,\ h \in H\,\}$. For $f \in i_H^G(W)$ and $g \in G$, set $(i_H^G\sigma(g)f)(x) = f(g^{-1}x)$. The representation $(i_H^G\sigma, i_H^G(W))$ is called the *representation of G induced from (σ, W)*.

LEMMA 10.4. *When $i_H^G(W)$ is given the compact-open topology, $(i_H^G\sigma, i_H^G(W))$ is a continuous representation of G.*

THEOREM 10.5. *(Frobenius Reciprocity) Let H be a closed subgroup of G. Let (π, V) and (σ, W) be continous representations of G and H, respectively. Then* $\mathrm{Hom}_G(\pi, i_H^G\sigma) \simeq \mathrm{Hom}_H(\mathrm{res}_H^G\pi, \sigma)$. *(Here,* $\mathrm{res}_H^G\pi$ *is the representation of H obtained by restriction of π to H.)*

PROOF. (Sketch) Let $A \in \mathrm{Hom}_G(\pi, i_H^G\sigma)$. Define $\eta : i_H^G(W) \to W$ by $\eta(f) = f(1)$. Check that $A \mapsto \eta \circ A$ is a map from $\mathrm{Hom}_G(\pi, i_H^G\sigma)$ to $\mathrm{Hom}_H(\mathrm{res}_H^G\pi, \sigma)$.

Next, suppose that $B \in \mathrm{Hom}_H(\mathrm{res}^G_H \pi, \sigma)$. Define $B' : V \to C(G,W)$ by $B'(v)(g) = B(\pi(g)^{-1}v)$, $v \in V$, $g \in G$. Then

$$B'(v)(gh) = B(\pi(gh)^{-1}v) = B(\pi(h)^{-1}\pi(g)^{-1}v) = \sigma(h)^{-1}B'(v)(g), \qquad g \in G,\, h \in H.$$

Hence $B'(v) \in i^G_H(W)$. Next, note that

$$B'(\pi(g)v)(x) = B(\pi(x^{-1}g)v) = B'(v)(g^{-1}x) = ((i^G_H\sigma)(g)B'(v))(x), \qquad g,\, x \in G,\ v \in V.$$

Thus the map $B \mapsto B'$ goes from $\mathrm{Hom}_H(\mathrm{res}^G_H \pi, \sigma)$ to $\mathrm{Hom}_G(\pi, i^G_H \sigma)$.

To complete the proof, check that the maps $A \mapsto \eta \circ A$ and $B \mapsto B'$ are inverses of each other. $\square$

For another approach to induction of representations, assume that (σ, W) is a unitary representation of a closed subgroup H of G. Let $L^2(G,W)$ be the completion of the space $C(G,W)$ with respect to the norm $\|f\| = \left(\int_G \|f(g)\|_W^2\, dg\right)^{1/2}$. Then, for the induced representation, we take the left regular representation on the space $\{\, f \in L^2(G,W) \mid f(gh) = \sigma(h)^{-1}f(g)\ \forall\, g \in G,\, h \in H \,\}$. This is the approach taken in [**R**].

CHAPTER 4

Representations of compact Lie groups

Throughout this chapter, G is a compact connected Lie group. The representations considered are finite-dimensional and continuous. Although outlines of a few proofs are included, most of the results are stated without proof. We focus on the parametrization of the irreducible representations in terms of dominant weights and the characters of the irreducible representations.

The Borel-Weil Theorem (and its generalization – the Borel-Weil-Bott Theorem), which provides concrete models for the irreducible representations of compact Lie groups, is not discussed. The text [**Se**] includes a section on the Borel-Weil Theorem.

There are many references on representations of compact Lie groups, for example, [**BD**], [**Ha**], [**Se**] and [**Si**].

The first section contains a summary of results about maximal tori in compact Lie groups and the statement of Weyl's Integral Formula.

In Section 2, we define weights of finite-dimensional representations and list a few of their properties.

At the beginning of Section 3, we recall some definitions related to Lie algebras and describe how to attach a (real) representation of the Lie algebra $\mathfrak{g}$ of G to a finite-dimensional (continuous) representation of G. Then we complexify to produce a complex representation of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$.

A choice of maximal torus T in G determines a *root space decomposition* of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, which is described in Section 4. This decomposition has the form $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{t}^{\mathbb{C}}$ is the complexification of the Lie algebra of T, Φ is a finite collection of linear functionals, called roots, and each $\mathfrak{g}_\alpha$ is a subspace of $\mathfrak{g}^{\mathbb{C}}$ determined by the root $\alpha \in \Phi$. Certain properties of the representation of $\mathfrak{g}^{\mathbb{C}}$ determined by a given finite-dimensional representation of G can be used to obtain information about the weights of the representation of G. In particular, each irreducible finite-dimensional representation has a uniquely defined "highest weight".

In Section 5, we comment on how Weyl's integral formula, together with information about weights of representations, is applied to obtain a formula for the character of an irreducible finite-dimensional representation of G. In addition, we discuss how to use the Peter-Weyl Theorem to see that every "dominant" weight occurs as the highest weight of some irreducible finite-dimensional representation.

1. Maximal tori and Weyl's Integral Formula

Proofs of results stated in this section can be found in Chapter IV of [**BD**] (and various other references).

DEFINITION 1.1. A Lie group is a *compact torus* if it is isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$ for some integer $n > 0$. (Here, n is the *dimension* of the torus.) A subgroup T of a Lie group is a *maximal torus* in a compact Lie group G if T is a compact torus and there is no other compact torus T' with $T \subsetneq T' \subset G$.

Since compact tori are connected, if T and T' are compact tori in G with $T \subsetneq T'$, then the dimension of T is strictly less than the dimension of T'. Hence maximal tori exist. A maximal torus in G is a maximal connected abelian subgroup of G.

DEFINITION 1.2. Let T be a maximal torus in G. Let $N_G(T)$ be the normalizer of T in G. That is $N_G(T) = \{\, g \in G \mid gtg^{-1} \in T \ \forall\, t \in T \,\}$. The group $W = N_G(T)/T$ is the *Weyl group* of T.

THEOREM 1.3. (1) *Any two maximal tori in G are conjugate.*
(2) *Every element of G is contained in a maximal torus.*
(3) *The Weyl group of a maximal torus is finite.*

EXAMPLE 1.4. For the group $G = U(n) = \{\, g \in GL_n(\mathbb{C}) \mid {}^t\bar{g}g = I_n \,\}$ (resp. $G' = SU(n) = U(n) \cap SL_n(\mathbb{C})$), the group of diagonal matrices in G (resp. G') is a maximal torus. In both cases, W is isomorphic to the symmetric group on n letters.

Because any two maximal tori in G are conjugate, we can see that if T_1 and T_2 are maximal tori in G, then the Weyl groups of T_1 and T_2 are isomorphic. For this reason, we refer to the Weyl group of a maximal torus in G as the Weyl group of G.

For the rest of the section, we fix a maximal torus T in G.

DEFINITION 1.5. let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. For $t \in T$ and $X \in \mathfrak{g}$, define $\mathrm{Ad}(t)(X) = tXt^{-1}$. Note that $\mathrm{Ad}(t)(X) = X$ for all $X \in \mathfrak{t}$ and $t \in T$.

Hence we may view $\mathrm{Ad}(t)$ (and hence also $\mathrm{Ad}(t^{-1}) - I$) as an element of $\mathrm{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{t})$. Let $D(t) = |\det(\mathrm{Ad}(t^{-1}) - I)_{\mathfrak{g}/\mathfrak{t}}|$, $t \in T$. (In fact, the absolute value is not needed here, as the endomorphism $\mathrm{Ad}(t^{-1}) - I$ of $\mathfrak{g}/\mathfrak{t}$ has no nonzero real eigenvalues.)

If $z_1, \dots z_n \in \mathbb{C}$, let $d(z_1, \dots, z_n)$ be the diagonal matrix whose (j,j) entry is equal to z_j, $1 \le j \le n$.

Example 1.6. For $G = SU(3) = \{\, g \in GL_3(\mathbb{C}) \mid {}^t\bar{g}g = I_3 \,\}$, the set T of diagonal matrices in G, that is, the matrices of the form $d(e^{2\pi i t_1}, e^{2\pi i t_2}, e^{-2\pi i(t_1+t_2)})$, where t_1, $t_2 \in \mathbb{R}$, is a maximal torus in G. The Lie algebra of G is $\mathfrak{g} = \{\, X \in M_3(\mathbb{C}) \mid {}^t\bar{X} = -X,\ \mathrm{tr}(X) = 0 \,\}$ and $\mathfrak{t}$ is the diagonal matrices in $M_3(\mathbb{C})$ with purely imaginary entries and trace zero. So $\mathfrak{g}/\mathfrak{t}$ can be identified with the (real) subspace of $\mathfrak{g}$ consisting of those matrices of the form

$$\begin{pmatrix} 0 & z_1 & z_2 \\ -\bar{z}_1 & 0 & z_3 \\ -\bar{z}_2 & -\bar{z}_3 & 0 \end{pmatrix}, \qquad \text{where } z_1,\ z_2,\ z_3 \in \mathbb{C}.$$

The following integral formula is an important tool in the proof of Weyl's character formula.

Theorem 1.7. *(Weyl's Integral Formula) Let $f \in C(G)$. Then*

$$|W| \int_G f(g)\, dg = \int_T \left(D(t) \int_G f(gtg^{-1})\, dg \right) dt.$$

If f is a class function, then

$$|W| \int_G f(g)\, dg = \int_T D(t)\, f(t)\, dt.$$

Lemma 1.8. *Let t_1, $t_2 \in T$. Then t_1 and t_2 are conjugate in G if and only if t_1 and t_2 are conjugate in $N_G(T)$.*

Definition 1.9. The group $N_G(T)$ acts on the space $C(T)$ (continuous complex-valued functions on T) as follows: Given $g \in N_G(T)$ and $f \in C(T)$, set ${}^g f(t) = f(g^{-1}tg)$, $t \in T$. When $g \in T$, ${}^g f = f$, so the action of $N_G(T)$ factors to an action of W on $C(T)$. Let $C(T)^W$ be the W-invariant functions in $C(T)$.

Definition 1.10. The *representation ring* $R(G)$ of G be the set of class functions of the form $\sum_i n_i \chi_i$, where $n_i \in \mathbb{Z}$ and χ_i is the character of a finite-dimensional representation of G. Let $R(T)^W$ be the W-invariant functions in $R(T)$.

Clearly, the restriction of a continuous class function on G to T belongs to $C(T)^W$.

LEMMA 1.11. (1) *The restriction map* $f \mapsto f \,|\, T$ *from* $C(G)$ *to* $C(T)^W$ *is onto.*
(2) *For* $f \in R(G)$, *the restriction* $f \,|\, T$ *belongs to* $R(T)^W$.
(3) *The map* $f \mapsto f \,|\, T$ *from* $R(G)$ *to* $R(T)^W$ *is one-to-one.*

REMARK 1.12. The map from $R(G)$ to $R(T)^W$ is actually an isomorphism.

2. Weights of finite-dimensional representations

Let (Π, V) be a finite-dimensional representation of a compact Lie group G. Let T be a maximal torus in G. The restriction $\mathrm{res}^G_T\sigma$ of σ to T is a direct sum of one-dimensional representations (these are also called characters) of T.

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ and $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Set $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}$.

The map $X \mapsto e^{2\pi i X}$ from $\mathfrak{t}_{\mathbb{R}}$ to T is a surjective homomorphism from $\mathbb{R}^d$ onto T, where d is the dimension of T. (Here, e denotes the usual matrix exponential map.) Let $\mathcal{K}$ be the kernel of this homomorphism. Then $\mathcal{K} \simeq \mathbb{Z}^d$ and $T \simeq \mathbb{R}^d/\mathcal{K}$.

EXAMPLE 2.1. Continuing with Example 1.6, recall that $\mathfrak{t} = \{\, d(it_1, it_2, -i(t_1+t_2)) \mid t_1,\, t_2 \in \mathbb{R}$. So we have $\mathfrak{t}_{\mathbb{R}} = \{\, d(t_1, t_2, -t_1 - t_2)) \mid t_1,\, t_2 \in \mathbb{R}$. For convenience, we set $X_{t_1,t_2} = d(t_1, t_2, -t_1 - t_2)$, $t_1,\ t_2 \in \mathbb{R}$. Here, $\mathcal{K} = \{\, X_{t_1,t_2} \mid t_1,\, t_2 \in \mathbb{Z}\,\}$. The complexfication $\mathfrak{g}^{\mathbb{C}}$ is equal to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ of 3×3 matrices whose trace is zero. Note that $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{sl}_3(\mathbb{R}) \oplus i\mathfrak{sl}_3(\mathbb{R})$, so $\mathfrak{sl}_3(\mathbb{C})$ is also the complexification of the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$. That is, $\mathfrak{g} = \mathfrak{su}(\mathfrak{z})$ and $\mathfrak{sl}_3(\mathbb{R})$ are real forms of $\mathfrak{sl}_3(\mathbb{C})$.

