REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

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1. Introduction

Let F be a field and n a positive integer. The set $G = GL_n(F)$ of invertible matrices with entries in the field F forms a group with respect to matrix multiplication. This is an example of a reductive group. If F is a finite field, then G is a finite group. As (complex) representations of finite groups are discussed in earlier courses, some results concerning representations of G in the case when F is a finite field will be assumed. In this course, we will be particularly interested in the case where F is a p-adic field (p-adic fields will be described in section 2). But we can ask what are the representations of G for any field F. A representation of G consists of a (possibly infinite dimensional) complex vector space V and a homomorphism π from G to the set of invertible linear operators on V. Depending on the field F, we restrict our attention to representations satisfying "extra" topological conditions.

We make a few remarks about the connection between representations of p-adic groups and automorphic forms. Suppose that F is a number field (a finite extension of the rational numbers). Then F has completions with respect to the various valuations on F. For example, if F is the rational numbers, the completion of F with respect to the standard valuation (absolute value) $|\cdot|_\infty$ is the real numbers. For each prime integer p, the completion of F with respect to the p-adic valuation $|\cdot|_p$ is the p-adic numbers $\mathbb{Q}_p$ (see section 2). The adelic ring $\mathbb{A}$ of $\mathbb{Q}$ is a certain subring of the direct product $\mathbb{R}\times\prod_p \mathbb{Q}_p$ of all of these completions. Similarly we can form the adelic ring $\mathbb{A}_F$ of a general number field F. Given a reductive group G which is defined over F, we can define the adelic group $G(\mathbb{A}_F)$ (e.g. $GL_n(\mathbb{A}_F)$). An automorphic form on $G(\mathbb{A}_F)$ is a complex-valued function on $G(\mathbb{A}_F)$ satisfying certain conditions (not discussed here). Now suppose that V is a subspace of the vector space of complex-valued functions on $G(\mathbb{A}_F)$, having the property that for $f \in V$ and $g \in G(\mathbb{A}_F)$, the function $x \mapsto f(xg)$ also belongs to V. Then we have a natural representation π on V: $(\pi(g)f)(x) = f(xg)$. If V happens to consist of automorphic forms, then π is said to be an automorphic representation of $G(\mathbb{A}_F)$. The group $G(\mathbb{A}_F)$ can be expressed as a restricted direct product of the groups $G(F_v)$, as F_v runs over the various completions of F. Also, the automorphic representation π decomposes into a tensor product of representations π_v, where π_v is a representation of $G(F_v)$. The fields F_v are called local fields, and the representations π_v are called the local components of π. It turns out that for all but finitely many completions, F_v is a p-adic field. So most of the local components of automorphic representations are representations of reductive p-adic groups.

(The other components are also important - they are representations of real or complex reductive groups).

Now we discuss harmonic analysis on topological groups. A topological group G is a group which is also a topological space and has the property that the map $(x,y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous. Assume that G is (the rational points of) a reductive group over a p-adic field: such a group is a locally compact topological group. There exists a measure μ on G which is G-invariant: $\mu(gS) = \mu(Sg) = \mu(S)$, for $g \in G$, and S any measurable subset of G. This measure μ is unique up to positive scalar multiples and is called Haar measure on G. Let $L^2(G)$ denote the set of complex valued functions on G which are square integrable relative to Haar measure. The representation R of G defined by $(R(y)f)(x) = f(xy)$, $x, y \in G$, $f \in L^2(G)$, is called the right regular representation. One of the main problems in harmonic analysis on G is to decompose the right regular representation, that is, to express it in terms of irreducible representations of G. Another major problem is to determine Plancherel measure. This involves character theory and Fourier transforms. Associated to an irreducible representation π of G (satisfying certain smoothness conditions) is a distribution Θ_π, the character of π. This distribution is a linear functional on some vector space $\mathcal{V}$ of complex valued functions on G. In the case of finite groups $\Theta_\pi(f) = \sum_{g \in G} \mathrm{trace}(\pi(g))\, f(g)$. In general, $\pi(g)$ is an operator on an infinite dimensional vector space and so does not have a trace - but if we replace the finite sum with integration with respect to Haar measure, and move the trace outside the integral, and make enough assumptions on π and $\mathcal{V}$, then we are taking the trace of an operator which has finite rank, so $\Theta_\pi(f)$ is defined. Let $\widehat{G}$ be the set of irreducible unitary representations of G. Given $f \in \mathcal{V}$, the Fourier transform $\widehat{f}$ of f is the complex valued function on $\widehat{G}$ defined by $\widehat{f}(\pi) = \Theta_\pi(f)$. Plancherel measure is a measure ν on $\widehat{G}$ having the property that

$$f(1) = \int_{\widehat{G}} \widehat{f}(\pi)\, d\nu(\pi), \qquad f \in \mathcal{V}.$$

This expression is called the Plancherel formula. A character Θ_π is an example of an invariant distribution on G: $\Theta_\pi(f^g) = \Theta_\pi(f)$ for all $g \in G$, where $f^g(x) = f(gxg^{-1})$. The map $f \mapsto f(1)$ is also an invariant distribution. Invariant distributions play an important role in harmonic analysis. Given an invariant distribution D, we can try to find a measure ν_D on $\widehat{G}$ such that

$$D(f) = \int_{\widehat{G}} \widehat{f}(\pi)\, d\nu_D(\pi), \qquad f \in \mathcal{V}.$$

Note that Plancherel measure is ν_D for $D(f) = f(1)$. Harmonic analysis on locally compact abelian groups and compact groups is well understood. Much work has been done for Lie groups, nilpotent, solvable, and semisimple Lie groups, etc. A fair amount is known for nilpotent and solvable p-adic groups. However, for reductive p-adic groups, there is very little known, and there are many open problems in harmonic analysis. For example, the characters of the irreducible representations of reductive groups over p-adic fields are not yet well understood.

In this course, we study some basic results in the theory of admissible representations of reductive p-adic groups. These results play important roles in applications to automorphic forms and harmonic analysis. As the structure of reductive algebraic groups is not discussed here, many of the results will be stated and proved for p-adic general linear groups. In these cases, the main ideas of the argument for arbitrary reductive p-adic groups is the same as that for general linear groups.

2. Valuations and local fields

The references for this part are the books by Bachman and Cassels. Let F be a field.

Definition. A *valuation* on F is a map $|\cdot|: F \to \mathbb{R}_{\geq 0}$ such that, for x, $y \in F$,

$$|x| = 0 \iff x = 0$$

$$|xy| = |x||y|$$

$$|x+y| \leq |x| + |y|$$

Definition. Two valuations $|\cdot|_1$ and $|\cdot|_2$ on F are said to be *equivalent* if $|\cdot|_1 = |\cdot|_2^c$ for some positive real number c.

Definition. A valuation $|\cdot|$ is *discrete* if there exists $\delta > 0$ such that $1-\delta < |a| < 1+\delta$ implies $|a| = 1$. That is, $\{\log|a| \mid a \in F^\times\}$ is a discrete subgroup of the additive group $\mathbb{R}$.

Example: Let p be a fixed prime integer. Given $x \in \mathbb{Q}^\times$, there exist unique integers m, n and r such that m and n are relatively prime, p does not divide m or n, and $x = p^r m/n$. Set $|x|_p = p^{-r}$ (and $|0|_p = 0$). This defines the p-adic valuation on $\mathbb{Q}$. This valuation is discrete. Observe that if $x = p^r m/n$ and $y = p^\ell m'/n'$, then

$$\begin{aligned} |x+y|_p &= \left|\frac{p^r mn' + p^\ell m'n}{nn'}\right|_p = |p^r mn' + p^\ell m'n|_p \\ &\leq \max\{p^{-r}, p^{-\ell}\} = \max\{|x|_p, |y|_p\}. \end{aligned}$$

Furthermore, if $|x|_p \neq |y|_p$, then $|x+y|_p = \max\{|x|_p, |y|_p\}$.

Definition. A valuation $|\cdot|$ is *non archimedean* if $|\cdot|$ (is equivalent to) a valuation which satisfies the ultrametric inequality: $|x+y| \leq \max\{|x|, |y|\}$, $x, y \in F$. Otherwise $|\cdot|$ is archimedean.

Example: If F is a finite field, then the only valuation on F is the trivial valuation. The field F has cardinality p^n for some prime p and some positive integer n, and so $x^{p^n} = 1$ for all $x \in F^\times$. Therefore $|x^{p^n}| = |x|^{p^n} = |1|$. But $|1|^2 = |1|$, which, since $1 \neq 0$, implies that $|1| = 1$. Hence $|x|$ is a root of unity which is also a positive real number. That is, $|x| = 1$ for all $x \in F^\times$.

Example: Let $|\cdot|$ be a non archimedean valuation on a field F. Suppose that x is transcendental over F and c is a positive real number. Given $f \in F[x]$, $f(x) = \sum_j a_j x^j$, $a_j \in F$, set $\|f\| = \|f\|_c = \max_j c^j |a_j|$. Then, if f, $g \in F[x]$ and $g \neq 0$, set $\|f/g\| = \|f\|\|g\|^{-1}$. This defines a valuation on the field $F(x)$ which coincides with $|\cdot|$ on F.

Example: Let $F(x)$ be as in the previous example. fix a real number γ such that $0 < \gamma < 1$ and a $\varphi \in F[x]$ which is irreducible over F. Given a non zero $h \in F(x)$, there exist f and $g \in F[x]$ relatively prime to φ and to each other such that $h = \varphi^r f/g$. Set $|h|_\varphi = \gamma^r$. This valuation is non archimedean.

Lemma. *The following are equivalent:*

(1) $|\cdot|$ is non archimedean

(2) $|x| \le 1$ for all x in the subring of F generated by 1

(3) $|\cdot|$ is bounded on the subring of F generated by 1

Proof. ($1 \implies 2$) $|1 + \cdots + 1| \le \max\{|1|, \ldots, |1|\} = 1$.

($2 \implies 1$) By the binomial theorem, (2), and the triangle inequality, we have

$$|x+y|^n = \left| \sum_k \binom{n}{k} x^k y^{n-k} \right| \le \sum_k |x|^k |y|^{n-k} \le (n+1)\max\{|x|, |y|\}^n.$$

Take nth roots and let $n \to \infty$. $\square$

Example: Consider the sequence $\{a_n\} \subset \mathbb{Q}$ defined by $a_1 = 4$, $a_2 = 34$, $a_3 = 334, \ldots, a_n = 33\cdots34$ (where 3 occurs $n-1$ times). If $m > n$, then $|a_m - a_n|_5 = |3\cdots30\cdots0|_5 = |10^n|_5 = 5^{-n}$. Thus $\{a_n\}$ is a Cauchy sequence with respect to the 5-adic valuation $|\cdot|_5$. Note that $|3a_n - 2|_5 = 5^{-n}$, so a_n has the (5-adic) limit $2/3$.

Example: Set $a_1 = 2$. Given an integer a_n such that $a_n^2 + 1 \equiv 0(5^n)$, choose an integer b such that $a_{n+1} = a_n + b5^n$ satisfies $a_{n+1}^2 + 1 = (a_n + b\,5^n)^2 + 1 \equiv 0(5^{n+1})$. The sequence $\{a_n\}$ is a Cauchy sequence which satisfies $|a_n^2 + 1|_5 \le 5^{-n}$. If the sequence has a limit L, then $|L^2 + 1|_5 = 0$. That is, $L^2 = -1$. Therefore this is an example of a (5-adic) Cauchy sequence which does not have a rational limit.

Theorem. *(Ostrowski) A valuation on $\mathbb{Q}$ is equivalent to the standard absolute value or to a p-adic valuation.*

The topology on F induced by a valuation $|\cdot|$ has as a basis for the open sets all sets of the form

$$U(a, \varepsilon) = \{\, b \mid |a - b| < \varepsilon \,\}, \qquad a \in F,\ \varepsilon \in \mathbb{R}_{>0}.$$

In fact, two valuations on F induce the same topology on F if and only if they are equivalent valuations.

Definition. A field K with valuation $\|\cdot\|$ is a *completion* of the field F with valuation $|\cdot|$ if $F \subset K$, $\|\cdot\| \,|\, F = |\cdot|$, K is complete with respect to $\|\cdot\|$, and K is the closure of F with respect to $\|\cdot\|$.

Note that the last condition is necessary. The complex numbers $\mathbb{C}$ are complete with respect to $|\cdot|_\infty$ and $\mathbb{C} \supset \mathbb{Q}$, but $\mathbb{C}$ is not the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$: $\mathbb{R}$ is the completion of $\mathbb{Q}$.

If p is a prime integer, the notation $\mathbb{Q}_p$ is used to denote the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$.

Definition. A *local field* F is a (non discrete) field F which is locally compact and complete with respect to a non trivial valuation.

We will see shortly that $\mathbb{Q}_p$ is a non archimedean local field. The next result tells us that (up to isomorphism), the real and complex numbers are the only archimedean local fields.

Theorem. *If F is complete with respect to an archimedean valuation $|\cdot|$, then F is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and $|\cdot|$ is equivalent to $|\cdot|_\infty$.*

From this point onward, we assume that $|\cdot|$ is a non trivial non archimedean valuation on F.

Definition: The ring $\mathfrak{o} = \{\, a \in F \mid |a| \leq 1 \,\}$ is called the (valuation)-*integers*.

The ideal $\mathfrak{p} = \{\, a \in F \mid |a| < 1 \,\}$ is a maximal ideal in $\mathfrak{o}$.

Definition: The *residue class field* of F is the field $\mathfrak{o}/\mathfrak{p}$.

Let $\overline{F}$ be the completion of F with respect to $|\cdot|$, with corresponding ring of integers $\overline{\mathfrak{o}}$ and maximal ideal $\overline{\mathfrak{p}}$.

