Mat 1197 - Representations of the mirabolic subgroup

February 9

Let

$$H = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid a \in F^\times,\, x \in F \right\},$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in F \right\},$$
$$S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in F^\times \right\}.$$

The group H is called the *mirabolic subgroup* of $GL_2(F)$. Let (π, V) be a smooth representation of N and let ϑ be a quasicharacter of N (that is, a one-dimensional smooth representation of N). Let

$$V(\vartheta) = \mathrm{Span}(\{\, \pi(n)v - \vartheta(n)v \mid n \in N,\ v \in V \,\}) \quad \text{and} \quad V_\vartheta = V/V(\vartheta).$$

When ϑ is trivial, we write $V(N)$ instead of $V(\vartheta)$ and V_N instead of V_ϑ.

1. Let (π, V) be a smooth representation of N and let $v \in V$. Show that $v \in V(\vartheta)$ if and only if there exists a compact open subgroup U of N such that $\int_U \vartheta(n)^{-1}\pi(n)v\, dn = 0$. (Here, dn is a Haar measure on N.)
2. If (π_j, V_j) are smooth representations of N, $1 \le j \le 3$ and $V_1 \to V_2 \to V_3$ an exact sequence of N-morphisms, show that there is a corresponding exact sequence at the level of the spaces $(V_j)_\vartheta$.
3. Suppose that ϑ is nontrivial. Show that the inclusion $V(N) \to V$ induces an isomorphism $V(N)_\vartheta \simeq V_\vartheta$.
4. Let (π, V) be a smooth representation of N. Prove that if $v \in V$ and $v \neq 0$, then there exists a quasicharacter ϑ of N such that $v \notin V(\vartheta)$.
5. Let (π, V) be a smooth representation of H. Suppose that $V_N = \{0\}$ and $V_\vartheta = \{0\}$ for some nontrivial quasicharacter ϑ. Prove that $V = \{0\}$. (*Hint*: Consider the action of S on the spaces $V(\vartheta)$ for ϑ nontrivial.)
6. Let ϑ be a nontrivial quasicharacter of N. Let $\pi = \mathrm{Ind}_N^H \vartheta$ and let V be the space of π. Let $\pi^c =$c-$\mathrm{Ind}_N^H \vartheta$ and let V^c be the space of π^c. Show that
   a) $V(N) = V^c(N) = V^c$ and $V/V^c(N) = \{0\}$.
   b) The map $f \mapsto f(1)$ induces isomorphisms $V_\vartheta \simeq \mathbb{C}$ and $V^c_\vartheta \simeq \mathbb{C}$.
7. Let ϑ be a nontrivial quasicharacter of N.
   a) Prove that c-$\mathrm{Ind}_N^H \vartheta$ is an irreducible representation of H.
   b) Prove that the contragredient (smooth dual) of c-$\mathrm{Ind}_N^H \vartheta$ is reducible.
   c) Prove that c-$\mathrm{Ind}_N^H \vartheta$ is not admissible.
8. Let (π, V) be a smooth representation of H and let ϑ be a nontrivial quasicharacter of N. Let $q_\vartheta : V \to V_\vartheta$ be the quotient map. Frobenius reciprocity gives an isomorphism $\mathcal{A} : \mathrm{Hom}_H(V, \mathrm{Ind}_N^H V_\vartheta) \simeq \mathrm{Hom}_N(V, V_\vartheta)$. Let $q_* = \mathcal{A}^{-1}(q_\vartheta)$. (That is, for $v \in V$, $q_*(v)$ is the function $h \mapsto q(\pi(h)v)$.) Prove that the H-morphism $q_* : V \to \mathrm{Ind}_N^H V_\vartheta$ induces an isomorphism $V(N) \simeq$c-$\mathrm{Ind}_N^H V_\vartheta$.
9. Let (π, V) be an irreducible smooth representation of H. Prove that exactly one of the following holds:
   (i) $\dim V = 1$ and there exists a quasicharacter χ of H such that $\pi(hn) = \chi(h)$ for all $n \in N$ and $h \in H$.
   (ii) V is infinite-dimensional and $\pi \simeq$c-$\mathrm{Ind}_N^H \vartheta$ for any nontrivial quasicharacter ϑ of N.

   Describe the spaces V_N and V_ϑ (for ϑ nontrivial) in both cases.