DEFINITION 2.2. The *weight lattice* $\mathcal{Y}$ is defined to be $\{\, \rho \in \mathfrak{t}^*_{\mathbb{R}} \mid \lambda(X) \in \mathbb{Z}\ \forall\ X \in \mathcal{K}\,\}$. Given $\rho \in \mathcal{Y}$, define $e_\rho(X) = e^{2\pi i\rho(X)}$, $X \in \mathfrak{t}_{\mathbb{R}}$.

For $\rho \in \mathcal{Y}$, since $e_\rho(X) = 1$ for all $X \in \mathcal{K}$, we may view e_ρ as a one-dimensional representation of the abelian group $\mathfrak{t}_{\mathbb{R}}/\mathcal{K}$. Since $T \simeq \mathfrak{t}_{\mathbb{R}}/\mathcal{K}$, we also view e_ρ as a one-dimensional representation of T.

DEFINITION 2.3. If (Π, V) is a finite-dimensional unitary representation of G and $\rho \in \mathcal{Y}$, set $V_\rho = \{\, v \in V \;|\Pi(e^{2\pi i X}) = e_\rho(X)v \mid X \in \mathfrak{t}_{\mathbb{R}}\,\}$.

We have $V = \oplus_{\rho\in\mathcal{Y}} V_\rho$. Since V is finite-dimensional, V_ρ is nonzero for finitely many ρ.

DEFINITION 2.4. The set $Y(\Pi) = \{\, \rho \in \mathcal{Y} \mid V_\rho \neq \{0\}\,\}$ is called the set of *weights* of Π. If $\rho \in Y(\Pi)$, let $m_\rho = \dim(V_\rho)$. We refer to m_ρ as the *multiplicity* of ρ in Π.

If $g \in G$, then conjugation by g preserves $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$. If $g \in N_G(T)$, then conjugation by g preserves $\mathfrak{t}$, $\mathfrak{t}^{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{R}}$. If g belongs to T, the restriction of conjugation by g to $\mathfrak{t}^{\mathbb{C}}$ is the identity map – that is g commutes with every matrix in $\mathfrak{t}^{\mathbb{C}}$. In Definition 1.9, we defined actions of $N_G(T)$ and $W = N_G(T)/T$ on $C(T)$.

If $\rho \in \mathcal{Y}$, and $g \in N_G(T)$, then, viewing e_ρ as an element of $C(T)$, we have already defined ${}^g e_\rho$ (see Definition 1.9). It follows from the definitions that ${}^g e_\rho(X) = e_\rho(g^{-1}Xg)$, $X \in \mathfrak{t}_{\mathbb{R}}$.

Definition 2.5. If $\rho \in \mathcal{Y}$ and $w = gT \in W$, we set $(w\rho)(X) = \rho(g^{-1}Xg)$, $X \in \mathfrak{t}_{\mathbb{R}}$.

Because $g\, e^{2\pi i X} g^{-1} = e^{2\pi i g X g^{-1}}$ for $X \in \mathfrak{g}$ and $g \in G$, it follows that if $g \in N_G(T)$, then $g\mathcal{K}g^{-1} = \mathcal{K}$.

Lemma 2.6. *Let $w \in W$ and $\rho \in \mathcal{Y}$, then $w\rho \in \mathcal{Y}$ and ${}^w e_\rho = e_{w\rho}$ for $w \in W$.*

Lemma 2.7. *If $w \in W$ and $\rho \in Y(\Pi)$, then $m_\rho = m_{w\rho}$.*

Proof. Choose $g \in G$ such that $w = gT$. For $t = e^{2\pi i X} \in T$ $(X \in \mathfrak{t}_{\mathbb{R}})$ and $v \in V_\rho$,

$$\Pi(t)\Pi(g)v = \Pi(g)(\Pi(g^{-1}tg)v = \Pi(g)e_\rho(g^{-1}Xg)v = e_{w\rho}(X)\Pi(g)v.$$

Hence $\Pi(g)V_\rho \subset V_{w\rho}$. Similarly, $\Pi(g^{-1})V_{w\rho} \subset V_\rho$. Thus $m_\rho = m_{w\rho}$. □

Example 2.8. In Example 1.6, $SU(3)$ is realized as a closed subgroup of $GL_3(\mathbb{C})$. We can define a 3-dimensional faithful representation Π of $SU(3)$ by $\Pi(g) = g$, $g \in SU(3)$. We call this the *self-representation*. Define $\rho_1(X_{t_1,t_2}) = t_1$ and $\rho_2(X_{t_1,t_2}) = t_2$, for t_1, $t_2 \in \mathbb{R}$. Since

$$\Pi(e^{2\pi i X_{t_1,t_2}}) = e^{2\pi i X_{t_1,t_2}} = d(e^{2\pi i t_1}, e^{2\pi i t_2}, e^{2\pi i(-t_1-t_2)}),$$

we see that $Y(\Pi) = \{\, \rho_1,\, \rho_2,\, -\rho_1 - \rho_2 \,\}$.

3. Relations between representations of compact Lie groups and complex Lie algebras

If G is a matrix Lie group occurring as a closed subgroup of $GL_n(\mathbb{C})$, then the Lie algebra $\mathfrak{g}$ of G is given by $\mathfrak{g} = \{\, X \in M_n(\mathbb{C}) \mid e^{tX} \in G\ \forall\, t \in \mathbb{R} \,\}$. For X, $Y \in M_n(\mathbb{C})$, the Lie algebra bracket $[X, Y]$ is defined by $[X, Y] = XY - YX$. If X,$Y \in \mathfrak{g}$, then $[X, Y] \in \mathfrak{g}$.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra.

(1) $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all X and $Y \in \mathfrak{g}$.

(2) A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all X and $Y \in \mathfrak{h}$.
(3) A subspace $\mathcal{I}$ of $\mathfrak{g}$ is an *ideal* if $[X, Y] \in \mathcal{I}$ whenever $X \in \mathfrak{g}$ and $Y \in \mathcal{I}$.
(4) The Lie algebra $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is nonabelian and $\{0\}$ and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$.
(5) The Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ does not contain any nonzero abelian ideals.
(6) The Lie algebra $\mathfrak{g}$ is *reductive* if $\mathfrak{g}$ is the direct sum of a semisimple subalgebra and an abelian subalgebra. (Equivalently, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$.)
(7) If $X \in \mathfrak{g}$, $\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}X(Y) = [X, Y]$, $Y \in \mathfrak{g}$.
(8) The *Killing form* is defined by $K(X, Y) = \mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y)$, X, $Y \in \mathfrak{g}$. It is a symmetric bilinear form on $\mathfrak{g}$.

Lemma 3.2. (1) *$K([X, Y], Z) = K(X, [Y, Z])$ for all X, Y, $Z \in \mathfrak{g}$.*
(2) *$K(gXg^{-1}, gYg^{-1}) = K(X, Y)$ for all X, $Y \in \mathfrak{g}$ and $g \in G$.*

Lemma 3.3. *(Cartan's criterion) A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.*

Lemma 3.4. *If G is a compact Lie group, then*
(1) *The Killing form is negative semidefinite.*
(2) *The Lie algebra $\mathfrak{g}$ of G is reductive.*

Proposition 3.5. *Let G and H be matrix Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then there exists a unique $\mathbb{R}$-linear map $d\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi(e^X) = e^{d\varphi(X)}$ for all $X \in \mathfrak{g}$. The map $d\varphi$ has the following additional properties:*
(1) *$d\varphi(gXg^{-1}) = \varphi(g)d\varphi(X)\varphi(g)^{-1}$, $g \in G$, $X \in \mathfrak{g}$.*
(2) *$d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$, X, $Y \in \mathfrak{g}$.*
(3) *$d\varphi(X) = d/dt\varphi(e^{tX})\,|_{t=0}$, $X \in \mathfrak{g}$.*

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let (Π, V) be a finite-dimensional unitary representation of G. Then $g \mapsto \Pi(g)$ is a continuous homomorphism from G to $U(n)$.

Proposition 3.6. *If G and H are Lie groups and $\varphi : G \to H$ is a continuous homomorphism, then φ is a homomorphism of Lie groups (that is, φ is smooth).*

Applying the above propositions to the homomorphism $g \mapsto \Pi(g)$, we have an $\mathbb{R}$-linear map $d\Pi : \mathfrak{g} \to \mathfrak{u}(n)$ such that $\Pi(e^X) = e^{d\Pi(X)}$ for all $X \in \mathfrak{g}$. Furthermore, $d\Pi([X, Y]) =$

$[d\Pi(X), d\Pi(Y)]$ for $X, Y \in \mathfrak{g}$. That is, $d\Pi$ is a real representation of the Lie algebra $\mathfrak{g}$. It is conventional to use the notation Π, rather than $d\Pi$, for the representation of $\mathfrak{g}$ determined by the group representation Π. Since $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, we can extend Π to a complex representation of the complex semisimple Lie algebra $\mathfrak{g}$. The notation Π will be used for this extension.

As discussed in the next section, properties of the representation Π of $\mathfrak{g}^{\mathbb{C}}$ can be used to analyze the weights $Y(\Pi)$.

REMARK 3.7. If $\rho \in \mathcal{Y}(\Pi)$ and $v \in V_\rho$, then $\Pi(X)v = \rho(X)v$, $X \in \mathfrak{t}_{\mathbb{R}}$.

EXAMPLE 3.8. Let Π be the self-representation of $SU(3)$. Then the representation of $\mathfrak{sl}_3(\mathbb{C})$ determined by Π satisfies $\Pi(X) = X$, $X \in \mathfrak{sl}_3(\mathbb{C})$.

4. Highest weight theorem

For simplicity, we assume that G is a semisimple compact Lie group. That is, we assume that the centre of G is finite. In this case, the Killing form (see Definition 3.1 and Lemma 3.3) is nondegenerate.

Let Π, T, $\mathcal{Y}$, $\mathcal{K}$, etc, be as in previous sections.

We define an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}^{\mathbb{C}}$ by $\langle X, Y \rangle = K(\bar{X}, Y)$, $X, Y \in \mathfrak{g}^{\mathbb{C}}$. If $X \in \mathfrak{t}_{\mathbb{R}}$, $\mathrm{ad}X$ is self-adjoint with respect to this inner product. So $\{ \mathrm{ad}X \mid X \in \mathfrak{t}_{\mathbb{R}} \}$ is a space of commuting self-adjoint operators on $\mathfrak{g}^{\mathbb{C}}$. Therefore the space $\mathfrak{g}^{\mathbb{C}}$ has a decomposition into simultaneous eigenspaces.

DEFINITION 4.1. A *root* is a nonzero $\alpha \in \mathfrak{t}_{\mathbb{R}}^*$ such that there exists a nonzero $Y \in \mathfrak{g}^{\mathbb{C}}$ with $\mathrm{ad}X(Y) = \alpha(X)Y$ for all $X \in \mathfrak{t}_{\mathbb{R}}$. Let Φ be the set of roots. If $\alpha \in \Phi$, the *root space* corresponding α is

$$\mathfrak{g}_\alpha := \{ Y \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}X(Y) = \alpha(X)Y \ \forall\, X \in \mathfrak{t}_{\mathbb{R}} \}.$$

There exist a unique $H_\alpha \in \mathfrak{t}_{\mathbb{R}}$, called a *root vector* for α, such that $\alpha(X) = K(H_\alpha, X)$ for all $X \in \mathfrak{t}_{\mathbb{R}}$.

PROPOSITION 4.2. (1) *If* $\alpha \in \Phi$, *then* $\dim \mathfrak{g}_\alpha = 1$.

(2) *If* α, β *and* $\alpha + \beta \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ *is a nonzero subset of* $\mathfrak{g}_{\alpha+\beta}$.

(3) *If* $\alpha + \beta \in \Phi$ *and* $\alpha + \beta \notin \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$.

(4) *If* $\alpha \in \Phi$, *then* $-\alpha \in \Phi$.

(5) $\mathrm{Span}_{\mathbb{R}}(\Phi) = \mathfrak{t}_R^*$

(6) $\mathrm{Span}_{\mathbb{R}}\{ H_\alpha \mid \alpha \in \Phi \} = \mathfrak{t}_{\mathbb{R}}$.

(7) *There exist elements* $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$, *such that* $[X_\alpha, X_{-\alpha}] = 2H_\alpha / K(H_\alpha, H_\alpha)$.
(8) $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$.
(9) *If* X, $Y \in \mathfrak{t}_{\mathbb{R}}$, *then* $K(X,Y) = \sum_{\alpha\in\Phi} \alpha(X)\alpha(Y)$.