Lemma. *The natural map $a + \mathfrak{p} \mapsto a + \overline{\mathfrak{p}}$ from $\mathfrak{o}/\mathfrak{p}$ to $\overline{\mathfrak{o}}/\overline{\mathfrak{p}}$ is an isomorphism.*

Proof. Let $\alpha \in \overline{\mathfrak{o}}$. By definition of completion, there exists $a \in F$ such that $|\alpha - a| < 1$, that is $\alpha - a \in \overline{\mathfrak{p}}$. Note that $|a| = |(a-\alpha) + \alpha| \leq \max\{|a-\alpha|, |\alpha|\} \leq 1$, so $a \in \mathfrak{o}$. □

Lemma. *The valuation $|\cdot|$ is discrete if and only if $\mathfrak{p}$ is a principal ideal.*

Proof. Suppose that $\mathfrak{p} = (\varpi)$. Then $|a| < 1$ implies that $a = \varpi b$ for some $b \in \mathfrak{o}$, and so $|a| \leq |\varpi|$. If $|a| > 1$, then $|a^{-1}| \leq |\varpi|$, so $|a| > |\varpi|^{-1}$. Thus $|\cdot|$ is discrete.

Now suppose that $|\cdot|$ is discrete. Then $\{\, |a| \; |a \in \mathfrak{p} \,\}$ attains its upper bound, say at $a = \varpi$. Now if $b \in \mathfrak{p}$, we have $|\varpi^{-1} b| \leq 1$. That is, $\varpi^{-1} b \in \mathfrak{o}$: $b = \varpi c$ for some $c \in \mathfrak{o}$. □

Definition: If $|\cdot|$ is discrete, then an element ϖ such that $\mathfrak{p} = (\varpi)$ is a *prime element* (or *uniformizer*) for the valuation.

Example: The residue class field of $\mathbb{Q}$ (with respect to $|\cdot|_p$), hence of $\mathbb{Q}_p$, is isomorphic to the finite field containing p elements. The integer p is a prime element. Note that the ring of valuation integers in $\mathbb{Q}$ is much bigger than the ordinary integers $\mathbb{Z}$. The ring of (valuation) integers in $\mathbb{Q}_p$ is called the p-adic integers and is often denoted by $\mathbb{Z}_p$.

Lemma. *Suppose F is complete and $a_n \in F$, $n \geq 0$. Then the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$.*

Proof. Suppose that $\lim_{n\to\infty} a_n = 0$. If $M > N$, then

$$\left|(\sum_{n=0}^{M} a_n) - (\sum_{n=0}^{N} a_n)\right| = |a_{N+1} + \cdots + a_M| \leq \max_{N+1 \leq n \leq M} |a_n|$$

implies that $\{\sum_{n=0}^{N} a_n\}$ is a Cauchy sequence. □

Lemma. *Suppose that F is complete with respect to a discrete (non archimedean) valuation $|\cdot|$. Let ϖ be a prime element, and $\mathcal{A} \subset \mathfrak{o}$ any set of coset representatives for $\mathfrak{o}/\mathfrak{p}$. Then $a \in \mathfrak{o}$ is uniquely of the form*

$$a = \sum_{n=0}^{\infty} a_n \varpi^n, \qquad a_n \in \mathcal{A}.$$

Further, any such series as above converges to an element $a \in \mathfrak{o}$.

Proof. The final statement follows from the previous lemma.

Let $a \in \mathfrak{o}$. There is exactly one $a_0 \in \mathcal{A}$ such that $a + \mathfrak{p} = a_0 + \mathfrak{p}$. Since $|a - a_0| < 1$, there exists $b_1 \in \mathfrak{o}$ such that $a = a_0 + b_1\varpi$. Suppose that we have $a_0, \ldots, a_N \in \mathcal{A}$ and $b_{N+1} \in \mathfrak{o}$ such that $a = a_0 + a_1\varpi + \cdots + a_N\varpi^N + b_{N+1}\varpi^{N+1}$. As $b_{N+1} \in \mathfrak{o}$, there exists exactly one $a_{N+1} \in \mathcal{A}$ such that $b_{N+1} - a_{N+1} \in \mathfrak{p}$. Thus there is a uniquely defined sequence $\{a_n\}$, $a_n \in \mathcal{A}$ such that $|a - \sum_{n=0}^{N} a_n\varpi^n| \leq |\varpi|^{N+1}$. □

In the example of the p-adic numbers, we often take $\mathcal{A} = \{0, 1, 2, \ldots, p-1\}$, and we then have an expression for each element of the p-adic integers $\mathbb{Z}_p$ in the form $\sum_{n=0}^{\infty} a_n p^n$, with $0 \leq a_n \leq p - 1$.

Corollary. *With hypotheses as in the lemma, suppose in addition that $0 \in \mathcal{A}$. Then every $a \in F^\times$ is uniquely of the form $a = \sum_{n=N}^{\infty} a_n\varpi^n$, for some integer N, with $a_N \neq 0$, and $a_n \in \mathcal{A}$.*

Corollary. *$\mathbb{Q}_p$ is uncountable.*

Lemma. *Let F be complete with respect to a discrete valuation $|\cdot|$ such that the residue class field $\mathfrak{o}/\mathfrak{p}$ is finite. Then $\mathfrak{o}$ is compact.*

Proof. Without loss of generality, we assume that $|\cdot|$ satisfies the triangle inequality. Then $|\cdot|$ makes $\mathfrak{o}$ into a metric space: $d(x, y) = |x - y|$ is a metric. For metric spaces, compactness is equivalent to sequential compactness. Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathfrak{o}$. It suffices to prove that such a sequence has a convergent subsequence. For each j, there exist $a_{jn} \in \mathcal{A}$, $n = 0, 1, \dots$ such that $a_j = \sum_{n=0}^{\infty} a_{jn}\varpi^n$. As $\mathcal{A}$ is finite, there exists $b_0 \in \mathcal{A}$ occurring as a_{j0} for infinitely many j. There exists $b_1 \in \mathcal{A}$ occurring as a_{j1} for an infinite number of those a_j satisfying $a_{j0} = b_0$. Continuing in this manner, we see that we get $b = \sum_{n=0}^{\infty} b_n\varpi^n$ as a limit of a subsequence of $\{a_j\}$. □

Corollary. *Let $|\cdot|$ be a non archimedean valuation on a field F. Then F is locally compact with respect to $|\cdot|$ if and only if the following all hold:*

(1) F is complete

(2) $|\cdot|$ is discrete

(3) the residue class field is finite.

Example: Let F be a finite field. Let ϖ be transcendental over F. The completion of $F(\varpi)$ with respect to the valuation defined using polynomial $\varphi(\varpi) = \varpi$ is the local field $F((\varpi))$ of formal Laurent series (and zero). This is an example of a local field of positive characteristic.

Definition. A *p-adic field* is a local non archimedean field.

Lemma. *A p-adic field of characteristic zero is a finite extension of $\mathbb{Q}_p$ for some prime p.*

Definition. Let E/F be an extension of finite degree. Given $\alpha \in E$, the *norm* of α, denoted $N_{E/F}(\alpha)$ is the determinant of the linear operator on the F-vector space E given by left multiplication by α.

Theorem. *Let E/F be an extension of finite degree. Assume that F is complete with respect to a non archimedean valuation $|\cdot|$. Then there exists a unique extension of $|\cdot|$ to E defined by $\|\alpha\| = |N_{E/F}(\alpha)|^{1/n}$. Furthermore, E is complete with respect to $\|\cdot\|$.*

It is clear that $\|\cdot\|$ satisfies the first two properties in the definition of valuation. If $a \in F$, then $N_{E/F}(a) = a^n$ implies that $\|a\| = |a|$. Verification of the ultrametric inequality is omitted, as is proof of uniqueness, and completeness (see Cassels, Local Fields).

Example: Let $E = \mathbb{Q}(\sqrt{2})$. Here, $\|\cdot\| = |N_{E/\mathbb{Q}}(\cdot)|_7^{1/2}$ is not a valuation on E. Note that

$$\begin{aligned}
\|3+\sqrt{2}\| &= \|3-\sqrt{2}\| = |7|_7^{1/2} = 7^{-1/2} \\
\|(3+\sqrt{2})+(3-\sqrt{2})\| &= \|6\| = 1
\end{aligned}$$

If $\|\cdot\|$ is a valuation, then $\|m\| = |m| \leq 1$ for $m \in \mathbb{Z}$ implies that $\|\cdot\|$ is non archimedean. Hence $\|(3+\sqrt{2})+(3-\sqrt{2})\| \leq \max\{\|3+\sqrt{2}\|, \|3-\sqrt{2}\|\} = 7^{-1/2} < 1$, which is a contradiction. If we are looking for a valuation on E extending $|\cdot|_7$, we can embed E in $\mathbb{Q}_7(\sqrt{2})$ and restrict the valuation on $\mathbb{Q}_7(\sqrt{2})$ given by the theorem to $\mathbb{Q}(\sqrt{2})$. In fact, 2 is a square in $\mathbb{Q}_7$ (left as an exercise: it follows from $3^2 = 9 \equiv 2(7)$). There exists $\alpha \in \mathbb{Z}_7$ such that $\alpha^2 = 2$ and $|\alpha - 3|_7 < 1$ (exercise). Sending $\sqrt{2}$ to α and restricting $|\cdot|_7$ to the image of $\mathbb{Q}(\sqrt{2})$ in $\mathbb{Q}_7$, results in a valuation having the property $\|2-\sqrt{3}\| = |\alpha - 3|_7 < 1$ and $\|2+\sqrt{3}\| = |\alpha+3|_7 = 1$. On the other hand, if we send $\sqrt{2}$ to $-\alpha$ we get a valuation $\|\cdot\|'$ on $\mathbb{Q}(\sqrt{2})$ such that $\|2-\sqrt{3}\|' = 1$ and $\|2+\sqrt{3}\|' < 1$. We have two inequivalent extensions of $|\cdot|_7$ from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{2})$.

Let F be a p-adic field. Note that we can replace a valuation on F by an equivalent valuation without changing the ring of integers $\mathfrak{o}$, or the maximal ideal $\mathfrak{p}$. Let q be the cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$. It is standard to use the valuation $|\cdot|_F$ on F which has the property that $|\varpi|_F = q^{-1}$. This will be called the *normalized valuation* on F. Let p be the characteristic of $\mathfrak{o}/\mathfrak{p}$. If F has characteristic zero, then F is a finite extension of $\mathbb{Q}_p$ and $q = p^f$ for some positive integer f. The normalized valuation is equal to $|N_{F/\mathbb{Q}_p}(\cdot)|_p$.

3. Smooth representations of locally compact totally disconnected groups

Definition. A *topological group* G is a group which is a topological space and has the property that the map $(x,y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous.

A p-adic field F is a topological group with respect to field addition. Given $j \in \mathbb{Z}$, define $\mathfrak{p}^j = \varpi^j \mathfrak{o} = \{\, x \in F \mid |x|_F \leq q^{-j} \,\}$. Note that $\mathfrak{p}^j$ is an open compact subgroup of F. The set $\{\, \mathfrak{p}^j \mid j \in \mathbb{Z} \,\}$ is a fundamental system of open neighbourhoods of zero in F. Furthermore, each quotient $F/\mathfrak{p}^j$ is countable.

Lemma. *(Chapter I, §1 of [S]) Let G be a topological group. The following are equivalent.*

(1) G is a profinite group - that is, G is a projective limit of finite groups.

(2) G is a compact, Hausdorff group in which the family of open normal subgroups forms a fundamental system of open neighbourhoods of the identity.

(3) G is a compact, totally disconnected, Hausdorff group.

Each of the subgroups $\mathfrak{p}^j$ of F is a profinite group. Because F is not compact, F is not a profinite group. However, F is a locally compact, totally disconnected, Hausdorff group.

Let n be a positive integer. Taking the topology on F determined by the valuation $|\cdot|_F$, the set $M_n(F) \simeq F^{n^2}$ of $n \times n$ matrices with entries in F is given the product topology. Our standard example will be $GL_n(F)$ where F is a p-adic field. Since $\det : M_n(F) \to F$ is a polynomial in the matrix entries, det is a continuous map. Because $GL_n(F)$ is the inverse image of $F^\times = F \backslash \{0\}$ (an open subset of F) relative to det, G is an open subset of $M_n(F)$. We give $GL_n(F)$ the topology it inherits as an open subset of $M_n(F)$.

Given $j \in \mathbb{Z}$, let $M_n(\mathfrak{p}^j)$ denote the subring of $M_n(F)$ consisting of matrices all of whose entries belong to $\mathfrak{p}^j$. For $j \in \mathbb{Z}$, $j \geq 1$, let

$$K_j = \{\, g \in GL_n(F) \mid g - 1 \in M_n(\mathfrak{p}^j) \,\}.$$

It is easy to see that K_j is an open compact subgroup of $GL_n(F)$. Set

$$K_0 = GL_n(\mathfrak{o}) = \{\, g \in GL_n(F) \mid g \in M_n(\mathfrak{o}),\ |\det(g)|_F = 1 \,\}$$

K_0 is also an open compact subgroup of $GL_n(F)$. It is a maximal compact subgroup of $GL_n(F)$. There are many other compact open subgroups of $GL_n(F)$ - for example the set of matrices $g = (g_{ij})$ such that $|g_{ii}|_F = 1$, $1 \leq i \leq n$, $g_{ij} \in \mathfrak{o}$ for $i > j$, and $g_{ij} \in \mathfrak{p}$ for $i < j$, is a compact open subgroup. Every open neighbourhood of the identity in $GL_n(F)$

contains a compact open subgroup - in particular, given such a neighbourhood U, if j is sufficiently large, then $K_j \subset U$. The family of compact open subgroups K_j, $j \geq 0$ forms a fundamental system of open neighbourhoods at the identity.

In general, if G is a locally compact, totally disconnected, Hausdorff group, then every open neighbourhood of the identity in G contains a compact open subgroup of G. Such groups are sometimes referred to as *locally profinite* groups. A compact open subgroup K of a locally compact, totally disconnected, Hausdorff group is a profinite group (we can see that K is the projective limit $\lim_{\longleftarrow} K/K'$, where K' ranges over the set of open normal subgroups of K). If $\mathbf{G}$ is a connected reductive linear algebraic group that is defined over F, then we say that the group of F-rational points $G = \mathbf{G}(F)$ is a *reductive p-adic group*. Reductive p-adic groups (and their closed subgroups) are locally compact, totally disconnected, Hausdorff groups. In addition to our family $GL_n(F)$, $n \geq 1$, of basic examples, the special linear groups: $SL_n(F) = \{ x \in M_n(F) \mid \det(x) = 1 \}$ and the symplectic groups $Sp_{2n}(F) = \{ x \in GL_{2n}(F) \mid {}^t x J x = J \}$, where $J = (J_{ij})$ is given by $J_{ij} = 0$ if $j \neq 2n+1-i$ and $J_{i,2n+1-i} = (-1)^{i-1}$, are examples of reductive p-adic groups.