EXAMPLE 4.3. Returning to our basic example $G = SU(3)$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_3(\mathbb{C})$, and $\mathfrak{t}_{\mathbb{R}} = \{ X_{t_1,t_2} \mid t_1,\, t_2 \in \mathbb{R} \}$, we can easily describe the roots, root vectors and root spaces. We can check that $K(X_{t_1,t_2}, X_{t_1,t_2})$ is 6 times the sum of the squares of the diagonal entries of X_{t_1,t_2}. This can be rewritten as $K(X_{t_1,t_2}, X_{t_1,t_2}) = 12(t_1^2 + t_1t_2 + t_2^2)$. Let $\alpha_1(X_{t_1,t_2}) = t_1 - t_2$ and $\alpha_2(X_{t_1,t_2}) = t_1 + 2t_2$. Then

$$\Phi = \{ \pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2) \}.$$

The root vectors for α_1 and α_2 are as follows: $H_{\alpha_1} = d(1,-1,0)/6$ and $H_{\alpha_2} = d(0,1,-1)/6$.

The matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

belong to the root spaces $\mathfrak{g}_{\alpha_1}$, $\mathfrak{g}_{\alpha_2}$ and $\mathfrak{g}_{\alpha_3}$, respectively. Their transposes belong to $\mathfrak{g}_{-\alpha_1}$, $\mathfrak{g}_{-\alpha_2}$ and $\mathfrak{g}_{-\alpha_3}$, respectively.

Let $X_0 \in \mathfrak{t}_{\mathbb{R}}$ be such that $\alpha(X_0) \neq 0$ for all $\alpha \in \Phi$. Let $\Phi^+ = \{ \alpha \in \Phi \mid \alpha(X_0) > 0 \}$ and $\Phi^- = \{ \alpha \in \Phi \mid \alpha(X_0) < 0 \}$.

DEFINITION 4.4. A root α is said to be *simple* if $\alpha \in \Phi^+$ and $\alpha \neq \beta + \gamma$ for roots β, $\gamma \in \Phi^+$. Let Δ be the set of simple roots.

EXAMPLE 4.5. For $G = SU(3)$, let $X_0 = d(1,0,-1)$. Then $\Phi^+ = \{ \alpha_1,\, \alpha_2,\, \alpha_1 + \alpha_2 \}$ and $\Delta = \{ \alpha_1,\, \alpha_2 \}$.

PROPOSITION 4.6. (1) *If* α, $\beta \in \Delta$ *are distinct then* $K(H_\alpha, H_\beta) < 0$.
(2) $\mathrm{Span}_{\mathbb{R}}(\Delta) = \mathfrak{t}_{\mathbb{R}}^*$.
(3) Δ *is linearly independent.*
(4) *If* $\alpha \in \Phi^+$, *then* $\alpha = \sum_{\alpha_i\in\Delta} n_i\alpha_i$, *where each* n_i *is a nonnegative integer.*

LEMMA 4.7. $\Phi \subset \mathcal{Y}$.

DEFINITION 4.8. A weight $\rho \in \mathcal{Y}$ is *dominant* if $\rho(H_{\alpha_i}) \geq 0\ \forall\, \alpha_i \in \Delta$. Let $\mathcal{Y}^d$ be the set of dominant weights in $\mathcal{Y}$.

EXAMPLE 4.9. For $G = SU(3)$, we have $\mathfrak{t}_{\mathbb{R}}^* = \mathrm{Span}_{\mathbb{R}}\{\, \alpha_1, \alpha_2 \,\}$, $\mathcal{Y} = \{\, k\alpha_1 + \ell\alpha_2 \mid k, \ell \in \mathbb{Z} \,\}$, and $\mathcal{Y}^d = \{\, k\alpha_1 + \ell\alpha_2 \mid k/2 \le \ell \le 2k,\ k, \ell \in \mathbb{Z} \,\}$. Note that $\alpha_1 + \alpha_2$ is the only dominant root. Among the weights ρ_1, ρ_2 and $-\rho_1 - \rho_2$ of the self-representation of $SU(3)$, only ρ_1 is dominant.

DEFINITION 4.10. Let $\rho_1, \rho_2 \in \mathcal{Y}$. We say that ρ_1 is *higher* than ρ_2 (or ρ_2 is *lower* than ρ_1) if $\rho_1 - \rho_2 = \sum_{\alpha_i \in \Delta} c_i \alpha_i$, where $c_i \ge 0$ for all i. The notation $\rho_1 \succcurlyeq \rho_2$ or $\rho_2 \preccurlyeq \rho_1$ is used. This defines a partial ordering on $\mathcal{Y}$.

EXAMPLE 4.11. The weights of the self-representation of $SU(3)$ satisfy $\rho_1 - \rho_2 = \alpha_1$ and $\rho_2 - (-\rho_1 - \rho_2) = \rho_1 + 2\,\rho_2 = \alpha_2$, so $\rho_1 \succcurlyeq \rho_2 \succcurlyeq -\rho_1 - \rho_2$.

If $\alpha \in \Phi$, let $Y_\alpha \in \mathfrak{g}_\alpha$. For example, we could take $Y_\alpha = X_\alpha$, where X_α is as in Proposition 4.2. The next result describes the images of the weight spaces V_ρ, $\rho \in Y(\Pi)$, under the operators $\Pi(Y_\alpha)$. In particular, if V_ρ is not a subspace of the kernel of $\Pi(Y_\alpha)$, then $\rho + \alpha \in Y(\Pi)$. This is used to obtain relations between various weights in $Y(\Pi)$.

PROPOSITION 4.12. *If $\alpha \in \Phi$, let $Y_\alpha \in \mathfrak{g}_\alpha$ and $\rho \in Y(\Pi)$, then $\Pi(Y_\alpha)V_\rho = \{0\}$ if $\rho + \alpha \notin Y(\Pi)$ and $\Pi(Y_\alpha)V_\rho \subset V_{\rho+\alpha}$ if $\rho + \alpha \in Y(\Pi)$.*

PROOF. Let $\rho \in Y(\Pi)$, $v \in V_\rho$, $X \in \mathfrak{t}_{\mathbb{R}}$, $\alpha \in \Phi$ and $Y_\alpha \in \mathfrak{g}_\alpha$. Then $t := e^{2\pi i X}$ belongs to T and

$$\Pi(t)\Pi(Y_\alpha)v = \Pi(t)\Pi(Y_\alpha)\Pi(t)^{-1}\Pi(t)v = \Pi(t)\Pi(Y_\alpha)\Pi(t)^{-1}e_\rho(X)v = e_\rho(X)\Pi(tY_\alpha t^{-1})v.$$

Here, we have used the relation from Proposition 3.5(1). Next, since $t = e^{2\pi i X}$, we have

$$tY_\alpha t^{-1} = e^{\mathrm{ad}\,2\pi i X}(Y_\alpha) = \sum_{n=0}^{\infty} (\mathrm{ad}\,2\pi i X)^n (Y_\alpha)/n!,$$

which, since $\mathrm{ad}\,X(Y_\alpha) = \alpha(X)Y_\alpha$, can be rewritten as $tY_\alpha t^{-1} = e^{2\pi i\alpha(X)}Y_\alpha = e_\alpha(X)Y_\alpha$. Hence

$$\Pi(t)\Pi(Y_\alpha)v = e_\rho(X)e_\alpha(X)\Pi(Y_\alpha)v = e_{\rho+\alpha}(X)\Pi(Y_\alpha)v.$$

That is, the vector $\Pi(Y_\alpha)v$ belongs to $V_{\rho+\alpha}$. $\square$

THEOREM 4.13. *Let (Π, V) be an irreducible unitary representation of G. Then there exists a unique dominant weight $\rho \in Y(\Pi)$ that is maximal among all weights in $Y(\Pi)$. Moreover,*

(1) *If $\alpha_i \in \Delta$, then $\rho + \alpha_i \notin Y(\Pi)$.*

(2) *Every weight in* $Y(\Pi)$ *has the form* $\rho - \sum_i n_i\alpha_i$, *with* $n_i \geq 0$.
(3) *Every weight in* $Y(\Pi)$ *belongs to the convex hull of the images of* ρ *under* W.
(4) $m_\rho = 1$.

The maximal weight ρ is called the *highest weight* of Π.

5. The Weyl character formula

CHAPTER 5

Representations of reductive groups

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over a field F and let $G = \mathbf{G}(F)$ be the F-rational points of $\mathbf{G}$. We will consider three cases:

- $F = \mathbb{R}$: In this case, G is a also a Lie group.
- F is a nonarchimedean local field of characteristic zero: F is a finite extension of the p-adic numbers $\mathbb{Q}_p$, where p is a prime. In this case, G is a locally profinite group.
- $F = \mathbb{F}_q$, a finite field. We say that G is a *finite group of Lie type.*

1. The Iwasawa decomposition

Let $F = \mathbb{R}$ or $\mathbb{Q}_p$ and let $G = SL_2(F)$. Let

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in F^\times \right\} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in F \right\}.$$

The group $B = A \ltimes N$ is called a *Borel* subgroup of G. The subgroup A is a maximal F-split torus in G. If $F = \mathbb{R}$, set

$$K = SO_2(\mathbb{R}) = \{\, g \in G \mid {}^t g\, g = I \,\} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

The subgroup K is a compact maximal torus in $SL_2(\mathbb{R})$, as well as a maximal compact subgroup of G. If $F = \mathbb{Q}_p$, let $K = SL_2(\mathbb{Z}_p)$ be the subgroup of G consisting of matrices with entries in $\mathbb{Z}_p$. The subgroup K is a maximal compact (open) subgroup of $SL_2(\mathbb{Q}_p)$.

Lemma 1.1. *(Iwasawa decomposition) $G = KB = BK$.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$.

Suppose that $F=\mathbb{R}$. Given $z\in\mathbb{C}$, define $g\cdot z=(az+b)/(cz+d)$. If the imaginary part $\Im(z)$ of z is positive, then $\Im(g\cdot z)>0$. Note that, for $a\in\mathbb{R}^\times$ and $x\in\mathbb{R}$,

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\cdot i = x+a^2\, i,$$

so $B\cdot i$ is the full upper half plane (complex numbers having positive imaginary part). Hence, given $g\in G$, there exists $g'\in B$ such that $g\cdot i=g'\cdot i$. Since $g'^{-1}g\cdot i=i$ and $K=\{\, h\in G\mid h\cdot i=i\,\}$, we see that $g\in BK$.

Suppose that $F=\mathbb{Q}_p$. Let g be as above. If $c=0$, then $g\in B$. If $|c|_p>|a|_p$, then

$$g=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-c & -d\\ a & b\end{pmatrix}.$$

Since $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ belongs to K and $|-c|_p>|a|_p$, it suffices to prove that $g\in KB$ whenever $|a|_p\geq |c|_p>0$. When $a\neq 0$,

$$g=\begin{pmatrix}1 & 0\\ a^{-1}c & 1\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d-a^{-1}bc\end{pmatrix},$$

which, when $|a|_p\geq|c|_p$, is clearly in KB. □

The Lie algebra of $SL_2(\mathbb{R})$ also has an Iwasawa decomposition: $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{R})=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$, where

$$\mathfrak{k}=\left\{\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\mid b\in\mathbb{R}\right\},\quad \mathfrak{a}=\left\{\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}\mid a\in\mathbb{R}\right\}\quad\text{and}\quad \mathfrak{n}=\left\{\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\mid x\in\mathbb{R}\right\}$$

are the Lie algebras of K, A and N, respectively.

If G is a matrix Lie group (closed subgroup of $GL_n(\mathbb{C})$), then G is stable under the involution $g\mapsto {}^t\bar{g}^{-1}$ and the subgroup K of fixed points of the involution is a maximal compact subgroup of G. Every maximal compact subgroup of G is conjugate to K. In this setting, the Iwasawa decomposition of G takes the form $G=P_0K=KP_0$, where P_0 is a minimal parabolic subgroup of G. (Parabolic subgroups will be discussed in a later section.)

In general, a reductive p-adic group contains a finite number of conjugacy classes of maximal compact subgroups. If G is a connected reductive p-adic group, there exists a maximal compact (open) subgroup K of G such that $G=P_0K=KP_0$ for any minimal parabolic subgroup of G. Here, K is referred to as a *good* maximal compact subgroup of

G. (*Note*: The adjective "connected" refers to the underlying algebraic group. In the p-adic topology, as discussed earlier in the course, the topological space G is totally disconnected.) It is worth noting that K is not itself a reductive p-adic group. That is, K is not the F-rational points of a reductive linear algebraic group defined over F. It is simply a particular compact open subgroup of G. (This contrasts with the Lie group setting, where K is a compact Lie group.)

2. Principal series representations of $SL_2(F)$

If G is a locally compact group, a *quasicharacter* of G is a continuous one-dimensional representation of G. A unitary quasicharacter is referred to as a *character*.

Let $G = SL_2(F)$, $F = \mathbb{R}$ or $\mathbb{Q}_p$, K, A, N, and B be as in Section 1. Given $a \in F^\times$, let $d(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Let χ be a quasicharacter of $F^\times$, which we may also view as a quasicharacter of A via the isomorphism given by $a \mapsto d(a)$. The notation $|\ |_F$ will be used for the given absolute value on $F^\times$ (that is, the usual one when $F = \mathbb{R}$ and the p-adic absolute value when $F = \mathbb{Q}_p$). Let

$$C(\chi) = \{\, f \in C(G) \mid f(d(a)nk) = |a|_F \chi(a)\, f(k) \mid \forall\, a \in F^\times,\, n \in N,\ k \in K\,\}.$$

We can see from the Iwasawa decomposition that a function in $C(\chi)$ is determined by its restriction to K. We define a norm on $C(\chi)$ by

$$\|f\| = \left(\int_K |f(k)|^2 \, dk \right)^{1/2}, \qquad f \in C(\chi).$$

Here, dk denotes Haar measure on the compact group K. The completion $\mathcal{H}(\chi)$ of $C(\chi)$ with respect to this norm is a Hilbert space. Let $I(\chi) = I_B^G(\chi)$ be the right regular representation of G on $\mathcal{H}(\chi)$.

Note that the function $d(a) \mapsto |a|_F \chi(a)$, which we will denote by $|\ |_F \chi$, is the internal tensor product of the quasicharacters $d(a) \mapsto |a|_F$ and $d(a) \mapsto \chi(a)$ of A. Extending $|\ |_F \chi$ trivially across N, we obtain a quasicharacter of $B = AN$. Strictly speaking, the representation ρ is induced from the quasicharacter $|\ |_F \chi$ of B. We will see below that the factor $|\ |_F$ is introduced so that $I(\chi)$ will be unitary whenever χ is unitary. This version of induction is called *normalized induction*.

Let $\delta_B(d(a)n) = |a|_F^2$, $a \in F^\times$, $n \in N$. The following lemma is a consequence of the Iwasawa decomposition and the fact that $\delta_B \,|\, B \cap K$ is trivial.

Lemma 2.1. *If $g = d(a_0)n_0k_0$, $a_0 \in F^\times$, $n_0 \in N$, $k \in K$, set $\delta_B(d(a_0)n_0k) = \delta_B(d(a_0))$. Then δ_B is a well-defined continuous function from G to the positive real numbers such that $\delta_B(d(a)ng) = \delta_B(d(a))$ for all $a \in F^\times$, $n \in N$ and $g \in G$.*

Since G is unimodular and $\delta_B \,|\, B$ is the modular function of B, we can use the extension δ_B defined in the above lemma to produce a quasi-invariant measure on the coset space $B\backslash G$. For x and $g \in G$, set $\sigma(Bg, x) = \frac{\delta_B(gx)}{\delta_B(g)}$. Then (see Theorem 5.11, Chapter 1), there exists a quasi-invariant measure $d\bar{g}$ on $B\backslash G$ associated to the ρ-function δ_B, having the property:

$$\int_{B\backslash G} \varphi(\bar{g})\, d(\bar{g}\cdot x) = \int_{B\backslash G} \sigma(\bar{g}, x)\, \varphi(\bar{g})\, d\bar{g}, \qquad x \in G,\ \varphi \in C_c(B\backslash G) = C(B\backslash G).$$

Lemma 2.2. *Let $V = \{\, f \in C(G) \mid f(bg) = \delta_B(b)\, f(g)\ \forall\ b \in B,\ g \in G \,\}$. For $g \in G$, $x \in G$ and $f \in V$, let $(R_g f)(x) = f(xg)$. Then*

1. *Given $f \in V$, the function $g \mapsto \delta_B(g)^{-1} f(g)$ factors to a continuous function on $B\backslash G$.*
2. *The map $\lambda : f \mapsto \int_{B\backslash G} (\delta_B^{-1} f)(\bar{g})\, d\bar{g}$ defines a G-invariant element of V^*: $\lambda(R_g f) = \lambda(f)$ for all $g \in G$ and $f \in V$.*

Proof. The first part of the lemma is a consequence of Lemma 2.1 and the definition of V. Let λ be as defined in the statement of the lemma. Then

$$\begin{aligned}\lambda(R_x f) &= \int_{B\backslash G} \delta_B(g)^{-1} f(gx)\, d\bar{g} = \int_{B\backslash G} \delta_B(gx^{-1})^{-1} f(g)\, d(\bar{g}\cdot x^{-1})\\ &= \int_{B\backslash G} \delta_B(gx^{-1})^{-1} f(g) \sigma(\bar{g}, x^{-1})\, d\bar{g} = \int_{B\backslash G} \delta_B(g)^{-1} f(g)\, d\bar{g} = \lambda(f).\end{aligned}$$

□

Lemma 2.3. *Let χ be a character of $F^\times$. Let V and λ be as in Lemma 2.2. Then*

1. *If $f_1,\ f_2 \in C(\chi)$, the function $g \mapsto f_1(g)\overline{f_2(g)}$ belongs to V.*
2. *Set $\langle\, f_1,\ f_2 \,\rangle = \lambda(f_1 \bar{f}_2)$. This defines a G-invariant nondegenerate hermitian form on $C(\chi)$.*

Proof. The lemma is a consequences of the definitions of V and $C(\chi)$, unitarity of χ, and Lemma 2.2. □

After verifying that, with appropriate normalizations of measures,

$$\int_{B\backslash G} (\delta_B^{-1} f)(\bar{g})\, d\bar{g} = \int_K f(k)\, dk, \qquad f \in V,$$

(we don't include the details here), we conclude that the Hilbert space $\mathcal{H}(\chi)$ is the completion of $C(\chi)$ with respect to $\langle\ ,\rangle$.

PROPOSITION 2.4. *If χ is a character of $F^\times$, then $I(\chi)$ is a unitary representation of $SL_2(F)$.*

REMARK 2.5. For $G = SL_2(\mathbb{F}_q)$, since all representations are unitary, we can use the usual definition of induced representation to define principal series representations of G. Define subgroups $A = \{\, d(a) \mid a \in \mathbb{F}_q^\times \,\}$, $N = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_q \,\}$ and $B = AN$. Given a character χ of $\mathbb{F}_q^\times$, let $I(\chi)$ be the right regular representation of G on the space of functions $f : G \to \mathbb{C}$ such that $f(d(a)ng) = \chi(d(a))f(g)$ for all $a \in \mathbb{F}_q^\times$, $n \in N$ and $g \in G$.

3. Parabolic subgroups of $Sp_4(F)$

Suppose that G is the F-rational points of a connected reductive linear algebraic F-split F-group $\mathbf{G}$. In particular, $\mathbf{G}$ has a maximal F-torus $\mathbf{A}_0$ that splits over F: $A_0 = \mathbf{A}_0(F)$ is isomorphic to $F^\times \times \cdots \times F^\times$. For example, $G = GL_n(F)$, $SL_n(F)$ and $Sp_{2n}(F)$ are split groups. For such groups, we can define Borel subgroups and other parabolic subgroups using the root system $\Phi = \Phi(G, A_0)$ of A_0 in G. We describe how this works for $G = Sp_4(F)$.

REMARK 3.1. When we refer to a parabolic subgroup of G, we mean a subgroup P of the form $\mathbf{P}(F)$, where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ and $\mathbf{P}$ is defined over the field F.

REMARK 3.2. In general, noncompact reductive groups contain several conjugacy classes of maximal tori. In particular, split reductive groups can contain compact maximal tori. This is the case for $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Q}_p)$ (see Examples 6.1 and 6.2.

Let

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Then $G = Sp_4(F) = \{\, g \in GL_4(F) \mid {}^t gJg = J\}$ and $\mathfrak{g} = \{X \in M_4(F) \mid {}^t XJ + JX = 0\}$.

The group $A_0 := \{\, d(a,\, b,\, b^{-1},\, a^{-1}) \mid a,\ b \in F^\times\}$ is (the F-rational points of) a maximal F-split torus in G. Given integers i and j, set $\chi_{(i,j)}(d(a,\, b,\, b^{-1},\, a^{-1})) = a^i b^j$. Let $\alpha = \chi_{(1,-1)}$ and $\beta = \chi_{(0,2)}$. Consider the action of G on $\mathfrak{g}$ by conjugation: $\operatorname{Ad} g(X) = gXg^{-1}$, $g \in G$, $X \in \mathfrak{g}$. When we consider conjugation by elements of A_0, we obtain a root space decomposition of $\mathfrak{g}$ relative to the eigenspaces of Ad A_0. If $(i,j) \in \mathbb{Z}$, $\chi_{(i,j)}$ is a root of A_0 if there exists a nonzero $X \in \mathfrak{g}$ such that $\operatorname{Ad} t(X) = \chi_{(i,j)}(t)X$ for all $t \in A_0$. In that case, the corresponding root space is

$$\mathfrak{g}_{\chi_{(i,j)}} = \{\, X \in \mathfrak{g} \mid \operatorname{Ad} t(X) = \chi_{(i,j)}(t) \ \forall\, t \in A_0 \,\}.$$

Let $\Phi = \Phi(G, A_0)$ be the set of roots of A_0 .. Then

$$\Phi = \{\, \pm\alpha,\ \pm\beta,\ \pm(\alpha+\beta),\ \pm(2\alpha+\beta\,\},$$

Because we often identify $\chi_{(i,j)}$ with the pair (i,j), we write $-\alpha$ instead of α^{-1} and $\alpha+\beta$ instead of $\alpha\beta$, etcetera.

The root spaces are easy to describe. If E_{ij} is the matrix in $M_4(F)$ with a one in the (i,j) entry and zeros elsewhere, then

$$\mathfrak{g}_\alpha = \operatorname{Span}_F(E_{12} - E_{34}), \quad \mathfrak{g}_{-\alpha} = \operatorname{Span}_F(E_{21} - E_{43}) \quad \mathfrak{g}_\beta = \operatorname{Span}_F E_{23}$$
$$\mathfrak{g}_{\alpha+\beta} = \operatorname{Span}_F(E_{13} + E_{24}), \quad \mathfrak{g}_{2\alpha+\beta} = \operatorname{Span}_F E_{14}, \quad \text{etcetera.}$$

For each root $\gamma \in \Phi$, there exists a root subgroup U_γ of G. This group is of the form $\mathbf{U}_\gamma(F)$ for an F-subgroup $\mathbf{U}_\gamma$ of $\mathbf{G}$, stable under conjugation by A_0 and has Lie algebra $\mathfrak{g}_\gamma$. The group G is generated by A_0 and the various root subgroups U_γ, $\gamma \in \Phi$.

A *base* for a root system Φ is a subset Δ of Φ having the property that each element of Φ has the form $\sum_{\alpha\in\Phi} n_\alpha \alpha$, where either the n_α are all nonnegative integers, or the n_α are all nonpositive integers. The elements of Δ are called the simple roots. In this example, $\Delta := \{\alpha,\ \beta\}$ is a base of Φ.

The Borel subgroup of G which corresponds to Δ (or equivalently to Φ^+) is the subgroup B of upper triangular matrices in G. The subgroup N of upper triangular matrices in G with ones on the diagonal is called the unipotent radical of B. The group N is generated by the root groups U_γ, $\gamma \in \Phi^+$ and the Lie algebra $\mathfrak{n}$ of N is $\sum_{\gamma\in\Phi^+} \mathfrak{g}_\gamma$. Clearly, $B = A_0 \ltimes N$. A standard parabolic subgroup is a parabolic subgroup of G that contains the Borel subgroup B. A parabolic subgroup of G is conjugate to some standard parabolic subgroup.

Each standard parabolic subgroup of G is attached to a subset of Δ. Fix a subset I of Δ. Let Φ_I be the set of $\gamma \in \Phi$ such that γ is an integral linear combination of elements of I. Then Φ_I forms a root system. Let N_I be the subgroup of N generated by the root groups U_γ such that $\gamma \in \Phi^+$ and $\gamma \notin \Phi_I$. The Lie algebra of N_I has the form

$$\mathfrak{n}_I = \oplus_{\gamma\in\Phi^+\setminus\Phi_I^+} \mathfrak{g}_\gamma,$$

where $\Phi_I^+ = \Phi_I \cap \Phi^+$. Let $A_I = (\cap_{\gamma\in I}\mathrm{Ker}\,\gamma)^0$, and $M_I = Z_{\mathbf{G}}(A_I)$ (the centralizer of A_I in G). (Here, $(\mathrm{Ker}\,\gamma)^0$ refers to the F-rational points of the identity component of the kernel of the rational character γ of the algebraic torus $\mathbf{A}_0$.) The set Φ_I coincides with the set of roots in Φ which are trivial on A_I. The group M_I is reductive and is generated by A_I and by the root groups U_γ, $\gamma \in \Phi_I$. The F-split torus A_I is the identity component of the centre of M_I, and $\Phi(M_I, A_0) = \Phi_I$. The Lie algebra of M_I is equal to

$$\mathfrak{m}_I = \mathfrak{a}_0 \oplus \bigoplus_{\gamma\in\Phi_I} \mathfrak{g}_\alpha$$

Here, $\mathfrak{a}_0$ is the Lie algebra of A_0. The group M_I normalizes N_I and $P_I = M_I \ltimes N_I$. The group M_I is called a Levi factor of P_I. It also called a Levi subgroup of G. If I and J are subsets of Δ then P_I is conjugate to P_J if and only if $I = J$. However, it is possible for M_I to be conjugate to M_J when $I \neq J$.