Definition. A representation (π, V) of G consists of a complex vector space V and a homomorphism π from G to the group of invertible linear operators on V. The representation will be denoted simply by π or by V where convenient.

Definition. A *subrepresentation* of (π, V) is an invariant subspace W (W is $\pi(g)$-invariant for all $g \in G$). A *quotient* of (π, V) is the natural representation of G on the quotient space V/W, where W is a subrepresentation of V. A *subquotient* of (π, V) is a quotient of a subrepresentation of π: If $V \subset V_1 \subset V_2$, where V_2 is a subrepresentation of a subrepresentation V_1 of V, then the natural representation of G on V_1/V_2 is a subquotient of V.

Definition. We say that a representation (π, V) of G is *irreducible* if $\{0\}$ and V are the only subrepresentations of π. We say that π is *reducible* if π is not irreducible.

Remark. If (π, V) is a representation of G and $v \in V$, the subspace $\mathrm{Span}(\{ \pi(g)v \mid g \in G \})$ of V is a subrepresentation of π that is nonzero whenever $v \neq 0$. Therefore, if (π, V) is irreducible and $v \in V$ is nonzero, then $V = \mathrm{Span}(\{ \pi(g)v \mid g \in G \})$.

Definition. A representation (π, V) of G is said to be *finitely generated* if there exists a finite subset $\{ v_1, \ldots, v_m \}$ of V such that

$$V = \mathrm{Span}(\{ \pi(g)v_j \; | g \in G, \; 1 \leq j \leq m. \}).$$

Definition. A representation (π, V) of G is said to be *semisimple* if V is a direct sum of irreducible subrepresentations of π.

Definition. If (π_1, V_1) and (π_2, V_2) are representations of G and $A : V_1 \to V_2$ is a linear transformation such that $\pi_2(g)A = A\pi_1(g)$ for all $g \in G$, we say that A *intertwines* π_1 and π_2, or that A is a *G-morphism*. The set of all such linear transformations that intertwine π_1 and π_2 will be denoted by $\mathrm{Hom}_G(\pi_1, \pi_2)$ or $\mathrm{Hom}_G(V_1, V_2)$. The notation $\mathrm{End}_G(\pi)$ or $\mathrm{End}_G(V)$ will be used for $\mathrm{Hom}_G(\pi, \pi) = \mathrm{Hom}_G(V, V)$.

Definition. If (π_1, V_1) and (π_2, V_2) are representations of G, we say that π_1 and π_2 are *equivalent* (or *isomorphic*) if $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains an invertible element. In that case, we write $\pi_1 \simeq \pi_2$.

Definition. If (π, V) is a representation of G, we say that π is *smooth* if $\mathrm{Stab}_G(v) = \{\, g \in G \mid \pi(g)v = v \,\}$ is open for every $v \in V$.

Let (π, V) be a representation of G. If K is a compact open subgroup of G let $V^K = \{\, v \in V \mid \pi(k)v = v \ \forall\, k \in k \,\}$. The representation (π, V) is smooth if and only if each $v \in V$ lies in V^K for some compact open subgroup K of G (here K depends on v). It is easy to see that subrepresentations, quotients and subquotients of smooth representations are smooth. Define $V^\infty = \{\, v \in V \mid \mathrm{Stab}_G(v) \text{ is open } \}$.

Lemma. *If (π, V) is a representation of G, then V^∞ is a smooth subrepresentation of V.*

Proof. It is clear that V^∞ is invariant under scalar multiplication. Suppose that v_1, $v_2 \in V^\infty$. There exists a compact open subgroup K of G such that $K \subset \mathrm{Stab}_G(v_1) \cap \mathrm{Stab}_G(v_2)$. Let $g \in \mathrm{Stab}_G(v_1 + v_2)$. Then gK is an open neighbourhood of g that lies in $\mathrm{Stab}_G(v_1 + v_2)$. Therefore $\mathrm{Stab}_G(v_1 + v_2)$ is open. That is, $v_1 + v_2 \in V^\infty$. It follows that V^∞ is a subspace of V.

Note that if $g \in G$ and $v \in V$, then $\mathrm{Stab}_G(\pi(g)v) = g\,\mathrm{Stab}_G(v)g^{-1}$. This implies that V^∞ is a subrepresentation of V. It is immediate from the definition of V^∞ that this subrepresentation is smooth. □

Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. If $\lambda \in V^*$ and $v \in V$, let $\langle\, \lambda, v \,\rangle$ be the value of the linear functional λ on the vector v. If $\lambda \in V^*$, define $\pi^*(g)\lambda \in V^*$ by

$$\langle\, \pi^*(g)\lambda, v \,\rangle = \langle\, \lambda, \pi(g)^{-1}v \,\rangle, \qquad v \in V.$$

Clearly (π^*, V^*) is a representation of V. Even if π is smooth, the dual representation π^* might not be smooth.

Definition. Let $\widetilde{\pi}$ be the subrepresentation $\widetilde{V} = V^{*,\infty}$ of π^*. This representation is called the *contragredient* (or *smooth dual*) of (π, V).

In general the representation $\widetilde{\widetilde{\pi}}$ is not equivalent to π. However, we will soon see that $\widetilde{\widetilde{\pi}} \simeq \pi$ whenever π is admissible.

Example: Let $\chi : F^\times \to \mathbb{C}^\times$ be a smooth one-dimensional representation of $F^\times$. Define $\pi(g) = \chi(\det g)$, $g \in GL_n(F)$. Then π is a smooth one-dimensional representation of $GL_n(F)$. It is easy to see that $\widetilde{\pi}(g) = \pi^*(g) = \chi(\det g)^{-1}$ for all $g \in GL_n(F)$. Because F is infinite, the derived group of $GL_n(F)$ (the closed subgroup generated by all commutators) is $SL_n(F)$. Any one-dimensional representation of $GL_n(F)$ must be trivial on the commutator subgroup, so must factor through the determinant. It follows that all smooth representations of $GL_n(F)$ have the above form (for some choice of χ).

Example: Let $C_c^\infty(G)$ be the space of functions $f : G \to \mathbb{C}$ such that the support of f is compact and f is locally constant. Given $f \in C_c^\infty(G)$, there exist compact open subgroups K_1 and K_2 of G such that $f(k_1 g k_2) = f(g)$ for all $k_1 \in K_1$, $k_2 \in K_2$ and $g \in G$. Of course, K_1 and K_2 depend on f. Note that $K = K_1 \cap K_2$ is a compact open subgroup of G and f is a finite linear combination of characteristic functions of double cosets in $K \backslash G / K$. For $f \in C_c^\infty(G)$ and $g \in G$, define

$$(\lambda(g)f)(x) = f(g^{-1}x), \quad \text{and} \quad (\rho(g)f)(x) = f(gx), \qquad x \in G.$$

Then $(\lambda, C_c^\infty(G))$ and $(\rho, C_c^\infty(G))$ are equivalent smooth representations of G. The map that takes f to the function $g \mapsto f(g^{-1})$ is an invertible element of $\mathrm{Hom}_G(\lambda, \rho)$.

Example: Let K be a fixed compact open subgroup of G. Let $V = \{\, f \in C_c^\infty(G) \mid f(kg) = f(g) \,\forall\, k \in K,\ g \in G\,\}$. The subrepresentation (π, V) of $(\rho, C_c^\infty(G))$ is a smooth representation of G and is said to be compactly induced from the trivial representation of K.

Lemma. *Let (π, V) be a nonzero smooth representation of G.*

(1) If π is finitely generated, then π has an irreducible quotient.

(2) π has an irreducible subquotient.

Proof. For the first part, consider the set of all proper subrepresentations W of V. This set is nonempty and closed under unions of chains (this uses V finitely generated). By Zorn's lemma, there is a maximal such W. By maximality of W, V/W is irreducible. Part (2) follows from part (1) since if v is a nonzero vector in V, part (1) says that the finitely generated subrepresentation $\mathrm{Span}(\{\,\pi(g)v \mid g \in G\,\})$ of π has an irreducible quotient. This irreducible quotient is an irreducible subquotient of V. $\square$

Lemma. *The following are equivalent:*

(1) (π, V) *is semisimple*

(2) For every invariant subspace $W \subset V$*, there exists an invariant subspace* $W^{\perp}$ *such that* $W \oplus W^{\perp} = V$*.*

Proof. (1) $\Longrightarrow$ (2) Consider the partially ordered set of subrepresentations $U \subset V$ such that $U \cap W = \{0\}$. This set is nonempty and closed under unions of chains, so Zorn's lemma implies existence of a maximal such U. Suppose that $W \oplus U \neq V$. Then since (π, V) is semisimple, there exists some irreducible submodule U' such that $U' \not\subset W \oplus U$. By irreducibility of U', $U' \cap (W \oplus U) = \{0\}$. This contradicts maximality of U.

(3) $\Longrightarrow$ (1) Consider the partially ordered set of direct sums of families of irreducible subrepresentations: $\sum_{\alpha} W_{\alpha} = \oplus_{\alpha} W_{\alpha}$. Zorn's lemma applies. Let $W = \oplus_{\alpha} W_{\alpha}$ be the direct sum for a maximal family. By (2), there exists a subrepresentation U such that $V = W \oplus U$. If $U \neq \{0\}$, by the previous lemma, there exists an irreducible subquotient: $U \supset U_1 \supset U_2$ such that U_1/U_2 is irreducible. By (2), $W \oplus U_2$ has a complement U_3: $V = W \oplus U_2 \oplus U_3$. Now

$$U_3 \simeq V/(W \oplus U_2) = (W \oplus U)/(W \oplus U_2) \simeq U/U_2 \supset U_1/U_2.$$

Identifying U_1/U_2 with an irreducible subrepresentation U_4 of U_3, we have $W \oplus U_4$ contradicting maximality of the family W_{α}.

$\square$

Remark. As we will see later, when G is noncompact, there are finitely generated smooth representations of G which are not semisimple.

Lemma. *Let* G *be as above. Assume in addition that* G *is compact. Then*

(1) Every smooth representation of G *is semisimple.*

(2) Given an irreducible smooth representation (π, V) *of* G*, there exists an open normal subgroup* N *of* G *such that* $\pi(n) = id_V$ *for all* $n \in N$*. (Note:* G/N *is finite since it is discrete and compact).*

Proof. Let $v \in V$. As $G/\mathrm{Stab}_G(v)$ is finite, the set $\{\,\pi(g)v \mid g \in G\,\}$. is finite. Thus $W = \mathrm{Span}(\{\,\pi(g)v \mid g \in G\,\})$ is finite dimensional. Set

$$N = \cap_{w \in G\cdot v} \mathrm{Stab}_G(w) = \cap_{g \in G/N}\, g\, \mathrm{Stab}_G(v) g^{-1}.$$

Note that N is an open normal subgroup of G and G/N is finite. The restriction of π to the subrepresentation W factors through G/N: $\pi(n)\,|\,W = id_W$, $n \in N$. Since representations of finite groups are semisimple, it follows that W is the direct sum of irreducible submodules of V.

We have shown that every vector in v belongs to a finite-dimensional semisimple subrepresentation of V. Let W' be a maximal element among the semisimple subrepresentations of V. If $W' \neq V$, let v be a vector in V that is not in W'. Let W'' be a finite-dimensional semisimple subrepresentation of V such that $v \in W''$. There is at least one irreducible subrepresentation $\dot{W}$ of W'' such that $\dot{W} \cap W' = \{0\}$. Because W'' is a proper subrepresentation of the semisimple subrepresentation $W' + \dot{W}$ of V, this contradicts maximality of W'. We conclude that $W' = V$ and V is semisimple.

Part (2) follows immediately, as whenever π is irreducible, $\mathrm{Span}(\{\,\pi(g)v \mid g \in G\,\}) = V$ for any nonzero vector v. □

The above result is used as follows. Given a smooth representation (π, V) of G, the restriction of π to a compact open subgroup K is a smooth representation of K, which by above is semisimple - that is, V is the direct sum of K-invariant subspaces, subspaces that are irreducible representations of K.

If K and K' are compact open subgroups of G and G/K is countable, then we can show that G/K' is countable as follows. The intersection $K \cap K'$ is a compact open subgroup of K, so $K/(K \cap K')$ is finite. The canonical surjective map from $G/(K \cap K')$ to G/K has a finite number of fibres. Thus $G/(K \cap K')$ is countable. This implies G/K' is countable.

From now on, we assume the following:

Hypothesis. G/K is countable for some (equivalently, any) compact open subgroup K of G.

If $G = GL_n(F)$, we can use the Cartan decomposition to show that G/K_0 is countable. Let ϖ be a prime element in F. Let D_n be the set of all diagonal matrices in $GL_n(F)$ of the form $\mathrm{diag}(\varpi^{m_1}, \varpi^{m_2}, \ldots, \varpi^{m_n})$ where the integers m_j satisfy $m_1 \geq m_2 \geq \cdots \geq m_n$. The Cartan decomposition says that G is the disjoint union of the K_0 double cosets of the form $K_0 d K_0$, $d \in D_n$. Note that $k_1 g K_0 = k_2 g K_0$ if and only if $k_2^{-1} k_1 \in g K_0 g^{-1} \cap K_0$. Thus the number of left K_0 cosets of K_0 in $K_0 g K_0$ is equal to $[K_0 : K_0 \cap g K_0 g^{-1}]$, which is finite. Because each K_0 double coset consists of finitely many left K_0 cosets and D_n is countable, we see that G/K_0 is countable.