When $G = Sp_4(F)$, apart from $G = P_\Delta$ and $B = P_\emptyset$, there are two standard parabolic subgroups, P_α and P_β, attached to the subsets $\{\alpha\}$ and $\{\beta\}$ of Δ, respectively.

If $I = \{\alpha\}$, then $\Phi_I = \{\,\pm\alpha\,\}$ and $\Phi^+ \setminus \Phi_I^+ = \{\,\beta,\, \alpha+\beta,\, 2\alpha+\beta\,\}$. We can describe the matrices in A_α, M_α and N_α:

$$A_\alpha = \mathrm{Ker}\ \alpha = \{\, d(a,\ a,\ a^{-1}\, a^{-1}) \mid a \in F^\times\},$$

$$M_\alpha = Z_G(A_\alpha) = \left\{ \begin{pmatrix} X & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tX^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \;\Big|\; X \in GL_2(F) \right\}$$

$$N_\alpha = \left\{ \begin{pmatrix} I_2 & Y \\ 0 & I_2 \end{pmatrix} \;\Big|\; Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_1 \end{pmatrix},\ y_j \in F \right\}$$

When $I = \{\,\beta\,\}$, $\Phi_I = \{\,\pm\beta\,\}$ and $\Phi^+ \setminus \Phi_I^+ = \{\,\alpha,\, \alpha+\beta,\, 2\alpha+\beta\,\}$.

$$A_\beta = \mathrm{Ker}\ \beta = \{\, d(a,\, 1,\, 1,\, a^{-1}) \mid a \in F^\times\,\}$$

$$M_\beta = Z_G(A_\beta) = \left\{ \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & d^{-1} \end{pmatrix} \;\Big|\; d \in F^\times,\, c_{11}c_{22} - c_{12}c_{21} = 1 \right\} \simeq SL_2(F) \times F^\times$$

$$N_\beta = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in F \right\}$$

When $F = \mathbb{R}$, that is, when G is a Lie group, parabolic subgroups may be written in a slightly different form. The Levi factor M_I of a standard parabolic subgroup P_I may be expressed as a direct product of two groups M_I' and A_I'. Here, A_I' is the subgroup of A_I of the form $(\mathbb{R}_+^\times)^\ell$ such that A_I/A_I' is finite (and a product of groups of order two). For example, for $G = Sp_4(\mathbb{R})$, M_α' is the set of elements in M_α such that $\det X = \pm 1$ (in the notation above) and $A_\alpha' = \{\, d(a, a, a^{-1}, a^{-1}) \mid a \in \mathbb{R}_+^\times \,\}$. An element of M_β belongs to M_β' if and only if $d = \pm 1$, and $A_\beta' = \{\, d(a, 1, 1, a^{-1}) \mid a \in \mathbb{R}_+^\times \,\}$.

Remark 3.3. Let $G = GL_n(F)$. The subgroup A_0 of diagonal matrices in G is a maximal split torus in G and the subgroup B of upper trangular matrices in G is a Borel subgroup of G with respect to a particular choice of base Δ of $\Phi = \Phi(G, A_0)$. The simple roots (that is, the elements of Δ) are of the form $d(a_1, \dots, a_n) \mapsto a_j a_{j+1}^{-1}$, $1 \le j \le n-1$. If $I \subset \Delta$, the standard parabolic subgroup $P_I = M_I \ltimes N_I$ is the smallest subgroup of G that contains M_I and B. In particular, $M_I \simeq GL_{n_1}(F) \times \cdots \times GL_{n_r}(F)$, where $n_1 + \cdots + n_r = n$ and each n_j is a positive integer. More generally, as we saw above for $G = Sp_4(F)$, a Levi factor of a parabolic subgroup of a reductive group is not necessarily a smaller rank version of the same type of group: $GL_2(F)$ occurs as a Levi factor of a parabolic subgroup of $Sp_4(F)$ and $GL_2(F)$ is not a symplectic group.

Although we have not used the Weyl group to define the standard parabolic subgroups, we include some comments about the Weyl group of A_0 for our example. The quotient group $W(A_0) = N_G(A_0)/A_0$ is called the Weyl group of A_0. The matrices

$$w_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad w_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

belong to $N_G(A_0)$. The distinct elements of the Weyl group $W(A_0)$ are represented by the matrices

$$\{\, I,\, w_\alpha,\, w_\beta,\, w_\alpha w_\beta,\, w_\beta w_\alpha,\, w_\beta w_\alpha w_\beta,\, w_\alpha w_\beta w_\alpha,\, (w_\beta w_\alpha)^2 \,\}$$

and $W(A_0)$ is isomorphic to the dihedral group of order 8.

REMARK 3.4. The matrix w_α (resp. w_β) belongs to $N_{M_\alpha}(A_0)$ (resp. $N_{M_\beta}(A_0)$) and represents the nontrivial element of the Weyl group of A_0 in M_α (resp. M_β.)

4. Parabolic subgroups and induction

Let $F = \mathbb{R}$ or $\mathbb{Q}_p$ (or a finite extension of $\mathbb{Q}_p$). Let G be the F-rational points of a connected reductive F-group and let P be a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$. The Iwawasa decomposition says that $G = BK = KB$ for B a minimal parabolic subgroup contained in P and some compact subgroup K of G. Since $B \subset P$, we have $G = PK = KP$.

The Levi factor M of P is the F-rational points of a connected reductive F-group. That is, it is the same kind of group as G. (For example, when $G = Sp_4(F)$, then M is isomorphic to one of $F^\times \times F^\times$, $GL_2(F)$, $SL_2(F) \times F^\times$ and $Sp_4(F)$.)

Let $\mathfrak{g}$ and $\mathfrak{n}$ be the Lie algebras of G and N. Let $g \mapsto \operatorname{Ad} g : \mathfrak{g} \to \mathfrak{g}$ be the usual adjoint representation of G: $\operatorname{Ad} g(X) = gXg^{-1}$, $g \in G$, $X \in \mathfrak{g}$. If $p \in P$, then $\operatorname{Ad} p : \mathfrak{n} \to \mathfrak{n}$. Set $\delta_P(p) = |\det(\operatorname{Ad} p) \,|\, \mathfrak{n}|_F$, $p \in P$. As in the principal series case, we use the function $\delta_P^{1/2}$ to normalize induction so that it takes unitary representations of M to unitary representations of G. (The arguments are virtually identical to those used in the principal series case. We simply replace the usual inner product on $\mathbb{C}$ with the appropriate inner product on the space of the representation of M.)

Let (σ, W) be a continuous representation of M in a Hilbert space W.

$$C(\sigma) = \{\, f \in C(G) \mid f(mnk) = \delta_P(mn)\, f(k) \mid \forall\ m \in N\, n \in N,\ k \in K \,\}.$$

We can see from the decomposition $G = PK$ that a function in $C(\sigma)$ is determined by its restriction to K. We define a norm on $C(\sigma)$ by

$$\|f\| = \left(\int_K \|f(k)\|_W^2 \, dk \right)^{1/2}, \qquad f \in C(\sigma).$$

Here, dk denotes Haar measure on the compact group K and $\|\ \|_W$ is the norm on W. The completion $\mathcal{H}(\sigma)$ of $C(\sigma)$ with respect to this norm is a Hilbert space. Let $I(\sigma) = I_P^G(\sigma)$ be the right regular representation of G on $\mathcal{H}(\sigma)$.

LEMMA 4.1. *If $g = m_0 n_0 k_0$, $m_0 \in M$, $n_0 \in N$, $k \in K$, set $\delta_P(m_0 n_0 k) = \delta_P(m_0)$. Then δ_P is a well-defined continuous function from G to the positive real numbers such that $\delta_P(mng) = \delta_P(m)$ for all $m \in M$, $n \in N$ and $g \in G$.*

Since G is unimodular and $\delta_P \,|\, P$ is the modular function of P, we can use the extension δ_P defined in the above lemma to produce a quasi-invariant measure on the coset space $P\backslash G$. For x and $g \in G$, set $\sigma(Pg, x) = \frac{\delta_P(gx)}{\delta_P(g)}$. Then (see Theorem 5.11, Chapter 1), there exists a quasi-invariant measure $d\bar{g}$ on $P\backslash G$ associated to the ρ-function δ_P, having the property:

$$\int_{P\backslash G} \varphi(\bar{g})\, d(\bar{g}\cdot x) = \int_{P\backslash G} \sigma(\bar{g}, x)\, \varphi(\bar{g})\, d\bar{g}, \qquad x \in G,\ \varphi \in C_c(P\backslash G) = C(P\backslash G).$$

LEMMA 4.2. *Let* $V = \{\, f \in C(G) \mid f(pg) = \delta_P(p)\, f(g)\ \forall\ p \in P,\ g \in G\,\}$. *For* $g \in G$, $x \in G$ *and* $f \in V$, *let* $(R_g f)(x) = f(xg)$. *Then*

(1) *Given* $f \in V$, *the function* $g \mapsto \delta_P(g)^{-1} f(g)$ *factors to a continuous function on* $P\backslash G$.

(2) *The map* $\lambda_P : f \mapsto \int_{P\backslash G} (\delta_P^{-1} f)(\bar{g})\, d\bar{g}$ *defines a* G*-invariant element of* V^*: $\lambda_P(R_g f) = \lambda_P(f)$ *for all* $g \in G$ *and* $f \in V$.

LEMMA 4.3. *Let* (σ, W) *be a unitary representation of* M, *with* M*-invariant inner product* $\langle\ ,\ \rangle_W$ *on* W. *Let* V *and* λ_P *be as above. Then*

(1) *If* $f_1,\ f_2 \in C(\sigma)$, *the function* $\varphi_{(f_1,f_1)} : g \mapsto \langle\, f_1(g),\ f_2(g)\,\rangle_W$ *belongs to* V.

(2) *Set* $\langle\, f_1,\ f_2\,\rangle = \lambda_P(\varphi_{(f_1,f_2)})$. *This defines a* G*-invariant nondegenerate hermitian form on* $C(\sigma)$.

After verifying that, with appropriate normalizations of measures,

$$\int_{P\backslash G} (\delta_P^{-1} f)(\bar{g})\, d\bar{g} = \int_K f(k)\, dk, \qquad f \in V,$$

we see that the Hilbert space $\mathcal{H}(\sigma)$ is the completion of $C(\sigma)$ with respect to $\langle\ ,\rangle$.

PROPOSITION 4.4. *If* (σ, W) *is a unitary representation of* M, *then* $I(\sigma)$ *is a unitary representation of* G.

5. Discrete series representations–general properties

Let (π, V) be a topologically irreducible unitary representation of a locally compact group G. If every matrix coefficient of π belongs to $L^2(G)$, we say that π is *square-integrable* or π belongs to the *discrete series.*

Let Z be the centre of G. If $z \in Z$, then $\pi(zg) = \pi(gz)$ for all $g \in G$. That is, $\pi(z) \in \mathrm{Hom}_G(\pi, \pi)$. Since π is topologically irreducible and unitary, according to Schur's Lemma, there exists a (nonzero) scalar $\omega(z)$ such that $\omega(z)^{-1}\pi(z)$ is the identity operator on V. By continuity of π, the map $z \mapsto \omega(z)$ is a quasicharacter of Z. Since π is unitary, ω is unitary – that is, ω is a character of the locally compact abelian group Z. The character ω is called the *central character* of π. For $v,\, w \in V$,

$$\langle\, \pi(zg)v,\, w\,\rangle = \omega(z)\langle\, \pi(g)v,\, w\,\rangle, \qquad g \in G,\, z \in Z.$$

In particular, the function $g \mapsto |\langle\, \pi(g)v,\, w\rangle|$ is left (and right) Z-invariant.

Lemma 5.1. *If G is a locally compact group with noncompact centre, then a nonzero matrix coefficient of a topologically irreducible unitary representation of G cannot belong to $L^2(G)$.*

Suppose that G is unimodular. (This is the case when G is the F-rational points of a reductive algebraic group over a locally compact field F, such as $F = \mathbb{R}$ or $F = \mathbb{Q}_p$). The centre Z of G is a locally compact abelian group, so Z is unimodular. Because G is also unimodular, by Theorem 5.8 of Chapter 1, there exists a G-invariant measure on the coset space G/Z. Fix a character ω of Z. We say that a function $f : G \to \mathbb{C}$ is compactly supported modulo Z if there exists a compact set C in G such that the support of f is a subset of $Z\,C = C\,Z$. Let

$$C_c(Z\backslash G, \omega) = \{\, g \in C(G) \mid f(zg) = \omega(z)f(g) \forall\; z \in Z,\, g \in G, \\ \text{such that } f \text{ is compactly supported modulo } Z\,\}.$$

Note that if $f_1,\, f_2 \in C_c(Z\backslash G, \omega)$, the function $f_1\bar{f}_2$ belongs to the space $C_c(Z\backslash G)$ of continuous compactly supported functions on $Z\backslash G$. We define an inner product on $C_c(Z\backslash G, \omega)$ by

$$\langle\, f_1,\, f_2\,\rangle = \int_{Z\backslash G} f_1(\bar{g})\overline{f_2(\bar{g})}\, d\bar{g}, \qquad f_1,\, f_2 \in C_c(Z\backslash G, \omega).$$

Let $L^2(Z\backslash G, \omega)$ be the completion of $C_c(Z\backslash G, \omega)$ with respect to the norm associated to this inner product. Recall that Proposition 4.1 of Chapter 2 shows that the left regular representation of G on the space $L^2(G/Z)$ is a continuous unitary representation of G. Of course, the right regular representation of G on $L^2(Z\backslash G)$ is also unitary. A similar argument can be used to see that the right regular representation ρ_ω of G on $L^2(Z\backslash G, \omega)$ is continuous and unitary.

Theorem 5.2. *Let G be a unimodular locally compact group with centre Z. Let (π, V) be a topologically irreducible unitary representation of G with central character ω. Then the following are equivalent*

(1) *π is equivalent to a subrepresentation of $(\rho_\omega, L^2(Z\backslash G, \omega))$.*
(2) *There exists a nonzero matrix coefficient of π that belongs to $L^2(Z\backslash G, \omega)$.*
(3) *Every matrix coefficient of π belongs to $L^2(Z\backslash G, \omega)$.*

We say that a representation (π, V) is *essentially square integrable* (or *square integrable modulo Z*) if the conditions of the theorem are satisfied. We may generalize our definition of discrete series representation to the square integrable setting.

Remark 5.3. Even though the matrix coefficients of an essentially square integrable representation with central character ω belong to $L^2(G, \omega)$ and are continuous functions on G, they do not generally belong to $C_c(Z\backslash G, \omega)$. As discussed below, there are certain families of essentially square-integrable representations of reductive p-adic groups whose matrix coefficients do belong to $C_c(Z\backslash G, \omega)$.

Theorem 5.4. *Let G be a unimodular locally compact group. Let (π, V) and (σ, W) be topologically irreducible essentially square integrable representations with the same central character ω. of G.*

(1) *If f_π and f_σ are matrix coefficients of π and σ, respectively, then $\int_{Z\backslash G} f_\pi(\bar{g})\overline{f_\sigma(\bar{g})}\, d\bar{g} = 0$.*
(2) *Suppose that v, v', u, $u' \in V$ and $\langle\ ,\ \rangle$ is a G-invariant inner product on V. Then there exists a positive real number $d(\pi)$, called the formal degree of π, such that*

$$\int_{Z\backslash G} \langle\, \pi(g)v,\, v'\,\rangle \overline{\langle\, \pi(g)u,\, u'\,\rangle}\, d\bar{g} = d(\pi)^{-1} \langle\langle\, v,\, u\,\rangle \overline{\langle\, v',\, u'\,\rangle}$$

Remark 5.5. The formal degree $d(\pi)$ depends on the choice of normalization of the G-invariant measure on G/Z.

6. Maximal Tori

We will see that (relative) discrete series of reductive Lie groups and certain kinds of discrete series representations of reductive p-adic groups are associated with characters of compact-mod-centre maximal tori. In this section, we describe examples of such tori.

A maximal torus T of G is said to be *compact-mod-centre* if T/Z is compact, where Z is the centre of G. (When $F = \mathbb{R}$, a compact-mod-centre maximal torus in G, or its identity component, is often referred to as a *relatively compact Cartan subgroup* of G.)

EXAMPLE 6.1. If $G = SL_2(\mathbb{R})$, the group $K = SO_2(\mathbb{R})$ is the fixed points of the Cartan involution and it is also a maximal torus in G. If $G = SL_3(\mathbb{R})$, then the group $K = SO_3(\mathbb{R})$ is the fixed points of the Cartan involution of G. The group

$$T = \left\{ \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

is a maximal torus in $SO_3(\mathbb{R})$. But T is not a maximal torus in $SL_3(\mathbb{R})$. In fact $SL_3(\mathbb{R})$ has rank two and

$$T' := \left\{ \begin{pmatrix} r\,\cos t & r\,\sin t & 0 \\ -r\sin t & r\cos t & 0 \\ 0 & 0 & r^{-1} \end{pmatrix} \mid r \in \mathbb{R}^\times,\, t \in \mathbb{R} \right\}$$

is a (noncompact) maximal torus of $SL_2(\mathbb{R})$ (of dimension 2) containing T. (Note that T' is a maximal torus in the proper Levi subgroup $SL_2(\mathbb{R}) \times \mathbb{R}^\times$ of $SL_3(\mathbb{R})$.) In fact, $SL_3(\mathbb{R})$ does not contain any compact maximal tori (see Remark 6.3).

Because p-adic fields have many extensions of varying degrees, there are plenty of maximal tori in reductive p-adic groups that are compact modulo the centre of the group.

EXAMPLE 6.2. Let p be an odd prime. There exists $\varepsilon \in \mathbb{Z}_p$ such that the image of ε in $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$ is a nonsquare. We can show that ε is a nonsquare in $\mathbb{Q}_p$. It is easy to see that p is not a square in $\mathbb{Q}_p$. (A nonzero element $x \in \mathbb{Q}_p$ satisfies $|x|_p = p^\ell$ for some integer ℓ and $|p|_p = p^{-1}$.) We have quadratic extensions $E = \mathbb{Q}_p(\sqrt{\varepsilon})$ and $L = \mathbb{Q}_p(\sqrt{p})$ of $\mathbb{Q}_p$. Let

$$T_\varepsilon = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{Q}_p,\ a^2 - b^2\varepsilon = 1 \right\}$$

$$T'_\varepsilon = \left\{ \begin{pmatrix} a & bp\varepsilon \\ bp^{-1} & a \end{pmatrix} \mid a,\, b \in \mathbb{Q}_p,\ a^2 - b^2\varepsilon = 1 \right\},$$

$$T_p = \left\{ \begin{pmatrix} a & bp \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{Q}_p,\ a^2 - b^2 p = 1 \right\}.$$

The subgroups T_ε, T'_ε and T_p are examples of compact maximal tori in $SL_2(\mathbb{Q}_p)$. No two of these subgroups are conjugate. (Note that T_ε is a closed subgroup of the maximal compact subgroup $SL_2(\mathbb{Z}_p)$ of $SL_2(\mathbb{Q}_p)$. The group T'_ε s a closed subgroup of the maximal compact subgroup $d(p)SL_2(\mathbb{Z}_p)d(p)^{-1}$, where $d(p) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q}_p)$. The group T_p is a closed subgroup of the intersection of those two maximal compact subgroups.) Let σ be the non-trivial element of the Galois group $\mathrm{Gal}(E/\mathbb{Q}_p)$ Then T_ε and T'_ε are isomorphic to the kernel of the norm map $N_{E/Q_p} : E^\times \to \mathbb{Q}_p^\times$: $N_{E/\mathbb{Q}_p}(x) = x\sigma(x)$, $x \in E^\times$. Similarly, T_p is isomorphic to the kernel of the norm map $N_{L/\mathbb{Q}_p}$. (There exist compact maximal tori in $SL_2(\mathbb{Q}_p)$ which are not conjugate to one of T_ε, T'_ε and T_p.)

Now suppose that n is an integer and $n \geq 3$. Suppose that E is an extension of $\mathbb{Q}_p$ of degree n. There are various ways to produce such extensions. For example, we can show that there exists a root of unity $\eta \in \overline{\mathbb{Q}}_p$ of order prime to p such that $\mathbb{Q}_p(\eta)$ is an extension of $\mathbb{Q}_p$ of degree n. This is an example of an unramified extension of $\mathbb{Q}_p$. Or, we can see that $x^n - p$ is irreducible over $\mathbb{Q}_p$. A root of this polynomial in $\overline{\mathbb{Q}}_p$ generates a degree n extension of $\mathbb{Q}_p$. This is an example of a totally ramified extension of $\mathbb{Q}_p$. Let β be a basis for E over $\mathbb{Q}_p$ (viewing E as an n-dimensional vector space over $\mathbb{Q}_p$). Given $x \in E^\times$, let x_β be the matrix of the invertible F-linear transformation ℓ_x of E given by multiplication by x. Then $x \mapsto x_\beta$ is an injection of $E^\times$ into $GL_n(\mathbb{Q}_p)$. The intersection of the image of this map with $SL_n(\mathbb{Q}_p)$ is isomorphic to the kernel of the norm map $N_{E/\mathbb{Q}_p} : E^\times \to \mathbb{Q}_p^\times$ and is a compact maximal torus in $SL_n(\mathbb{Q}_p)$.

If T is a compact-mod-centre maximal torus in a reductive group over a local field, then T cannot lie inside any proper Levi subgroup of G. (Note that a maximal torus of a proper Levi subgroup of G contains the centre of the Levi subgroup and the centre of the Levi subgroup contains an F-split torus that is not compact modulo the centre of G.) We often refer to a maximal torus that is compact-mod-center as an *elliptic* maximal torus.

Remark 6.3. Of course, the maximal torus $SO_2(\mathbb{R})$ of $SL_2(\mathbb{R})$ is obtained from the quadratic extension $\mathbb{C}$ of $\mathbb{R}$ in the same way that the maximal tori T_ε, T'_ε and T_p of $SL_2(\mathbb{Q}_p)$ are obtained from quadratic extensions $\mathbb{Q}_p(\sqrt{\varepsilon})$ and $\mathbb{Q}_p(\sqrt{p})$ of $\mathbb{Q}_p$. The lack of degree n extensions of $\mathbb{R}$ for $n \geq 3$ is what prevents $SL_n(\mathbb{R})$ from having compact maximal tori.

We may also consider tori in reductive groups over finite fields. In this context, we say that a maximal torus is *elliptic* if it does not lie an any proper Levi subgroup of G. Elliptic maximal tori always exist – for $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$, this is connected with the fact that, given any positive integer n, there exists a degree n extension of $\mathbb{F}_q$.

7. Discrete series representations of reductive Lie groups

Let G be a real reductive Lie group. In this case, G has (relative) discrete series representations if and only if G has a compact-mod-centre maximal torus. subgroup). Suppose that such a torus T exists. Then T is unique up to conjugacy. Let $\widehat{T}$ be the group of unitary characters of T. The "complex Weyl group" W (see below) acts on $\widehat{T}$. Let

$$\widehat{T}' = \{\, \chi \in \widehat{T} \mid w \cdot \chi \neq \chi \ \forall\ w \in W,\ w \neq 1 \,\}.$$

The finite group $W_G(T) := N_G(T)/T$ also acts on $\widehat{T}$ and is a subgroup of the complex Weyl group W. (When G is compact $W_G(T)$ is equal to the complex Weyl group.)

Let $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{t}^{\mathbb{C}}$ be the complexifications of the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ of G, K and T, respectively. By assumption, $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra (Lie algebra of a maximal torus) of $\mathfrak{g}^{\mathbb{C}}$ and of $\mathfrak{k}^{\mathbb{C}}$. Let $\Phi_G = \Phi(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and $\Phi_K = \Phi(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ be the root systems of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ (respectively) with respect to $\mathfrak{t}^{\mathbb{C}}$. The complex Weyl group is the Weyl group of the root system Φ_G.

Example 7.1. Let $G = Sp_4(\mathbb{R})$ (realized as in Section 3). Then

$$T := \left\{ \begin{pmatrix} \cos t_1 & 0 & 0 & \sin t_1 \\ 0 & \cos t_2 & \sin t_2 & 0 \\ 0 & -\sin t_2 & \cos t_2 & 0 \\ -\sin t_1 & 0 & 0 & \cos t_2 \end{pmatrix} \mid t_1,\, t_2 \in \mathbb{R} \right\}$$

is a compact maximal torus in the group K of fixed points of the Cartan involution of G, as well as a compact maximal torus in G. Hence G has discrete series representations. In this example, K is isomorphic to $U(2)$ and is realized as $K = \{\, g \in G \mid {}^t g g = I \,\}$. The Lie algebra $\mathfrak{k}$ of K is realized as follows:

$$\mathfrak{k} = \{\, X \in M_4(\mathbb{R}) \mid {}^t X = -X \text{ and } J^{-1} X J = X \,\} \ = \left\{ \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & b_3 & b_1 \\ -b_1 & -b_3 & 0 & -a \\ -b_2 & -b_1 & a & 0 \end{pmatrix} \mid a,\, b_j \in \mathbb{R} \right\}$$

The Lie algebra $\mathfrak{t}$ of T has the form

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & 0 & 0 & t_1 \\ 0 & 0 & t_2 & 0 \\ 0 & -t_2 & 0 & 0 \\ -t_1 & 0 & 0 & 0 \end{pmatrix} \mid t_j \in \mathbb{R} \right\}.$$

The root system $\Phi_G = \Phi(\mathfrak{sp}_4(\mathbb{C}), \mathfrak{t}^{\mathbb{C}})$ contains eight roots and the complex Weyl group is a dihedral group of order eight (see Section 3 for information on the Weyl group of

$Sp_4(\mathbb{C})$). The root system Φ_K contains two roots and has a Weyl group of order two. Viewing characters of $T = SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ as tensor products $\chi_1 \otimes \chi_2$ where each χ_j is a character of $SO_2(\mathbb{R})$,

$$\widehat{T}' = \{\, \chi_1 \otimes \chi_2 \mid \chi_j^2 \neq 1,\ \chi_1 \neq \chi_2^{\pm 1} \,\}$$

For convenience, we will assume that G is semisimple. The discrete series representations of G are parametrized by $\widehat{T}'$. If $\chi,\ \chi' \in \widehat{T}'$, then the corresponding discrete series representations are equivalent if and only if there exists $w \in W_G(T)$ such that $\chi' = w\chi$.