Proof of the Cartan decomposition for $GL_2(F)$: Let $g \in G$. Note that permuting the rows or columns of g amounts to multiplying g on the right or the left by a permutation matrix that lies in K_0. Therefore, there exist k_1 and $k_2 \in K_0$ such that $k_1 g k_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \neq 0$ and $|a| \geq \max\{|b|, |c|, |d|\}$. Note that the matrix $k_3 = \begin{pmatrix} 1 & 0 \\ -a^{-1}c & 1 \end{pmatrix}$ belongs to K_0 and

$$k_3 k_1 g k_2 = \begin{pmatrix} a & b \\ 0 & d - a^{-1}bc \end{pmatrix}.$$

Next, let $k_4 = \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix}$. Then $k_4 \in K_0$,

$$k_3 k_1 g k_2 k_4 = \begin{pmatrix} a & 0 \\ 0 & d - a^{-1}bc \end{pmatrix}, \quad \text{and} \quad |a| \geq |d - a^{-1}bc|.$$

Any nonzero element of F can be expressed as the product of ϖ^ℓ ($\ell \in \mathbb{Z}$) and an element of $\mathfrak{o}^\times$. Hence there exists a diagonal matrix d_0 in K_0 such that $d_0 k_3 k_1 g k_2 k_4 \in D_2$. Let $d_{m,n}$, $d_{k,\ell} \in D_2$ be such that the diagonal entries are ϖ^m and ϖ^n, with $m \geq n$, and ϖ^ℓ and ϖ^ℓ, respectively, with $m \geq n$ and $k \geq \ell$. Suppose that $d_{m,n}$ and $d_{k,\ell}$ belong to the same K_0 double coset. From $|\det(d_{m,n})| = q - m + n = |\det(d_{k,\ell})|$, we find that $m + n = k + \ell$. A simple calculation shows

$$[K_0 : K_0 \cap d_{m,n} K_0 d_{m,n}^{-1}] = (q+1)q^{n-m-1},$$

Thus $n - m - 1 = \ell - k - 1$. We conclude that $n = \ell$ and $m = k$.

Lemma. *If (π, V) is an irreducible smooth representation of G, then* $\dim V$ *is (at most) countable.*

Proof. Observe that smoothness of π implies that if $v \in V$, then $\{\, \pi(g)v \mid g \in G \,\} \simeq G/K$ for some open subgroup K of G. Thus the set $\{\, \pi(g)v \mid g \in G \,\}$ is countable. We have already remarked that irreducibility of π guarantees that if $v \neq 0$, then $V = \mathrm{Span}(\{\, \pi(g)v \mid g \in G \,\})$. It follows that $\dim(V)$, is countable. □

Schur's Lemma. *Let (π, V) be an irreducible smooth representation of G. Then* $\mathrm{End}_G(V) = \mathbb{C}$. *That is, every operator which intertwines π with itself is a scalar multiple of the identity.*

Proof. To begin, we show that the dimension of $\mathrm{End}_G(V)$ is at most countable. Let $v \in V$, $v \neq 0$. Since π is irreducible, $V = \mathrm{Span}(\pi(g)v \mid g \in G\,\})$. Therefore any $A \in \mathrm{End}_G(V)$ is determined by $A(v)$. This implies that the map $\mathrm{End}_G(V) \to V$ given by $A \mapsto A(v)$ is one-to-one. According to the previous lemma, V is countable. Thus $\mathrm{End}_G(V)$ is countable.

The null space and range of any element of $\mathrm{End}_G(V)$ are subrepresentations of V. If $A \in \mathrm{End}_G(V)$ is nonzero, then the range of A is nonzero, so irreducibility of π implies that A is onto. The null space of A is then a proper subrepresentation of V, which by irreducibility of π, must be zero. Hence any nonzero element of $\mathrm{End}_G(V)$ is invertible. That is, $\mathrm{End}_G(V)$ is a division ring.

Let $A \in \mathrm{End}_G(V)$. The set $\mathbb{C}(A)$ is commutative and all nonzero elements are invertible. Thus $\mathbb{C}(A)$ is a field. If A is transcendental over $\mathbb{C}$, then the set $\{\, 1/(A-c) \mid c \in \mathbb{C}\,\}$ is linearly independent and uncountable. Thus $\dim_{\mathbb{C}}(\mathbb{C}(A))$ is uncountable. The contradicts countability of $\dim(\mathrm{End}_G(V))$. We conclude that $\mathbb{C}(A)$ must be an algebraic extension of $\mathbb{C}$. Hence $\mathbb{C}(A) = \mathbb{C}$. $\square$

Remark. If G is compact and (π, V) is smooth, then $\mathrm{End}_G(\pi)$ is one-dimensional if and only if π is irreducible. However, if G is not compact, it is possible to have $\dim(\mathrm{End}_G(\pi)) = 1$ for π reducible.

Corollary. *Let Z be the centre of G. If (π, V) is an irreducible smooth representation of G, there exists a smooth one-dimensional representation χ_π of Z such that $\pi(z) = \chi_\pi(z) I_V$, $z \in Z$. (Here, I_V is the identity operator on V).*

Proof. If $z \in Z$, then $\pi(gz) = \pi(zg)$ for all $g \in G$. This says that $\pi(z) \in \mathrm{End}_G(\pi)$. By Schur's Lemma, there exists $\chi_\pi(z) \in \mathbb{C}^\times$ such that $\pi(z) = \chi_\pi(z) I_V$. Because π is a representation, $\chi_\pi(z_1 z_2) = \chi_\pi(z_1)\chi_\pi(z_2)$ for z_1, $z_2 \in Z$. Finally, let $v \in V$ be nonzero. There exist a compact open subgroup K of G such that $v \in V^K$. From $v = \pi(z)v = \chi_\pi(z)v$, $z \in Z \cap K$, we see that $\chi_\pi(z) = 1$ for all $z \in Z \cap K$. This says that χ_π is a smooth representation of Z. $\square$

Remark: If (π, V) is a smooth irreducible representation of G, the the quasicharacter (that is, the one-dimensional smooth representation) χ_π of Z is called the *central quasicharacter* (or *central character*) of π.

Corollary. *If G is abelian, then every irreducible smooth representation of G is one-dimensional.*

Definition. A smooth representation (π, V) is *admissible* if for every compact open subgroup K of G, the space $V^K = \{\, v \in V \mid \pi(k)v = v \ \forall k \in K\,\}$ is finite dimensional.

Lemma. *Let K be a compact open subgroup of G. A smooth representation (π, V) of G is admissible if and only if every irreducible representation of K occurs finitely many times in V.*

Proof. If an irreducible representation τ of K occurs with infinite multiplicity in $\pi \,|\, K$, let N be as in the above lemma. Then $\dim_{\mathbb{C}}(V^N) = \infty$ and π is not admissible.

Now suppose that the multiplicity of every irreducible representation of K in $\pi \,|\, K$ is finite. Let K' be a compact open subgroup of G. Let $K'' = \cap_{g \in K/(K\cap K')} g(K' \cap K)g^{-1}$. Then K'' is a compact open subgroup and $V^{K''} \supset V^{K'}$. Thus it suffices to show $V^{K''}$ is finite dimensional. Note that K'' is a normal subgroup of K. An irreducible representation τ of K has nonzero K''-fixed vectors if and only if $\tau(k)$ is the identity for all k in K''. Thus the dimension of $V^{K''}$ is equal to the sum as τ varies over irreducible representations of K which factor through K/K'' of the product of the multiplicity of τ in $\pi \,|\, K$ times the degree of τ. (The degree of τ is the dimension of the space of τ). By assumption, the multiplicities are finite. As K/K'' is finite, there are finitely many τ which factor through K/K''. Thus $\dim(V^{K''}) < \infty$. □

Example. Let (π, V) be the smooth representation of $GL_n(F)$ compactly induced from the trivial representation of the open compact subgroup K_0 of $GL_n(F)$ (see the third example on page 14). If $g \in G$, let f_g be the characteristic function of the double coset $K_0 g K_0$. Let S be a set of representatives for the distinct such double cosets. Then $\{\, f_g \mid g \in S \,\}$ is linearly independent and $V^{K_0} = \mathrm{Span}(\{\, f_g \mid g \in S \,\})$. It is easy to see that S is infinite. (Of course, we know from the Cartan decomposition that we could take $S = D_n$.) Therefore $\dim(V^{K_0})$ is infinite and π is not admissible. Clearly the trivial representation of K_0 is irreducible and admissible. This example shows that a representation that is compactly induced from an irreducible admissible representation of a compact open subgroup of G might not be admissible.

Lemma. *If (π, V) is admissible, then $(\widetilde{\pi}, \widetilde{V})$ is admissible. Furthermore, $\widetilde{\widetilde{\pi}} \simeq \pi$.*

Proof. Let K be a compact open subgroup of G. Let W be a K-invariant complement for V^K: $V = V^K \oplus W$. Let $v^* \in \widetilde{V}^K$. Then $v^* \,|_W : W \to \mathbb{C}$ intertwines the representation of K on W and the trivial representation of K on $\mathbb{C}$. But W is direct sum of nontrivial representations of K. If $\mathrm{Hom}_K(W, \mathbb{C})$ were nonzero, then the trivial representation of K would occur as a subrepresentation of W. Thus $v^* \,|_W = 0$. That is, $v^* \in (V^K)^*$, which is finite dimensional, by admissibility of π. We have shown $\widetilde{V}^K \subset (V^K)^*$. Hence $\dim_{\mathbb{C}}(\widetilde{V}^K) < \infty$.

Given $\lambda \in (V^K)^*$, extend λ to an element of V^* by setting λ equal to zero on W. This extension belongs to $\widetilde{V}^K$. Hence $\widetilde{V}^K = (V^K)^*$. This implies that $V^K \simeq \widetilde{\widetilde{V}}^K$. Let K

vary to get $V \simeq \widetilde{\widetilde{V}}$. It follows that the natural map $V \to \widetilde{\widetilde{V}}$, which is easily seen to lie in $\mathrm{Hom}_G(\pi, \widetilde{\widetilde{\pi}})$, is invertible. That is, $\widetilde{\widetilde{\pi}} \simeq \pi$. $\square$

4. Haar measure, convolution, and characters of admissible representations

Given a complex vector space V, let $C_c^\infty(G,V)$ denote the set of V valued functions on G which are compactly supported and locally constant.

Lemma. *Let $f \in C_c^\infty(G,V)$. Then there exists a compact open subgroup K such that f is right K-invariant.*

Proof. Choose a compact subset C of G such that $f(x)=0$ for all $x \notin C$. As f is locally constant, for each $x \in C$, there exists a compact open subgroup K_x such that $f \,|\, xK_x$ is constant. Then there exist $x_1,\dots,x_n \in C$ such that $C \subset \cup_j x_j K_{x_j}$. Let $K = \cap_j K_{x_j}$. □

Corollary. *$C_c^\infty(G,V) = \cup_K C_c(G/K,V)$, as K ranges over all compact open subgroups of G.*

Corollary. *$C_c^\infty(G,V) = C_c^\infty(G)\otimes V$.*

Proof. If $f_i \in C_c^\infty(G)$ and $v_i \in V$, $1 \le i \le m$, the element of $C_c^\infty(G,V)$ that corresponds to the element $\sum_{i=1}^m f_i \otimes v_i$ of $C_c^\infty(G,V)$ is the function $g \mapsto \sum_{i=1}^m f(g_i)v_i$. □

Definition. A positive Borel measure m on G which satisfies $m(gS)=m(S)$ for every measurable subset S of G and $g \in G$ (that is, m is invariant under left translation) is called *left Haar measure* on G.

A key fact is that left Haar measure exists and is unique up to positive scalar multiples. Basic properties of Haar measure are that nonempty open sets have positive measure and compact sets have finite measure. For groups of the type we are considering, we can define a left Haar measure easily. Fix a compact open subgroup K_0. Set $m(K_0)=1$. Suppose K is another compact open subgroup. Then, by left invariance of m, we have $m(K_0\cap K) = [K_0 : K_0\cap K]^{-1}$. Hence $m(K) = [K : K\cap K_0]\, m(K_0\cap K) = [K : K\cap K_0][K_0 : K\cap K_0]^{-1}$. Setting $m(gK)=m(K)$, for all $g\in G$ and open compact subgroups K then determines m.

Definition. Let $f \in C_c^\infty(G,V)$. Choose a compact open subgroup K such that f is right K-invariant: $f \in C_c(G/K,V)$. Then $\int_G f(g)dg := \sum_{x\in G/K} f(x)\, m(K)$ (this is independent of the choice of K).

Example. For Haar measure dx on F (it is both a left and right Haar measure, as F is abelian), we normalize so that $\mathfrak{o}$ has volume one. Then $\mathfrak{p} = \varpi\mathfrak{o}$ has volume $[\mathfrak{o}:\mathfrak{p}]^{-1} = q^{-1}$. If $j\in\mathbb{Z}$, then the volume of $\mathfrak{p}^j$ is q^{-j}. Recall that we have normalized the valuation on F

so that $|\varpi|_F = q^{-1}$. Thus dx has the property that $d(ax) = |a|_F\, dx$, $a \in F^\times$. As $F^\times$ is an open subset of F, dx restricts to a measure on $F^\times$, but is not a Haar measure on $F^\times$. The measure $|x|_F^{-1} dx$ is a Haar measure on $F^\times$.

Example. For a Haar measure on the abelian group $M_n(F)$ we take the product measure $dx = \prod_{i,j} dx_{ij}$ where dx_{ij} is the Haar measure on F. To get a Haar measure on $GL_n(F)$, we take $|\det(x)|_F^{-n}\, dx$, where dx is the restriction of the measure on $M_n(F)$ to the open subset $GL_n(F)$. This is both a left and a right Haar measure (exercise).

Exercise. Let

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in F^\times, b \in F \right\}.$$

Find a left Haar measure and a right Haar measure.

Let $d_\ell x$ be a left Haar measure on G. Then, for a fixed $y \in G$, $d_\ell(xy)$ is another left Haar measure, so must be of the form $\Delta_G(y)\, d_\ell x$ for some $\Delta_G(y) \in \mathbb{R}_{>0}$. In fact, $\Delta_G : G \to \mathbb{R}^{>0}$ is a continuous homomorphism. It is a simple matter to check that $\Delta_G(x)^{-1}\, d_\ell(x)$ is a right Haar measure on G. The measure $d_\ell(x^{-1})$ is also a right Haar measure.

Lemma. *$d_\ell(x^{-1}) = \Delta_G(x^{-1})\, d_\ell x$.*

Proof. Let K be a compact open subgroup. Since the measure of Kg is equal to the measure of K for all $g \in K$, it follows that $\Delta_G \,|\, K \equiv 1$. Because both $d_\ell(x^{-1})$ and $\Delta_G(x^{-1})\, d_\ell(x)$ are right Haar measures, there exists a positive real number c such that $d_\ell(x^{-1}) = c\, \Delta_G(x^{-1})\, d_\ell(x)$. Comparing the integrals of the characteristic function of K with respect to each of these measures, we find that $c = 1$. $\square$

Definition. The group G is said to be *unimodular* if $\Delta_G \equiv 1$. (That is, every left Haar measure is also a right Haar measure.)

Reductive groups (e.g. $GL_n(F)$, $Sp_{2n}(F)$) are always unimodular, but they have closed that are not unimodular: for example, the group B mentioned above is a subgroup of $GL_2(F)$ which is not unimodular.

For the time being, we assume that G is unimodular. Fix a choice dx of Haar measure on G. Given f_1, $f_2 \in C_c^\infty(G)$, define a function $f_1 * f_2 : G \to \mathbb{C}$ by

$$(f_1 * f_2)(g) = \int_G f_1(x)\, f_2(x^{-1}g)\, dx, \qquad g \in G.$$

The operation $*$ is called *convolution*.

If G_1 and G_2 are locally profinite groups, then an element $\sum_i f_i \otimes \dot{f}_i$ of $C_c^\infty(G_1) \otimes C_c^\infty(G_2)$ gives a function on $G_1 \times G_2$ as follows:

$$(g_1, g_2) \mapsto \sum_i f_i(g_1)\, \dot{f}_i(g_2).$$

This defines an isomorphism from $C_c^\infty(G_1) \otimes C_c^\infty(G_2)$ and $C_c^\infty(G_1 \times G_2)$. Here, we will apply this remark in the case $G_1 = G_2 = G$.

Lemma. *If $f_1,\ f_2 \in C_c^\infty(G)$, then $f_1 * f_2 \in C_c^\infty(G)$.*

Proof. The function $(x, g) \mapsto f_1(x)\, f_2(x^{-1}g)$ is in $C_c^\infty(G \times G)$, so it is a linear combination of functions of the form $(x, g) \mapsto h_1(x)\, h_2(g)$, $h_1,\ h_2 \in C_c^\infty(G)$. Clearly, $g \mapsto \int_G h_1(x)\, h_2(g)\, dx = h_2(g)$ belongs to $C_c^\infty(G)$. It follows that $g \mapsto \int_G f_1(x)\, f_2(x^{-1}g)\, dx$ is a linear combination of functions in $C_c^\infty(G)$, hence belongs to $C_c^\infty(G)$. $\square$

Lemma. *If $f_j \in C_c^\infty(G)$, $1 \le j \le 3$, then $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$.*

Proof.

$$\begin{aligned} f_1 * (f_2 * f_3)(g) &= \int_G f_1(x) \int_G f_2(y) f_3(y^{-1}x^{-1}g)\, dy\, dx = \int_G f_1(x) \int_G f_2(x^{-1}y)\, f_3(y^{-1}g)\, dy\, dx \\ &= \int_G \left(\int_G f_1(x)\, f_2(x^{-1}y)\, dx \right) f_3(y^{-1}g)\, dy = \int_G (f_1 * f_2)(y)\, f_2(y^{-1}g)\, dy. \end{aligned}$$

We see that $C_c^\infty(G)$ (with the product given by convolution) is an associative $\mathbb{C}$-algebra. That is, $C_c^\infty(G)$ is a ring and a complex vector space, such that $c(f_1 * f_2) = (cf_1) * f_2 = f_1 * (cf_2)$ for all $c \in \mathbb{C}$, and $f_1,\ f_2 \in C_c^\infty(G)$. This algebra is noncommutative when G is noncommutative and has no unit element in general.

If K is a compact open subgroup K of G, let $m(K)$ be the measure of K relative to the chosen Haar measure on G. Define e_K to be the function vanishing off K and taking the value $m(K)^{-1}$ on K. Clearly, $e_K \in C_c^\infty(G)$.

Lemma. *Let K be a compact open subgroup of G. Then*

(1) $e_K * e_K = e_K$. (That is, e_K is an idempotent.)

(2) If $f \in C_c^\infty(G)$, then $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $k \in K$ and $g \in G$.

(3) If $f \in C_c^\infty(G)$, then $f * e_K = f$ if and only if $f(g) = f(gk)$ for all $k \in K$ and $g \in G$.

(4) The set $e_K * C_c^\infty(G) * e_K$ is a subalgebra of $C_c^\infty(G)$, with unit element e_K.

Proof. For part (1), note that

$$(e_K * e_K)(g) = \int_G e_K(x)\, e_K(x^{-1}g)\, dx = m(K)^{-1} \int_K e_K(x^{-1}g)\, dk$$

$$= \begin{cases} m(K)^{-2} \int_K dx = m(K)^{-1}, & \text{if } g \in K \\ 0, & \text{if } g \notin K. \end{cases} = e_K(g).$$

Next, note that if f is left K-invariant, we have

$$(e_K * f)(g) = m(K)^{-1} \int_K f(g)\, dx = m(K)^{-1} f(g) m(K) = f(g), \qquad g \in G.$$

If $g \in G$ and $k \in K$, then, changing variables and using the fact that e_K is left K-invariant, we see that

$$\begin{aligned}(e_K * f)(kg) &= \int_G e_K(x)\, f(x^{-1}kg)\, dx = \int_G e_K(kx)\, f(x^{-1}g)\, dx \\ &= \int_G e_K(x)\, f(x^{-1}g)\, dx = (e_K * f)(g).\end{aligned}$$

Thus $e_K * f$ is left K-invariant for all $f \in C_c^\infty(G)$. From this it follows that $e_K * f = f$ implies that f is left K-invariant.

Part (3) is proved similarly, and part (4) then follows immediately. □

Definition. Let (π, V) be a smooth representation of G. Given $f \in C_c^\infty(G)$, the operator $\pi(f) : V \to V$ is defined by:

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg, \qquad v \in V.$$

Here dg denotes left Haar measure on G. (Note that $g \mapsto f(g)\pi(g)v$ is in $C_c^\infty(G, V)$.)

Let $f_1,\ f_2 \in C_c^\infty(G)$. Then

$$\begin{aligned}\pi(f_1)\pi(f_2)v &= \int_G f_1(h)\pi(h) \left(\int_G f_2(g)\pi(g)v\, dg \right) dh \\ &= \int_G \left(\int_G f_1(h) f_2(h^{-1}g)\, dh \right) \pi(g)v\, dg \\ &= \pi(f_1 * f_2)v, \qquad v \in V,\end{aligned}$$

Thus the map $f \mapsto \pi(f)$ is an algebra homomorphism from $C_c^\infty(G)$ to $\mathrm{End}_{\mathbb{C}}(V)$. That is, V is a $C_c^\infty(G)$-module.

Lemma. *$\pi(e_K)$ is a projection of V onto V^K. The kernel of $\pi(e_K)$ is $V(K) := \mathrm{Span}\{\, \pi(k)v - v \mid k \in K,\, v \in V \,\}$.*

Proof. Let τ be an irreducible representation of K. If $v \in V$ is nonzero and such that $\pi(k)v = \tau(k)v$ for all $k \in K$, then $\pi(e_K)v = m(K)^{-1} \int_K \tau(k)v\, dk$. Let N be an open normal subgroup of K such that $\tau(n) = Id$ for all $n \in N$. Then $\pi(e_K)v = m(K)^{-1}m(N) \sum_{x \in K/N} \tau(x)v$, which, since τ factors to an irreducible representation of the finite group K/N, equals zero if τ is nontrivial, and equals $m(K)^{-1}m(N)[K : N]v = v$ if τ is trivial (that is, if $v \in V^K$). Verification that $V(K)$ is the direct sum of all nontrivial irreducible smooth K-subrepresentations of V is left to the reader. $\square$

Because π is smooth, each $v \in V$ satisfies $\pi(e_K)v = v$ for some compact open subgroup K (pick K such that $v \in V^K$), so whenever V is nonzero, V is a nondegenerate $C_c^\infty(G)$-module. (That is, given a nonzero $v \in V$, there exists $f \in C_c^\infty(G)$ such that $\pi(f)v \neq 0$.)

Lemma. *Let (π, V) and (π', V') be smooth representations of G.*

(1) A subspace W of V is a subrepresentation if and only if W is $\pi(f)$-invariant for all $f \in C_c^\infty(G)$.

(2) A linear map $A : V \to V'$ belongs to $\mathrm{Hom}_G(\pi, \pi') = \mathrm{Hom}_G(V, V')$ if and only if $\pi'(f)A = A\pi(f)$ for all $f \in C_c^\infty(G)$.

Proof. If $v \in V$, there exists a compact open subgroup K of G such that $v \in V^K$. Let $f_{g,K} = \lambda(g^{-1})e_K$ (that is, $f_{g,K}$ is equal to $m(K)^{-1}$ on gK and zero elsewhere). Note that $\pi(f)v = \pi(g)v$. It is also useful to note that, given $v \in V$ and $f \in C_c^\infty(G)$, if a compact open subgroup K of G is sufficiently small, then there exist $g_1, \ldots, g_m \in G$ such that

$$\pi(f)v = m(K) \sum_{i=1}^{m} f(g_i)\, \pi(g_i)v.$$

The details of the proof of the lemma are left to the reader. $\square$

Remark. Any nondegenerate $C_c^\infty(G)$-module is associated to a unique smooth representation of G. The category of nondegenerate $C_c^\infty(G)$-modules is the same as the category of smooth representations of G with intertwining maps.

Lemma. *A smooth representation (π, V) of G is admissible if and only if $\pi(f)$ has finite rank for every $f \in C_c^\infty(G)$.*

Proof. ($\Leftarrow$) Let K be a compact open subgroup of G. By above, $\pi(e_K)$ is a projection onto V^K. Therefore $\dim_{\mathbb{C}}(V^K)$, being the rank of $\pi(e_K)$, is finite. Hence π is admissible.

($\Rightarrow$) Let $f \in C_c^\infty(G)$. Choose a compact open subgroup K such that f is right K-invariant. Then $\pi(f) = \sum_{x \in G/K} f(x)\, m(K)\, \pi(x)\pi(e_K)$ (since f is compactly supported, $f(x) = 0$ for all but finitely many $x \in G/K$). By admissibility of π, $\dim_{\mathbb{C}}(V^K)$, that is, the rank of $\pi(e_K)$, is finite. It follows that the rank of $\pi(f)$ is finite. $\square$

Lemma. *Let (π_j, V_j) be smooth representations of G. Suppose that $V_1 \to V_2 \to V_3$ is an exact sequence of G-morphisms, then for every compact open subgroup K, the sequence $V_1^K \to V_2^K \to V_3^K$ is also exact.*

Proof. Suppose $v \in V_2^K$ has image 0 in V_3. There exists $v_1 \in V_1$ having image v in V_2. The vector $\pi_1(e_K)v_1$ belongs to V_1^K and has image $\pi_2(e_K)v = v$. $\square$

Proposition. *If (π_j, V_j) are smooth representations of G, and the sequence*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is exact, then π_2 is admissible if and only if π_1 and π_3 are admissible.

Proof. It follows from the preceding lemma that if K is a compact open subgroup of G, then

$$0 \to V_1^K \to V_2^K \to V_3^K \to 0$$

is exact.

Suppose that π_2 is admissible. Let K be a compact open subgroup of G. From exactness of the above sequence, we see that V_1^K is isomorphic to a subspace of V_2^K, so $\dim(V_1^K) \le \dim(V_2^K) < \infty$. Using exactness again, we see that the image of V_2^K in V_3 is equal to V_3^K. That is, V_3^K is isomorphic to a quotient of V_2^K. It follows that $\dim(V_3^K) \le \dim(V_2^K) < \infty$. It now follows that π_1 and π_3 are admissible.

Now suppose that π_1 and π_3 are admissible. The image of V_2^K is V_3^K, which is finite-dimensional. The kernel of the map from V_2^K to V_3^K is finite-dimensional, (since it's isomorphic to V_1^K). It follows that $\dim(V_2^K) = \dim(V_1^K) + \dim(V_3^K) < \infty$. $\square$

Let $\mathbb{G}$ be a finite group and let (π, V) be a finite-dimensional representation of $\mathbb{G}$ Recall that the character χ_π of π is the function

$$\chi_\pi(g) = \operatorname{trace} \pi(g), \qquad g \in \mathbb{G}.$$

The character χ_π is a class function: $\chi_\pi(hgh^{-1}) = \chi_\pi(g)$ for all g and $h \in \mathbb{G}$. If π and π' are equivalent representations, then $\chi_\pi = \chi_{\pi'}$. If $\pi_1, \dots, \pi_m$ are pairwise inequivalent irreducible representations of $\mathbb{G}$, then the functions χ_{π_j}, $1 \le j \le m$, are linearly independent.

We may define a linear functional on the space $C(\mathbb{G})$ of complex-valued functions on $\mathbb{G}$ as follows:

$$\Theta_\pi(f) = \sum_{g \in \mathbb{G}} f(g)\, \chi_\pi(g) = \operatorname{trace} \pi(f), \qquad f \in C(\mathbb{G}),$$

where $\pi(f)v = \sum_{g \in \mathbb{G}} f(g)\, \pi(g)\, v$, $v \in V$.

If G is a nonabelian locally profinite group many irreducible smooth representations of G are infinite-dimensional, so we cannot evaluate the trace of the invertible operator $\pi(g)$, $g \in G$. However, if π is an admissible representation, we can define a linear functional on $C_c^\infty(G)$ that is analogous in some ways to the character of a finite-dimensional representation of a finite group.