There is a connection with the irreducible representations of K. If we have a unitary representation (π, V) of G we can restrict the representation to K. Then we can consider the space of K-finite vectors in V: the vectors belonging to finite-dimensional K-subrepresentations of V (see Definition 7.4 of Chapter 3). The space of K-finite vectors is dense in V (Corollary 7.5 of Chapter 3). A K-type of π is defined to be an irreducible unitary representation σ of K that occurs in the space of K-finite vectors. Equivalently, the space V^σ of σ-isotypic vectors in V is nonzero (see Definition 9.8 of Chapter 3). According to the theory of finite-dimensional irreducible representations of compact Lie groups (see Theorem 4.13, Chapter 4), each K-type of π has a highest weight. The parametrization of discrete series representations is expressed in terms of the highest weights of the K-types of the representations.

For convenience, we assume that G is semisimple. Take $\chi \in \widehat{T}'$. As in Chapter 4, via the isomorphism $X + \mathcal{K} \mapsto e^{2\pi i X}$ from $\mathfrak{t}_{\mathbb{R}}/\mathcal{K} \mapsto T$, where $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}$, χ corresponds to an element λ of $\mathfrak{t}_{\mathbb{R}}^*$ which is trivial on $\mathcal{K}$, that is, an element of the weight lattice $\mathcal{Y}$ (see Definition 2.2, Chapter 4). Perhaps λ is not dominant. Since the $W_G(T)$-orbit of λ contains exactly one dominant weight, after replacing χ by a suitable element in this orbit, we may assume that λ is dominant. In the current setting, it may be convenient to view λ as an element of $(\mathfrak{t}^{\mathbb{C}})^*$ (recall that $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t}_{\mathbb{R}} \oplus i\mathfrak{t}_{\mathbb{R}}$). Each root space $\mathfrak{g}_\alpha^{\mathbb{C}}$, $\alpha \in \Phi_G$ either lies inside $\mathfrak{k}^{\mathbb{C}}$ or has trivial intersection with $\mathfrak{k}^{\mathbb{C}}$. This means that the root system Φ_K is a subset of Φ_G. In defining dominant elements of the weight lattice $\mathcal{Y}^d$, we fixed a choice Φ_K^+ of positive roots in Φ_K. As in Definition 4.1 of Chapter 4, for each $\alpha \in \Phi_G$, let $H_\alpha \in \mathfrak{t}_{\mathbb{R}}$ be such that $\alpha(X) = K(H_\alpha, X)$ for all $X \in \mathfrak{t}_{\mathbb{R}}$. (Here, K is the Killing form.) Since λ is dominant, we have $\lambda(H_\alpha) > 0$ for $\alpha \in \Phi_K^+$. The fact that $\chi \in \widehat{T}'$ implies that $\lambda(H_\alpha) \neq 0$ for all $\alpha \in \Phi_G$. There is a unique choice of positive roots Φ_G^+ such that $\lambda(H_\alpha) > 0$ for all $\alpha \in \Phi_G^+$. Let δ_G and δ_K be the half-sums of the positive roots in Φ_G^+ and Φ_K^+, respectively.

For $\mu \in \mathcal{Y}^d$, let σ_μ be the irreducible representation of K with highest weight μ.

A proof of the following theorem can be found in [**K1**] (see Theorems 9.20 and 12.21).

THEOREM 7.2. *(Harish-Chandra) Assume that G is semisimple and $\delta_G \in \mathcal{Y}$. Let π_λ be the discrete series representation corresponding to λ. Let $\Lambda = \lambda + \delta_G - 2\,\delta_K$. Then*

(1) *The isotypic subspace V^{σ_Λ} is nonzero and the multiplicity of σ_Λ in V^{σ_Λ} equals one.*
(2) *If $\mu \in \mathcal{Y}^d$ occurs as the highest weight of a K-type of π, then $\mu = \Lambda + \sum_{\alpha \in \Phi_G^+} n_\alpha \alpha$ for nonnegative integers n_α.*

Two such representations π_λ are equivalent if and only if their parameters λ are conjugate under $W_G(T)$.

There is an additional property of the discrete series π_λ, which, together with the two properties stated above in the theorem, characterizes the representation. By a version of Schur's Lemma, the centre $Z(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ acts on the space K-finite vectors in the space of π_λ via scalar operators. This results in a $\mathbb{C}$-algebra homomorphism from $Z(\mathfrak{g})$ to $\mathbb{C}$, which is associated to a Weyl group orbit in $(\mathfrak{t}^{\mathbb{C}})^*$. This is called the *infinitesimal character* of π. The additional property of π_λ is that the infinitesimal character of π_λ is the Weyl group orbit of λ.

If (π, V) is a continuous representation of G in a Hilbert space V, for each $f \in C_c(G)$, we define a linear operator $\pi(f)$ on V by $\pi(f)v = \int_G f(g)\pi(g)v\,dg$, $v \in V$. Assume that $\pi(f)$ is of trace class for all $f \in C_c(G)$ (for example, this is the case if π is irreducible and unitary). Then the map $f \mapsto \pi(f)$ is a distribution on G (that is, a continuous linear functional on $C_c(G)$). This distribution is called the *(global) character* of π. It is realized by a locally integrable function Θ_π on G:

$$\operatorname{tr}\pi(f) = \int_G f(g)\,\Theta_\pi(g)\,dg, \qquad f \in C_c(G).$$

Harish-Chandra parametrized the discrete series representations and constructed their characters as locally integrable functions. On the compact Cartan subgroup T, the character Θ_{π_λ} of the discrete series representation π_λ has a formula that is analogous to the Weyl character formula for compact Lie groups. (The character also has nonzero values on the noncompact Cartan subgroups.)

8. Discrete series representations of reductive p-adic groups

Let G be a (nonabelian) connected reductive p-adic group. In this case, G has several conjugacy classes of compact-mod-centre maximal tori and there are discrete series representations associated with characters of such tori. In addition, there are discrete series representations that are not associated with characters of compact-mod-centre maximal tori.

Let (π, V) be a continuous representation of G. We say that a vector $v \in V$ is *smooth* if there exists a compact open subgroup K of G such that $\pi(k)v = v$ for every $k \in K$. This is equivalent to saying that $\mathrm{Stab}_G(v) := \{\, g \in G \mid \pi(g)v = v \,\}$ is open. (Note that here K is any compact open subgroup of G – it is not the compact open subgroup in the Iwasawa decomposition of G.) We can show that V^∞ is a subspace of V and V^∞ is dense in V. Recall that (see Proposition 3.1 of Chapter 3) that $V^\infty = V$ whenever π is finite-dimensional.

For the next example, we need the definition of cuspidal representation of a finite group of Lie type. Let $H = \mathbf{H}(\mathbb{F}_q)$ be a finite group of Lie type. An irreducible representation $(\sigma, \mathcal{V})$ of H is *cuspidal* if

$$\mathcal{V}^N := \{\, v \in \mathcal{V} \mid \sigma(n)v = v \ \forall\ n \in N \,\} = \{0\}$$

for any unipotent radical of a proper parabolic subgroup of H. (Recall that a parabolic subgroup P is a semidirect product of the form $P = M \ltimes N$.) It can be shown that σ is cuspidal if it does not occur as an irreducible consituent of any representation of the form $\mathrm{Ind}_P^G \tau$, where $P = M \ltimes N$ is a proper parabolic subgroup of H and τ is a finite-dimensional representation of M.

Example 8.1. Let $K = SL_2(\mathbb{Z}_p)$. The maximal normal pro-p-subgroup of K consists of the set of matrices $k \in K$ such that all entries of $k - I$ belong to $p\mathbb{Z}_p$. The factor group K/K^u is isomorphic to $SL_2(\mathbb{F}_p)$. Let $(\sigma, \mathcal{V})$ be an irreducible cuspidal representation of $SL_2(\mathbb{F}_p)$. Let

$$\begin{aligned}\mathcal{V}^G = \{\, f : G \to \mathcal{V} \mid f(kg) = \sigma(k)\, f(g),\ \forall\ k \in K,\ g \in G,\ \mathrm{supp} f \subset \cup_j K g_j,\\ \text{for a finite subset } \{g_j\} \text{ of } G \,\}.\end{aligned}$$

Let $\mathrm{ind}_K^G \sigma$ be the right regular representation of G on $\mathcal{V}^G$. Then $\mathrm{ind}_K^G \sigma$ is an irreducible smooth representation of G. Let φ_σ be a matrix coefficient of σ. Define φ'_σ on G by

$$\varphi'_\sigma(g) = \begin{cases} \varphi_\sigma(g) & \text{if } g \in K \\ 0, & \text{if } g \notin K \end{cases}$$

Let $v \in \mathcal{V}$ and $\lambda \in \mathcal{V}^*$ be such that $\varphi_\sigma(k) = \lambda(\sigma(k)v)$ for $k \in K$. Define $f_v : G \to \mathcal{V}$ by

$$f_v(g) = \begin{cases} \sigma(g)v & \text{if } g \in K \\ 0, & \text{if } g \notin K \end{cases}$$

Then $f_v \in \mathcal{V}^G$. Define $\Lambda \in \mathcal{V}^{G*}$ by $\Lambda(f) = \lambda(f(1))$, $f \in \mathcal{V}^G$. Then we can show that $\Lambda(\mathrm{ind}_K^G(g)f_v) = \varphi'_\sigma(g)$, $g \in G$. Thus φ'_σ is a matrix coefficient of $\mathrm{ind}_K^G \sigma$. By definition, φ'_σ is a compactly supported continuous function. In particular, $\varphi'_\sigma \in L^2(G)$. The representation $\mathrm{ind}_K^G \sigma$ is an example of a supercuspidal representation of G.

Since G and K are unimodular (G is reductive and K is compact), we have a G-invariant measure $d\bar{g}$ on $K\backslash G$. The representation σ is unitary. Let $\langle\ ,\ \rangle_\sigma$ be a K-invariant inner product on $\mathcal{V}$. The function $g \mapsto \langle f_1(g),\, f_2(g)\rangle_\sigma$ factors to a function in $C_c(K\backslash G)$. Set

$$\langle f_1,\, f_2\rangle = \int_{K\backslash G} \langle f_1(\bar{g}),\, f_2(\bar{g})\rangle_\sigma d\bar{g}, \qquad f_1,\, f_2 \in \mathcal{V}^G.$$

Using G-invariance the measure, we can see that the inner product is G-invariant. Hence the completion of $\mathcal{V}^G$ with respect to this inner product is a unitary representation of G.

A smooth representation π of G is *supercuspidal* if every matrix coefficient of π is compactly supported modulo the centre Z of G. It follows that a unitarizable supercuspidal representation of G has the property that its matrix coefficients belong to $L^2(G)$.

LEMMA 8.2. *Let π is an irreducible smooth representation of G, then π is supercuspidal if and only if some nonzero matrix coefficient of π is compactly supported modulo Z.*

Let $f : G \to \mathbb{C}$ be a locally constant function such that the support of f is compact modulo Z. Then f is a *cusp form* if $\int_N f(gn)\, dn = 0$ for all $g \in G$ and all unipotent radicals of proper parabolic subgroups of G. (Here, dn denotes a Haar measure on N.)

LEMMA 8.3. *If π is an irreducible supercuspidal representation of G, then all matrix coefficients of π are cusp forms.*

REMARK 8.4. We may also define cusp forms on finite groups of Lie type. It is easy to see that a matrix coefficient of a cuspidal representation of a finite group of Lie type is a cusp form. For this reason, we may view supercuspidal representations of reductive p-adic groups as analogues of cuspidal representations of finite groups of Lie type.

As we saw above, some supercuspidal representations are obtained via induction from inflations of cuspidal representations of finite groups of Lie type. However, there are many supercuspidal representation that are induced from representations of compact-mod-centre subgroups where the inducing representations are not related to cuspidal representations of finite groups of Lie type. There exist families of supercuspidal representations which arise from characters of elliptic (compact-mod-centre) maximal tori.