Given $f \in C_c^\infty(G)$ and $g \in G$, set $f^g(x) = f(gxg^{-1})$, $x \in G$. Clearly, $f^g \in C_c^\infty(G)$.

Definition. A *distribution* on G is defined to be a linear functional on $C_c^\infty(G)$. A distribution D on G is said to be *G-invariant* (or *invariant*) if $D(f^g) = D(f)$ for all $g \in G$.

Definition. Let (π, V) be an admissible representation of G. The *character* of π is the distribution Θ_π on G defined by $\Theta_\pi(f) = \operatorname{trace}(\pi(f))$, $f \in C_c^\infty(G)$.

Even though characters of admissible representations of reductive p-adic groups are not defined the same way as characters of representations of finite groups, they do have some properties that are analogous to properties of characters of representations of finite groups.

It is easy to see that Θ_π is an invariant distribution on G. If π and π' are equivalent (admissible) representations of G, then $\Theta_\pi = \Theta_{\pi'}$. As shown in Section 12 of these notes, if $\pi_1, \dots, \pi_m$ are pairwise inequivalent irreducible admissible representations of G, then the distributions $\Theta_{\pi_1}, \dots, \Theta_{\pi_m}$ are linearly independent.

In [HC'], Harish-Chandra proved that if G is a reductive p-adic group and F has characteristic zero, then the character Θ_π of an irreducible admissible representation π of G is realized by integration against a function on G: there exists a function $\Theta_\pi : G \to \mathbb{C}$ such that $\Theta_\pi(f) = \int_G f(g)\, \Theta_\pi(g)\, dg$ for all $f \in C_c^\infty(G)$. This function is a class function: $\Theta_\pi(gxg^{-1}) = \Theta_\pi(x)$ for all x and $g \in G$. (We remark the function Θ_π is usually viewed as a function on the regular set in G. Since the regular set is an open dense subset of G, we may consider Θ_π as a function on G by setting $\Theta_\pi(g) = 0$ for all elements g in the complement of the regular set - this complement is a set of measure zero.)

Part of the Langlands' program involves parametrizing the conjectural L-packets of admissible representations. Each L-packet should consist of finitely many (equivalence classes of) irreducible admissible representations, say $\pi_1, \ldots, \pi_m$ of G One of the key properties that such an L-packet is expected to satisfy is that a specific linear combination $\sum_{i=1}^{m} c_i \, \Theta_{\pi_i}$ is a stable distribution.

5. Induced representations - general results

The results of this section are valid for a locally compact totally disconnected, Hausdorff topological group G. In particular, we do not require that G is a reductive p-adic group. We remark that if H is a closed subgroup of G, then H is a topological group satisfying the same conditions as G - that is, G is a locally compact, totally disconnected, Hausdorff group.

Let (σ, W) be a smooth representation of a closed subgroup H of G. The representation of G induced from σ acts by right translation on the space $\mathrm{Ind}_H^G(W)$ of functions $f : G \to W$ such that $f(hg) = \sigma(h)\, f(g)$, for all $h \in H$ and $g \in G$, and such that f is right K-invariant for some compact open subgroup K (here, K depends on f). The representation and will be denoted by $\mathrm{Ind}_H^G(\sigma)$ Note that $\mathrm{Ind}_H^G(\sigma)$ is smooth by definition. If we restrict to the subspace c-$\mathrm{Ind}_H^G(W)$ of functions $f \in \mathrm{Ind}_H^G(\sigma)$ that have the additional property that the support of f is compact modulo H (the image of the support of f in the topological space $H\backslash G$ is compact), this subrepresentation is denoted by c-$\mathrm{Ind}_H^G(\sigma)$. This is called compact induction.

Note that the map $f \mapsto f(1)$ from $\mathrm{Ind}_H^G(W)$ to W belongs to $\mathrm{Hom}_H(\mathrm{Ind}_H^G(\sigma), \sigma)$:

$$(\mathrm{Ind}_H^G(\sigma)(h)f)(1) = f(h) = \sigma(h) f(1), \qquad h \in H.$$

The restriction of this map to c-$\mathrm{Ind}_H^G(\sigma)$ clearly belongs to $\mathrm{Hom}_H(\text{c-}\mathrm{Ind}_H^G(\sigma), \sigma)$.

Let K be a compact open subgroup of G. If $f \in (\mathrm{Ind}_H^G(W))^K$ and $g \in G$, then

$$f(g) = f(gk) = f(gkg^{-1}g) = \sigma(gkg^{-1})\, f(g), \quad \forall k \in K \cap g^{-1}Hg.$$

It follows that

$$(*) \qquad f(g) \in W^{H \cap gKg^{-1}} \text{ whenever } f \in (\mathrm{Ind}_H^G(W))^K.$$

This will be used in several places below.

Proposition. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Then*

(1) If $H\backslash G$ is compact and σ is admissible, then c-$\mathrm{Ind}_H^G(\sigma) = \mathrm{Ind}_H^G(\sigma)$ is admissible.

(2) (Frobenius reciprocity) If (π, V) is a smooth representation, then composition with the map $f \mapsto f(1)$ induces an isomorphism of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma))$ with $\mathrm{Hom}_H(\pi, \sigma)$,

Remark: As seen in the example of c-$\mathrm{Ind}_{K_0}^{GL_n(F)}\mathbf{1}$, if $H\backslash G$ is noncompact, then c-$\mathrm{Ind}_H^G(\sigma)$ might not be admissible when σ is irreducible and admissible.

Proof. For (1), assume that σ is admissible and $H\backslash G$ is compact. Let K be a compact open subgroup of G. As g runs over a set of coset representatives for G/K, the images in $H\backslash G$ of the double cosets HgK form an open cover. By compactness of $H\backslash G$, this open cover has a finite subcover. This means that there are finitely many disjoint H-K double cosets in G. Choose a finite set X such that $G = HXK$.

Let $f \in (\mathrm{Ind}_H^G(W))^K$. Let $x \in X$. Because $H \cap xKx^{-1}$ is a compact open subgroup of H and σ is admissible, $\dim(W^{H\cap xKx^{-1}}) < \infty$. Let $W_0 = \sum_{x\in X} W^{H\cap xKx^{-1}}$. Then W_0 is finite sum of finite-dimensional vector spaces, so $\dim(W_0) < \infty$. Applying (*) with $g = x$, we see that $f(X) \subset W_0$ for all $f \in (\mathrm{Ind}_H^G(W))^K$. The map $f \mapsto f\,|\,X$ from $(\mathrm{Ind}_H^G(W))^K$ to the space $C(X, W_0)$ of functions from X to W_0 is one-to-one. Because X is finite and $\dim(W_0) < \infty$, we have $\dim C(X, W_0) < \infty$. Thus $\dim((\mathrm{Ind}_H^G(W))^K) < \infty$.

For (2), let $\mathcal{A} \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma))$. Composition of $f \mapsto f(1)$ with $\mathcal{A}$ is the map $v \mapsto (\mathcal{A}v)(1)$ from V to W. If $h \in H$ and $v \in V$, then, using the fact that $\mathcal{A} \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma))$, followed by $\mathcal{A}v \in \mathrm{Ind}_H^G(W)$, we see that

$$(\mathcal{A}(\pi(h)v))(1) = (\mathrm{Ind}_H^G(\sigma)(h)\mathcal{A}v)(1) = (\mathcal{A}v)(h) = \sigma(h)((\mathcal{A}v)(1)).$$

Thus the map $v \mapsto (\mathcal{A}v)(1)$ belongs to $\mathrm{Hom}_H(\pi, \sigma)$.

Suppose that $(\mathcal{A}v)(1) = 0$ for all $v \in V$. Then

$$(\mathcal{A}v)(g) = (\mathrm{Ind}_H^G(\sigma)(g)\mathcal{A}v)(1) = (\mathcal{A}\pi(g)v)(1) = 0$$

for all $g \in G$. Thus $\mathcal{A}v$ is identically zero for all $v \in V$. That is, $\mathcal{A} = 0$. Hence the map taking $\mathcal{A}$ to $v \mapsto (\mathcal{A}v)(1)$ is one to one.

To see that this map is onto, given $A \in \mathrm{Hom}_H(\pi, \sigma)$, define a function A^G from V to the space of functions from G to W by $A^G(v)(g) = A(\pi(g)\,v)$, $v \in V$. Note that

$$A^G(v)(hg) = A(\pi(hg)v) = A(\pi(h)\pi(g)v) = \sigma(h)A(\pi(g)v) = \sigma(h)(A^G(v)(g)), \qquad h \in H,\ g \in G.$$

Let $v \in V$. Then $v \in V^K$ for some compact open subgroup K of G, so

$$A^G(v)(gk) = A(\pi(gk)v) = A(\pi(g)v) = A^G(v)(g), \qquad g \in G,\ k \in K.$$

Hence $A^G(v)$ is right K-invariant. We have shown that $A^G(v) \in \mathrm{Ind}_H^G(W)$ for every $v \in V$.

To finish, we check that A^G maps to A. Our map takes A^G to the function $v \mapsto A^G(v)(1)$. By definition of A^G, if $v \in V$, then $A^G(v)(1) = A(\pi(1)v) = A(v)$. $\square$

Remark: Suppose that H is an open (and closed) subgroup of G and (σ, W) is a smooth representation of H. Given $w \in W$, define $f_w \in \mathrm{Ind}_H^G(W)$ by $f_w(h) = \sigma(h)w$ for $h \in H$ and $f_w(g) = 0$ if $g \notin H$. We can show that

(1) The map $w \mapsto f_w$ is an H-isomorphism from W to the H-subrepresentation $\{ f \in \mathrm{Ind}_H^G(W) \mid \mathrm{supp}\, f \subset H \}$ of $\mathrm{Ind}_H^G(\sigma)$.

(2) The map from $\mathrm{Hom}_G(\text{c-Ind}_H^G(\sigma), \pi) \to \mathrm{Hom}_H(\sigma, \pi)$ that takes $\mathcal{A}$ to the function $w \mapsto \mathcal{A} f_w$ is an isomorphism.

Lemma. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Let $\delta_{H\backslash G}(h) = \Delta_H(h)^{-1}\Delta_G(h)$, $h \in H$. (Here, Δ_G and Δ_H are the modular functions of G and H, respectively.) There exists a linear functional $I_{H\backslash G}$ on c-$\mathrm{Ind}_H^G(\delta_{H\backslash G})$ having the following two properties:*

(1) $I_{H\backslash G}(\rho(g)f) = I_{H\backslash G}(f)$ for all $f \in$c-$\mathrm{Ind}_H^G(\delta_{H\backslash G})$ and all $g \in G$. (Here, $\rho(g)$ is right translation by g.)

(2) If $g \in G$, K is a compact open subgroup of G, and $f \in$c-$\mathrm{Ind}_H^G(\delta_{H\backslash G})^K$ is supported on the double coset HgK, then $I_{H\backslash G}(f)$ is a positive multiple of $f(g)$.

The linear functional $I_{H\backslash G}$ is unique up to positive scalar multiples.

Proof. Given $f \in C_c^\infty(G)$, define

$$\dot{f}(g) = \int_H \Delta_G(h^{-1}) f(hg)\, d_\ell h, \qquad g \in G.$$

If $h_1 \in H$, then, because $d_\ell h = \Delta_H(h_1)^{-1} d_\ell(hh_1)$, we have that

$$\begin{aligned}\dot{f}(h_1 g) = &\int_H \Delta_G(h^{-1}) f(hh_1 g)\, d_\ell h = \int_H \Delta_H(h_1^{-1})\Delta_G(h_1 h_2^{-1}) f(h_2 g)\, d_\ell h_2 \\ =&\Delta_H(h_1)^{-1}\Delta_G(h_1)\dot{f}(g) = \delta_{H\backslash G}(h_1)\dot{f}(g), \qquad g \in G.\end{aligned}$$

Since the support of f is compact, the support of $\dot{f}$ is compact modulo H. If K is a compact open subgroup of G such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$, then $\dot{f}(gk) = \dot{f}(g)$ for all $g \in G$ and $k \in K$. Therefore $\dot{f} \in$c-$\mathrm{Ind}_H^G(\delta_{H\backslash G})$.

Next, verify that if K is a compact open subgroup of G, $\varphi \in (\text{c-Ind}_H^G(\delta_{H\backslash G}))^K$, and φ is supported in HgK for a fixed $g \in G$, then $\dot{f} = \varphi$ for $f \in C_c^\infty(G)$ defined by $f(gk) = m(H \cap gKg^{-1})\varphi(gk)$, $k \in K$, and $f(x) = 0$ for $x \notin gK$. It follows that $f \mapsto \dot{f}$ is onto at the level of K-fixed vectors. This implies that the map is onto.

In order to define $I_{H\backslash G}$, we show that $f \mapsto \int_G f(g)\, \Delta_G(g)^{-1}\, d_r g$ factors through the map $f \mapsto \dot{f}$.

Let $f \in C_c^\infty(G)$. Choose a compact open subgroup K of G such that $f \in (C_c^\infty(G))^K$. The function f is a finite linear combination of characteristic functions of right K cosets in G. Grouping together right K cosets that lie in the same H-K double coset, we may write $f = \sum_i f_i$, where the sum is finite, $Hg_iK \neq Hg_jK$ if $i \neq j$ and $f_i \in (C_c^\infty(G))^K$ is supported in Hg_iK. Because $\dot{f}_i$ is supported in Hg_iK, we see that $\dot{f} = 0$ if and only if $\dot{f}_i = 0$ for all i.

This allows us to reduce to the case where $f \in (C_c^\infty(G))^K$ is supported in HgK for a fixed $g \in G$. In this case, there exist finitely many $h_j \in H$ such that the support of f lies in the (disjoint) union of the cosets h_jgK. Note that if $h \in H$, then $hg \in h_jgK$ if and only if $h \in h_j(H \cap gKg^{-1})$. Modular functions are trivial on compact subgroups, so Δ_G and $\delta_{H\backslash G}$ are trivial on $H \cap gKg^{-1}$. It follows that

$$\dot{f}(g) = \sum_j \Delta_G(h_j)^{-1} f(h_jg)\, m_H(H \cap gKg^{-1}).$$

Next, we evaluate $\int_G f(x)\Delta_G(x)^{-1}\, d_\ell x$ for the same function f:

$$\begin{aligned}
\int_G f(x)\Delta_G(x)^{-1}\, d_\ell x &= \sum_j \int_K f(h_jg)\, \Delta_G(h_jg)^{-1} d_\ell(h_jgk) = \sum_j \int_K f(h_jg)\, \Delta_G(h_jg)^{-1} d_\ell k \\
&= m(K)\Delta_G(g)^{-1} \sum_j \Delta_G(h_j)^{-1} f(h_jg) \\
&= m(K)\, m_H(H \cap gKg^{-1})^{-1} \Delta_G(g)^{-1} \dot{f}(g).
\end{aligned}$$

This shows that $\dot{f} = 0$ implies $\int_G f(x)\Delta_G(x)^{-1}\, d_\ell x = 0$ for K-fixed functions. Since all functions in $C_c^\infty(G)$ are smooth, we see that $\dot{f} = 0$ implies that $\int_G f(x)\Delta_G(x)^{-1}\, d_\ell x$ for all $f \in C_c^\infty(G)$. If $f \in C_c^\infty(G)$, define $I_{H\backslash G}(\dot{f}) = \int_G f(x)\Delta_G(x)^{-1}\, d_\ell x$. Since we have already observed that the map $f \mapsto \dot{f}$ is onto, this gives a well-defined linear functional on $\text{c-Ind}_H^G \delta_{H\backslash G}$.

To finish the proof, supose that $g \in G$, K is a compact open subgroup of G, and f satisfies $f(x) = 0$ if $x \notin gK$, $f(gk) = f(g)$ for all $k \in K$. Then $\dot{f}(g) = m_H(H \cap gKg^{-1}) f(g)$ and $I_{H\backslash G}(f) = m_G(K) m_H(H \cap gKg^{-1})^{-1} \Delta_G(g)^{-1} \dot{f}(g)$. The second part of the lemma follows. The verification that $I_{H\backslash G}$ is unique up to scalar multiples is left as an exercise.

□

Proposition. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Let $\delta_{H\backslash G}(h) = \Delta_H(h)^{-1}\Delta_G(h)$, $h \in H$. Let $\pi =$c-$\mathrm{Ind}_H^G(\sigma)$. Then $\widetilde{\pi} \simeq \mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_{H\backslash G})$.*

Sketch of proof. Details to be added. Take the usual pairing $\langle \cdot, \cdot \rangle_W$ on $\widetilde{W} \times W$. Set $A(\widetilde{w} \otimes w) = \langle \widetilde{w}, w \rangle_W$. Now view $\widetilde{W} \otimes W$ as the space of $(\widetilde{\sigma} \otimes \delta_{H\backslash G}) \otimes \sigma$, and view $\mathbb{C}$ as the space of $\delta_{H\backslash G}$. Then $A \in \mathrm{Hom}_H((\widetilde{\sigma} \otimes \delta_{H\backslash G}) \otimes \sigma, \delta_{H\backslash G})$.

Given $\varphi \in$ c-$\mathrm{Ind}_H^G(\sigma)$ and $\Phi \in \mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_{H\backslash G})$, define $\langle \Phi, \varphi \rangle_* : G \to \mathbb{C}$ by

$$\langle \Phi, \varphi \rangle_*(g) = A(\Phi(g) \otimes \phi(g)) = \langle \Phi(g), \varphi(g) \rangle_W, \qquad g \in G.$$

Verify that $\langle \Phi, \varphi \rangle_* \in$ c-$\mathrm{Ind}_H^G(\delta_{H\backslash G})$.

Let $I_{H\backslash G}$ be as in the preceding lemma. Given $\Phi \in \mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_{H\backslash G})$, define $\mathcal{A}\Phi \in$ (c-$\mathrm{Ind}_H^G(\sigma))\tilde{}$ by

$$\langle \mathcal{A}\Phi, \varphi \rangle = I_{H\backslash G}(\langle \Phi, \varphi \rangle_*), \qquad \varphi \in \mathrm{c-Ind}_H^G(\sigma).$$

Check that $\mathcal{A}\Phi$ is smooth and properties of $I_{H\backslash G}$ can be used to show that

$$\mathcal{A} \in \mathrm{Hom}_G(\mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_{H\backslash G}), (c - \mathrm{Ind}_H^G(\sigma))\tilde{}\,).$$

The second step is to show that $\mathcal{A}$ is an isomorphism at the level of K-fixed vectors. Let K be a compact open subgroup of G.

Let S be a set of coset representatives for $H\backslash G/K$. For each $g \in S$, let β_g be a basis of $W^{H \cap gKg^{-1}}$. Let $w \in \beta_g$. Define

$$\varphi_{g,w}(x) = \begin{cases} \sigma(h)w, & \text{if } x = hgk,\ h \in H,\, k \in K \\ 0, & \text{if } x \notin HgK. \end{cases}$$

The support of $\varphi_{g,w}$ is HgK, which is compact modulo H. Also, $\varphi_{g,w}$ is invariant under right translation by elements of K. Therefore $\varphi_{g,w} \in (\text{c-}\mathrm{Ind}_H^G \sigma)^K$. We have already observed (see equation (*) near the beginning of the section) that any $\Phi \in (\mathrm{Ind}_H^G \sigma)^K$ satisfies $\Phi(g) \in W^{H \cap gKg^{-1}}$ for all $g \in G$. Therefore $\{\, \varphi_{g,w} \mid g \in S,\, w \in \beta_g \,\}$ spans (c-$\mathrm{Ind}_H^G \sigma)^K$. Because this set is linearly independent, it is a basis of (c-$\mathrm{Ind}_H^G \sigma)^K$.

If $g \in S$, let β_g^* be the basis of $(W^{H \cap gKg^{-1}})^*$ that is dual to the basis β_g. For $g \in S$ and $\tilde{w} \in \beta_g^*$, define

$$\Phi_{g,\tilde{w}}(x) = \begin{cases} \delta_{H\backslash G}(h)\widetilde{\sigma}(h)\tilde{w}, & \text{if } x = hgk,\ h \in H,\, k \in K \\ 0, & \text{if } x \notin HgK. \end{cases}$$

The function $\Phi_{g,\tilde{w}}$ belongs to (c-$\mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_{H\backslash G}))^K$. Let $\Phi \in \mathrm{Ind}_H^G(\widetilde{\sigma} \otimes \delta_{H\backslash G}))^K$. Fix $g \in S$. The restriction $\Phi \,|\, HgK$ is compactly supported modulo H and $\Phi(g) \in \mathrm{Span}(\beta_g^*)$. Thus $\Phi \,|\, HgK$ is a finite linear combination of functions of the form $\Phi_{g,\tilde{w}}$, $\tilde{w} \in \beta_g^*$.

If $g \in S$ and $w \in \beta_g$, then $\langle \Phi, \varphi_{g,w} \rangle_*(x) = 0$ whenever $x \notin HgK$, since $\varphi_{g,w}(x) = 0$ whenever $x \notin HgK$. Therefore

$$\langle \Phi, \varphi_{g,w} \rangle_* = \langle \Phi \,|\, HgK, \varphi_{g,w} \rangle_*.$$

It follows that

$$\langle \Phi, \varphi_{g,w} \rangle_{H\backslash G} = I_{H\backslash G}(\langle \Phi, \varphi_{g,w} \rangle_*) = I_{H\backslash G}(\langle \Phi \,|\, HgK, \varphi_{g,w} \rangle_*)$$

is a linear combination of $\langle \Phi_{g,\tilde{w}}, \varphi_{g,w} \rangle_{H\backslash G}$, $\tilde{w} \in \beta_g^*$.

Let g_1, $g_2 \in S$, $w \in \beta_{g_1}$ and $\tilde{w} \in \beta_{g_2}^*$. If $g_1 = g_2$, then

$$\langle \Phi_{g_1,\tilde{w}}, \varphi_{g_1,w} \rangle_*(hg_1k) = \delta_{H\backslash G}(h) \langle \tilde{w}, w \rangle_W, \qquad h \in H,\, k \in K,$$

and $\langle \Phi_{g_1,\tilde{w}}, \varphi_{g_1,w} \rangle_*$ is supported on Hg_1K. It follows that $\langle \Phi_{g_1,\tilde{w}}, \varphi_{g_1,w} \rangle_{H\backslash G}$ is equal to $c\langle \tilde{w}, w \rangle_W$ for some positive real number c. We have that

$$\langle \Phi_{g_1,\tilde{w}}, \varphi_{g_2,w} \rangle_{H\backslash G} = \begin{cases} 0, & \text{if } g_1 \neq g_2 \\ c\langle \tilde{w}, w \rangle_W, & \text{if } g_1 = g_2. \end{cases}$$

It follows that the pairing $\langle \cdot, \cdot \rangle_{H\backslash G}$ is nondegenerate at the level of K-fixed vectors for each compact open subgroup K of G. $\square$

Lemma. *Let (σ, W) be a smooth representation of a closed subgroup H of G. Suppose that* c-$\mathrm{Ind}_H^G(\sigma)$ *is admissible. Then* c-$\mathrm{Ind}_H^G(\sigma) = \mathrm{Ind}_H^G(\sigma)$.

Proof. Let $V = \mathrm{Ind}_H^G(W)$ and $V_c =$c-$\mathrm{Ind}_H^G(W)$. Let K be a compact open subgroup of G. Let $S_K \in H\backslash G/K$ be the set of double cosets HgK such that there exists $\varphi \in V^K$ such that $\varphi(g) \neq 0$. Suppose that $HgK \in S_K$, $\varphi \in V^K$ and $\varphi(g) \neq 0$. Define $\varphi^{(g)} : G \to W$ by $\varphi^{(g)}(x) = \varphi(x)$ if $x \in HgK$ and $\varphi^{(g)}(x) = 0$ if $x \notin HgK$. Then $\varphi^{(g)} \in V_c^K$.

Suppose that $Hg_jK \in S_K$ are distinct, $1 \leq j \leq \ell$. Choose $\varphi_j \in V^K$ such that $\varphi_j(g_j) \neq 0$. Then $\{\, \varphi_j^{(g_j)} \mid 1 \leq j \leq \ell \,\}$ is a linearly independent subset of V_c^K. Thus $\dim(V_c^K) \geq \ell$. If V_c is admissible, then $\dim(V_c^K)$ is finite and $\ell \leq \dim(V_c^K)$. This implies that the set S_K must be finite. Hence, if $\varphi \in V^K$, the support of φ lies inside the union of finitely many double cosets in $H\backslash G/K$. It follows that φ is compactly supported modulo H - that is, $\varphi \in V_c^K$. We have shown that $V^K = V_c^K$ for all compact open subgroups K. Thus $V = V_c$. $\square$

Definition. A representation (π, V) is *unitary* if there exists a G-invariant (positive definite Hermitian) inner product on V.

Remark. Strictly speaking, since we are not assuming that V is a Hilbert space, it would be more accurate to refer to a representation π as in the definition as *unitarizable*. Then, after completing V with respect to the given inner product, we obtain a unitary representation of G in a Hilbert space.

Lemma. *If (π, V) is an admissible unitary representation of G, then π is semisimple.*

Proof. Let W be a subrepresentation of π. Let $W^\perp$ be the orthogonal complement of W relative to a G-invariant inner product $(\cdot,\cdot)$ on V. Let $v \in W^\perp$. Then, if $w \in W$ and $g \in G$,

$$(w, \pi(g)v) = (\pi(g)^{-1}w, v) \in (W, W^\perp) = 0,$$

since $\pi(g)^{-1}w \in W$. Therefore $W^\perp$ is a subrepresentation of V.

Fix a compact open subgroup K of G. We can show that the inner product $(\cdot,\cdot)$ restricts to a K-invariant inner product on V^K. Note that nondegeneracy of the restriction follows from the property

$$(*) \qquad (v_1, \pi(e_K)v_2) = (\pi(e_K)v_1, \pi(e_K)v_2) = (v_1, v_2), \qquad v_1 \in V^K,\ v_2 \in V.$$

By definition, $W^K = W \cap V^K$. Let U be the orthogonal complement of W^K in V^K. Since π is admissible, the vector space V^K is finite-dimensional and we have $V^K = W^K \oplus U$. We can use (*) to show that $U = W^\perp \cap V^K$. Thus we have $V^K = W^K \oplus (W^\perp \cap V^K)$. Hence, by smoothness of π, $V = W \oplus W^\perp$. It now follows from an earlier lemma that π is semisimple. □

The representation $(\bar\pi, \bar V)$ conjugate to (π, V) is defined as follows. As a set $\bar V = V$, but $c \cdot v = \bar c v$, for $c \in \mathbb{C}$ and $v \in \bar V$. For $g \in G$ and $v \in V$, $\bar\pi(g)v = \pi(g)v$.

Lemma. *If (π, V) is admissible and unitary, then $(\bar\pi, \bar V)$ is equivalent to $(\widetilde\pi, \widetilde V)$.*

Proof. Let $(\cdot,\cdot)$ be a G-invariant inner product on V. Define $A : \bar V \to \widetilde V$ by $Av = (\cdot, v)$. Check that A is linear and intertwines $\bar\pi$ with $\widetilde\pi$. Earlier, we saw that for any compact open subgroup K, $\widetilde V^K = (V^K)^*$. To see that $A(\bar V^K) = (V^K)^*$, use the fact that $(\cdot,\cdot)|_{V^K}$ is nondegenerate. □

Lemma. *Suppose that (π, V) is irreducible, smooth and admissible. Then a G-invariant positive definite inner product on V is unique up to scalar multiples.*

Proof. Let $(\cdot,\cdot)_j$, $j = 1, 2$, be two such inner products. Define $A_j : \bar V \to \widetilde V$ by $A_j(v) = (\cdot, v)_j$, $v{\in}V_j$. It follows from (the proof of) the preceding lemma that A_2 is invertible.