EXAMPLE 8.5. Let $T = T_\varepsilon$ be as in Section 6. Let χ be a character ot T such that $\chi \,|\, T \cap K^u$ is nontrivial, where K^u is as in the previous example. There is a way to define a compact open subgroup K_χ of $SL_2(\mathbb{Z}_p)$ that is normalized by T and an irreducible smooth representation κ_χ of the compact open subgroup TK_χ such that the representation $\mathrm{ind}_{TK_\chi}^G \kappa_\chi$ is an irreducible smooth representation of $SL_2(\mathbb{Q}_p)$. This induced representation is supercuspidal.

As seen above, we may construct unitary discrete series representations from unitarizable supercuspidal representations. Although we don't take the time to discuss examples, there are some discrete series representations of reductive p-adic groups which are not obtained from supercuspidal representations (and are not associated with characters of maximal tori).

9. Maximal tori and representations of finite groups of Lie type

In this section, q is a power of a prime p, $\mathbf{G}$ is a connected reductive $\mathbb{F}_q$-group, and $G = \mathbf{G}(\mathbb{F}_q)$. If $\mathbf{T}$ is a maximal $\mathbb{F}_q$-torus in $\mathbf{G}$ we will refer to $T := \mathbf{T}(\mathbb{F}_q)$. as a maximal torus in G. The so-called "Deligne-Lusztig construction" associates class functions on G to characters of maximal tori in G. Certain of these class functions are, up to sign, equal to characters of irreducible representations of G. The book of Carter ([**Car**]) is a good basic reference for this (and for other information about the representation theory of reductive groups over finite fields).

Let $\ell \neq p$ be a prime. The ℓ-adic cohomology groups with compact support play a role in the Deligne-Lusztig construction. Suppose that X is an algebraic variety over the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Each automorphism of X induces a nonsingular linear map $H_c^i(X, \overline{\mathbb{Q}}_\ell) \to H_c^i(X, \overline{\mathbb{Q}}_\ell)$. This makes $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ a module for the group of automorphisms of X. If g is an automorphism of X of finite order, the "Lefschetz number of g on X" $\mathcal{L}(g, X)$ is defined to be $\sum_i (-1)^i \mathrm{trace}(g, H_c^i(X, \overline{\mathbb{Q}}_\ell))$. It is known that $\mathcal{L}(g, X)$ is an integer that is independent of ℓ.

Let $T = \mathbf{T}(\mathbb{F}_q)$ be a maximal torus in G and let Let $\mathbf{B}$ be a Borel subgroup (minimal parabolic subgroup) of $\mathbf{G}$ that contains $\mathbf{T}$. Note that $\mathbf{B}$ is not necessarily defined over $\mathbb{F}_q$. If $\mathbf{N}$ is the unipotent radical of $\mathbf{B}$, then $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$. Let Fr be the Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. (That is, $\mathrm{Fr}(x) = x^q$ for every $x \in \overline{\mathbb{F}}_q$.) The notation Fr will also be used for the action of Fr on $\mathbf{G}$, $\mathbf{T}$, etc. Define $L(g) = g^{-1}\mathrm{Fr}(g)$, $g \in \mathbf{G}$. This map is called *Lang's map*. Note that $G = \mathbf{G}^{\mathrm{Fr}} = L^{-1}(1)$. The set $\widetilde{X} = L^{-1}(\mathbf{N})$ is an affine algebraic variety. The group G acts on $\widetilde{X}$ by left multiplication: if $g \in G$ and $x \in \widetilde{X}$, then

$L(gx) = x^{-1}g^{-1}\mathrm{Fr}(g)\mathrm{Fr}(x) = x^{-1}\mathrm{Fr}(x) = L(x) \in \mathbf{N}$, so $gx \in \widetilde{X}$. The group T acts on $\widetilde{X}$ by right multiplication: if $x \in \widetilde{X}$ and $t \in T$, then $L(xt) = t^{-1}x^{-1}\mathrm{Fr}(x)\, t \in t^{-1}\mathbf{N}t = \mathbf{N}$. These actions commute with each other. Thus $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ is a left G-module and a right T-module such that $(gv)t = g(vt)$ for $g \in G$, $t \in T$ and $v \in H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$.

Let $\widehat{T}$ be the set of characters (one-dimensional representations) of the maximal torus T. Let $\theta \in \widehat{T}$. If n is the order of T and $t \in T$, then $\theta(t)$ is an nth root of unity, so $\theta(t)$ is an algebraic integer. Because $\overline{\mathbb{Q}}_\ell$ contains the algebraic numbers, we can view θ as a homomorphism from T to $\overline{\mathbb{Q}}_\ell^\times$. Let $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)_\theta$ be the T-submodule of $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ on which T acts by the character θ. Define $R_{T,\theta} = R^G_{T,\theta} : G \to \overline{\mathbb{Q}}_\ell$ by

$$R_{T,\theta}(g) = \sum_{i \geq 0} (-1)^i \mathrm{trace}(g, H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)_\theta), \qquad g \in G.$$

We say that the character θ is in *general position* if $w \cdot \theta \neq \theta$ for all nontrivial elements w of the Weyl group $N_G(T)/T$ of T in G.

Properties of $R_{T,\theta}$:

(1) $R_{T,\theta}$ is a "generalized character" (an integral linear combination of characters of irreducible representations of G).
(2) $R_{T,\theta}(g) = |T| \sum_{t \in T} \theta(t^{-1})\, \mathcal{L}((g,t), \widetilde{X})$, $g \in G$.
(3) $R_{T,\theta}$ is independent of the choice of Borel subgroup $\mathbf{B}$ that has $\mathbf{T}$ as Levi factor.
(4) If θ is in general position, then, up to sign, $R_{T,\theta}$ is the character of an irreducible representation of G.
(5) If T is elliptic (that is, T does not lie in the Levi factor of a proper $\mathbb{F}_q$-parabolic subgroup of G) and θ is in general position, then, up to sign, $R_{T,\theta}$ is the character of an irreducible cuspidal representation of G.

Remark 9.1. In general, there exist irreducible representations of G whose characters are not of the form $\pm R_{T,\theta}$ for some T and θ, with θ in general position.

10. Harish-Chandra's Philosophy of Cusp Forms

Harish-Chandra's "philosophy of cusp forms" describes similarities between the following four theories:

- Representation theory of finite groups of Lie type
- Representation theory of reductive Lie groups
- Representation theory of reductive p-adic groups

- Theory of automorphic forms

The philosophy says that there are certain basic representations from which all other representations are constructed. In the first three cases, the basic representations are cuspidal, supercuspidal, and discrete series representations, respectively, and the construction is parabolic induction. In the fourth case, the basic representations are cuspidal automorphic representations and the construction is Eisenstein series. (We will not be discussing the fourth case here.) Much of the time, though not always, the construction produces irreducible representations. When the construction produces a representation which is not irreducible, some work may be involved in determining the irreducible subquotients of the representation.

An irreducible representation of a finite group G of Lie type is cuspidal or is a constituent of a representation that is parabolically induced from a cuspidal representation of a proper Levi subgroup of G.

When working with representations of reductive Lie groups and reductive p-adic groups, most results are valid in the context of admissible representations.

A smooth representation (π, V) of a reductive p-adic group is *admissible* if the space V^K of K-fixed vectors in V is finite-dimensional for every compact open subgroup K of G. An irreducible smooth representation of a reductive p-adic group is admissible. An irreducible smooth representation of a reductive p-adic group G is supercuspidal or occurs as a subquotient of a representation that is parabolically induced from an irreducible supercuspidal representation of a proper Levi subgroup of G.

Let G be a reductive Lie group and let K be group of fixed points of the Cartan involution of G. A continuous representation (π, V) of G in a Hilbert space V is *admissible* if for every irreducible continuous representation σ of K, the dimension of the σ-isotypic subspace V is finite-dimensional. (That is, the representation σ occurs finitely many times in the decomposition of the restriction of π to K.) Any irreducible unitary representation of G is admissible. We say that a parabolic subgroup $P = M \ltimes N$ is *cuspidal* if M has relative discrete series representations. Roughly speaking, the Langlands classification of irreducible admissible representations of G is stated as follows. An irreducible admissible representation is a quotient of a representation obtained via parabolic induction from a representation of the form $\chi \otimes \sigma$, where σ, resp. χ, is a relative discrete series representation, resp. quasicharacter, of a Levi factor of a cuspidal parabolic subgroup of G. For a precise statement, see

Theorem 14.92 of [**K1**]. (In fact we also have to allow some other representations, known as "limits of discrete series" on the Levi subgroups.)

Bibliography

Introductions to representation theory

[Bo] A. Borel, *Properties and linear representations of Chevalley groups*, in *Seminar on Algebraic Groups and Related Finite Groups*, Springer Lecture Notes in Math. **131** (1970), 1–55.

[BD] T. Brocker and T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, **98**, Springer-Verlag, 1985.

[BH] C.J. Bushnell and G. Henniart, *The local Langlands conjecture for* $GL(2)$, A Series of Comprehensive Studies in Mathematics **335** Springer-Verlag 2006. (Chapter 1: Introduction to smooth representations of locally profinite groups; Chapters 2–4: Representations of $GL(2)$ over finite and p-adic fields.)

[C] W. Casselman, *Introduction to the theory of admissible representations of* p*-adic reductive groups*, Unpublished manuscript, 1974. http://www.math.ubc.ca/people/faculty/cass/research.html.

[FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics, **129**, Springer-Verlag, New York, 1991. (Representations of finite groups; Finite-dimensional representations of Lie groups and Lie algebras.)

[Ha] Brian C. Hall, *Lie Groups, Lie Algebras, and Representations, An Elementary Introduction*, Graduate Texts in Mathematics **222**, 2003, Springer.

[H] *Introduction to Lie Algebras and Representation Theory*, James E. Humphreys, Graduate Texts in Mathematics, **9**, Springer-Verlag.

[K1] Anthony W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, 1986.

[K2] Anthony W. Knapp, *Lie Groups: Beyond an Introduction*, Birkhauser 1996.

[M] F. Murnaghan, Mat 1197 Course Notes: Representations of reductive p-adic groups, available at www.math.toronto.edu/ murnaghan.

[R] A. Robert, *Introduction to the Representation Theory of Compact and Locally Compact Groups*, London Mathematical Society Lecture Note Series **80**, Cambridge University Press, 1983. (Main reference for the first part of the course.)

[Se] M. Sepanski, *Compact Lie Groups*, Graduate Texts in Mathematics, **235**, Springer-Verlag, 2007.

[Si] B. Simon, *Representations of finite and compact groups*, Graduate Studies in Mathematics, **10**, American Mathematical Society, 1996.

Topological groups and measure theory

[B] N. Bourbaki, *Integration I, Chapters 1–6*, Translated from the original French edition (1959–1965), Springer-Verlag, 2004.

[Hal] P. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc. New York, 1950.

[HR] E. Hewitt and K. Ross, *Abstract Harmonic Analysis. Vol. I, Structure of topological groups, integration, theory, group representations*, *Grundlehren der Mathematischen Wissenshaften*, **115**, Springer-Verlag, 1979.

[Hu] T. Husain, *Introduction to Topological Groups*, W.B. Saunders Co., 1966.

[L] L. Loomis, *An Introduction to Abstract Harmonic Analysis*, D. Van Nostrand Company, Inc., 1953.

[Ma] N. Markley, *Toopological groups: an introduction*, Wiley, 2010.

[P] L.S. Pontryagin, *Topological Groups*, second edition, Gordon and Breach, 1966.

Profinite groups and local fields

[Ba] G. Bachman, *Introduction to p-adic numbers and valuation theory*, Academic Press, London, 1964.

[Cas] J.W.S. Cassels, *Local Fields*, London Mathematical Society Student Texts. **3**, Cambridge University Press, 1986.

[S] S. Shatz, *Profinite groups, arithmetic, and geometry*, Annals of Mathematics Studies **67** Princeton University Press, 1972.

Representations of reductive p-adic groups

[BZ1] I.N. Bernstein and A.V. Zelevinsky, *Representations of the group $GL(n,F)$, where F is a non-archimedean local field*, Russian Math. Surveys **31:3** (1976), 1–68.

[BZ2] I.N. Bernstein and A.V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. scient. Éc. Norm. Sup. 4^e série, **10** (1977), 441–472.

[Z] A. Zelevinsky, *Induced representations of p-adic reductive groups II: Irreducible representations of $GL(n)$*, Ann. Sci. Ec. Norm. Sup. **3** (1980), 165–210.

Other references

[Car] R. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Pure and Applied Mathematics, Wiley, 1985.

[Ka] R. Kallman, *A characterization of uniformly continuous unitary representations of connected locally compact groups*, Michigan Math. J. **16** (1969), 257–263.

[KT] E. Kaniuth and K. Taylor, *Induced representations of locally compact groups*, Cambridge Tracts in Mathematics, Cambridge University Press, 2013.

[K] S. Kantorovitz, *Introduction to modern analysis*, Oxford Graduate Texts in Mathematics, **8**, Oxford University Press, 2003.

[Sg] I.M. Singer, *Uniformly continuous representations of Lie groups*, Annals of Math. **56** (1952), 242–247.