As $A_2^{-1} \circ A_1$ belongs to $\mathrm{End}_G(\bar{\pi})$ and $\bar{\pi}$ is irreducible, it follows from Schur's lemma that $A_2^{-1} \circ A_1$ is scalar. $\square$

Proposition. *Let (σ, W) be a unitary smooth representation of a closed subgroup H of G. Then the representation* $\text{c-Ind}_H^G(\sigma \otimes \delta_{H\backslash G}^{1/2})$ *is unitary.*

Proof. Let $(\cdot,\cdot)_W$ be an H-invariant inner product on W. Given $\varphi_1, \varphi_2 \in$c-$\mathrm{Ind}_H^G(\sigma \otimes \delta_{H\backslash G}^{1/2})$, define $(\varphi_1, \varphi_2)_* \in$c-$\mathrm{Ind}_H^G(\delta_{H\backslash G})$ by

$$(\varphi_1, \varphi_2)_*(g) = (\varphi_1(g), \varphi_2(g))_W, \qquad g \in G.$$

Then define $(\varphi_1, \varphi_2)_V = I_{H\backslash G}((\varphi_1, \varphi_2)_*)$. To finish, verify that this defines a G-invariant inner product on $\text{c-Ind}_H^G(\sigma \otimes \delta_{H\backslash G}^{1/2})$. $\square$

We remark that it can happen that a representation $\text{c-Ind}_H(\sigma)$ may be nonunitary and have an irreducible unitary subquotient. Some discrete series representations are unitary quotients of nonunitary induced representations.

There are two kinds of induced representations that play central roles in the theory of admissible representations of reductive p-adic groups. One kind, which will be discussed in Section 7, are representations induced from irreducible smooth representations of open subgroups that are compact modulo the centre of the group G. Here, the inducing representation σ is finite dimensional. The other kinds are representations induced from smooth representations of parabolic subgroups. These will be discussed in Section 6.

6. Parabolic induction and Jacquet modules

Some of the results in this section are stated and proved for $G = GL_n(F)$, where F is a nonarchimedean local field. These results have analogues for other connected reductive p-adic groups, such as $SL_n(F)$ and $Sp_{2n}(F)$. The general ideas of the proofs for connected reductive p-adic groups are basically the same as for $GL_n(F)$.

We start with a description of the parabolic subgroups of $G = GL_n(F)$. Suppose that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a partition of n: $r \in \mathbb{Z}$, $r \geq 1$, $\alpha_j \in \mathbb{Z}$, $\alpha_j > 0$, $1 \leq j \leq r$, such that $\sum_{j=1}^r \alpha_j = n$. Let M_α be the subgroup $\prod_{j=1}^r GL_{\alpha_j}(F)$:

$$M_\alpha = \begin{pmatrix} GL_{\alpha_1}(F) & 0 & \cdots & 0 \\ 0 & GL_{\alpha_2}(F) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL_{\alpha_r}(F) \end{pmatrix}$$

Let N_α be the subgroup

$$N_\alpha = \begin{pmatrix} I_{\alpha_1} & * & \cdots & * \\ 0 & I_{\alpha_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{\alpha_r} \end{pmatrix}.$$

The group N_α is a nilpotent p-adic group and M_α is a connected reductive p-adic group. Note that M_α normalizes N_α. Set $P_\alpha = M_\alpha \ltimes N_\alpha$.

Definition. A *Levi subgroup* M of G is a subgroup of G that is conjugate to M_α for some partition α of n.

Definition. A *parabolic subgroup* P of G is a subgroup of G that is conjugate to P_α for some partition α of n. Choose $x \in G$ such that $x^{-1}Px = P_\alpha$. The subgroup $M = xM_\alpha x^{-1}$ is called a *Levi component* or *Levi factor* of P, and the *unipotent radical* N of P is $xN_\alpha x^{-1}$. The decomposition $P = MN = M \ltimes N$ is called a *Levi decomposition* of P. The subgroup $B = P_{(1,\ldots,1)}$ is called a *standard Borel subgroup* of G, and its conjugates are called *Borel subgroups*. A parabolic subgroup of the form P_α is called a *standard parabolic subgroup*.

Remark: If G is an arbitrary connected reductive p-adic group and P is a parabolic subgroup of G, then $P = M \ltimes N$ for some connected reductive p-adic group M (a Levi factor of P) and N is the unipotent radical of P, that is, N is the maximal normal connected nilpotent subgroup of P. In the case $G = GL_n(F)$, M is a direct product of general linear groups. For other reductive p-adic groups G, M is not a direct product of the same type of group as G. For example, $GL_n(F)$ occurs as the Levi factor of a maximal parabolic subgroup of $Sp_{2n}(F)$.

Recall that $K_0 = GL_n(\mathfrak{o})$ is the compact open subgroup of $GL_n(F)$ consisting of matrices whose entries are in $\mathfrak{o}$ and whose determinant is in $\mathfrak{o}^\times$.

Lemma. *(Iwasawa decomposition) Let P be a parabolic subgroup of $G = GL_n(F)$. Then $G = K_0P = PK_0$.*

Proof. First, suppose that $G = K_0B$, where $B = P_{(1,\dots,1)}$. If $x \in G$, then $x = kb$ for some $k \in K_0$ and $b \in B$, so

$$K_0(xBx^{-1}) = K_0k(bBb^{-1})k = K_0kBk = K_0Bk = Gk = G.$$

If $x \in G$ and $P = xP_\alpha x^{-1}$, then, because $P_\alpha \supset B$, we have that

$$K_0P \supset K_0xBx^{-1} = G.$$

Thus $G = K_0P$. Inversion then gives $G = PK_0$.

To finish, we need to show that $G = K_0B$. Let $g = (g_{ij}) \in G$. At least one entry in the first column of g is nonzero, since g is invertible. If we interchange the ith and jth row of g we get a matrix kg, where $k \in K$ (k is obtained from the identity matrix I by interchanging the ith and jth rows of I). It follows that there exists $k \in K_0$ such that the matrix $h = kg$ has the property that $h_{11} \neq 0$ and $|h_{11}|_F \geq \cdots \geq |h_{n1}|_F$. If $h_{n1} \neq 0$, let k_1 be the matrix obtained by adding $-h_{n1}h_{n2}^{-1}$ times the $n-1$st row of I to the nth row of I. Then $|-h_{n1}h_{n2}^{-1}|_F \leq 1$, so $k_1 \in K_0$. The matrix $y = k_1kg$ is obtained from kg by adding $-h_{n1}h_{n2}^{-1}$ times the $n-1$st row of kg to the nth row of kg. The matrix y satisfies $y_{j1} = h_{j1}$ for $1 \leq j \leq n-1$ and $y_{n1} = 0$. By performing analogous row operations on y, we can produce $k' \in K_0$ such that $z = k'y$ satisfies $z_{11} \neq 0$ and $z_{j1} = 0$ for $2 \leq j \leq n$.

If $n = 2$, we see that $z \in B$. Since $z \in K_0g$, this implies that $g \in K_0B$. It follows that $g \in {k'}^{-1}k_1^{-1}k^{-1}B = K_0B$.

If $n \geq 3$, let $z_{n-1} \in GL_{n-1}(F)$ be the matrix obtained from z by deleting the first row and column of z. By induction, there exist $k_{n-1} \in GL_{n-1}(\mathfrak{o})$ and b_{n-1} in the standard Borel subgroup of $GL_{n-1}(F)$ such that $z_{n-1} = k_{n-1}b_{n-1}$. It follows that

$$z = \begin{pmatrix} 1 & 0 \\ 0 & k_{n-1} \end{pmatrix} \begin{pmatrix} z_{11} & * \\ 0 & b_{n-1} \end{pmatrix} \in K_0B.$$

Since $z \in K_0g$, this implies $g \in K_0B$. $\square$

Remark: Arbitrary connected reductive p-adic groups may have more than one conjugacy class of maximal compact subgroups. In general, $PK \neq G$ for a maximal compact subgroup K and parabolic subgroup P. However, there exists a maximal compact subgroup K_0 of G such that $G = K_0P = PK_0$ holds for all parabolic subgroups P of G.

It follows from the Iwasawa decomposition $G = PK_0 = K_0P$ that $P\backslash G$ is compact for any parabolic subgroup P of G. Thus $\text{c-Ind}_P^G\sigma = \text{Ind}_P^G\sigma$ for any smooth representation σ of P.

A reductive p-adic group G is unimodular. Thus if H is a closed subgroup of G, the function $\delta_{H\backslash G} : H \to \mathbb{R}^\times$ defined in the previous section is equal to Δ_H^{-1}. If P is a parabolic subgroup of G, we will use the notation δ_P instead of $\delta_{P\backslash G}$.

Lemma. *Let P be a parabolic subgroup of G. Let $I_{P\backslash G} \in (\text{Ind}_P^G(\delta_P))^*$ be as in the previous section. Then Haar measures on G, P and K_0 can be normalized so that $I_{P\backslash G}(\dot{f}) = \int_{K_0} \dot{f}(k)\, dk$ for all $\dot{f} \in \text{Ind}_P^G(\delta_P)$.*

Proof. If $f \in C_c^\infty(G)$, define $\dot{f} \in \text{Ind}_P^G(\delta_P)$ by

$$\dot{f}(g) = \int_P f(pg) d_\ell p, \qquad g \in G.$$

If $\varphi \in C^\infty(K_0)$, define $\varphi^\sharp \in \text{Ind}_{P\cap K_0}^{K_0}\mathbf{1}$ by

$$\varphi^\sharp(k) = \int_{P\cap K_0} \varphi(p_0k) dp_0, \qquad k \in K_0.$$

The maps $f \mapsto \dot{f}$ and $\varphi \mapsto \varphi^\sharp$ are onto.

Recall from Section 5 that $I_{P\backslash G} \in (\text{Ind}_P^G(\delta_P))^*$ satisfies:

$$I_{P\backslash G}(\dot{f}) = \int_G f(g)dg \ \forall f \in C_c^\infty(G)$$
$$I_{P\backslash G}(\rho(g)\dot{f}) = I_{P\backslash G}(\dot{f}) \ \forall\, g \in G$$

Similarly, there exists $I_{(P\cap K_0)\backslash K_0} \in (\text{Ind}_{P\cap K_0}^{K_0}(\mathbf{1}))^*$ that satisfies:

$$I_{(P\cap K_0)\backslash K_0}(\varphi^\sharp) = \int_{K_0} \varphi(k)\, dk \ \forall f \in C^\infty(K_0)$$
$$I_{(P\cap K_0)\backslash K_0}(\rho(k)\varphi^\sharp) = I_{(P\cap K_0)\backslash K_0}(\varphi^\sharp) \ \forall\, k \in K_0$$

Define a map $R : \text{Ind}_P^G(\delta_P) \to \text{Ind}_{P\cap K_0}^{K_0}(\mathbf{1})$ by $R(\dot{f})(k) = \dot{f}(k)$, $k \in K$. Note that $P \cap K_0$ is a compact subgroup of P, so $\delta_P(k) = \Delta_P^{-1}(k) = 1$ for all $k \in P \cap K_0$. It is easy to use the Iwahori decomposition $G = PK$ to show that R is an isomorphism. Clearly, $R \in \mathrm{Hom}_K(\mathrm{Ind}_P^G(\delta_P), \mathrm{Ind}_{P\cap K_0}^{K_0}(\mathbf{1}))$.

A left Haar measure on P restricts to a Haar measure on the compact open subgroup $P \cap K_0$. It follows from uniqueness of Haar measure that if dp_0 is the Haar measure on $P \cap K_0$ used in the definition of $I_{(P\cap K_0)\backslash K_0}$, the restriction of the measure $d_\ell p$ used in the definition of $I_{P\backslash G}$ to $P \cap K_0$ is equal to $c\, dp_0$ for some positive real number c.

Let $\varphi \in C^\infty(K_0)$. Viewing φ as an element of $C_c^\infty(G)$, we have, for $p \in P$ and $k \in K$,

$$\begin{aligned}\dot\varphi(pk) &= \delta_P(p)\dot\varphi(k) = \delta_P(p)\int_P \varphi(p_1 k) d_\ell p_1\\ &= c\,\delta_P(p)\int_{P\cap K_0}\varphi(p_0 k)\,dp_0 = c\,\delta_P(p)\varphi^\sharp(k) = c\,R^{-1}(\varphi^\sharp)(pk).\end{aligned}$$

Thus $\dot\varphi = c\,R^{-1}(\varphi^\sharp)$ for each $\varphi \in C^\infty(K_0)$.

Let $\varphi \in C^\infty(K_0)$. Then

$$\begin{aligned}I_{P\backslash G}(R^{-1}(\varphi^\sharp)) &= I_{P\backslash G}(c\,\dot\varphi) = c\,I_{P\backslash G}(\dot\varphi) = c\int_G \varphi(g)\,dg\\ &= cb\int_{K_0}\varphi(k)\,dk = cb\,I_{(P\cap K_0)\backslash K_0}(\varphi^\sharp),\end{aligned}$$

where b is the positive real number such that the restriction of dg to K_0 is equal to $b\,dk$.

If $\varphi \in C^\infty(K_0)$, then

$$\begin{aligned}\int_{K_0}\varphi^\sharp(k)\,dk &= \int_{K_0}\int_{K_0\cap P}\varphi(p_0k)\,dp_0\,dk\\ &= \int_{K_0\cap P}\left(\int_{K_0}\varphi(p_0k)dk\right)\,dp_0 = m_{K_0\cap P}(K_0\cap P)\int_{K_0}\varphi(k)\,dk.\end{aligned}$$

Given $f \in C_c^\infty(G)$, let $\varphi \in C^\infty(K_0)$ be such that $\varphi^\sharp = R(\dot f)$.

$$\begin{aligned}I_{P\backslash G}(\dot f) &= I_{P\backslash G}(R^{-1}(\varphi^\sharp)) = cb\,I_{(P\cap K_0)\backslash K_0}(\varphi^\sharp) = cb\int_{K_0}\varphi(k)\,dk\\ &= cb\,m_{K_0\cap P}(K_0\cap P)\int_{K_0}\varphi^\sharp(k)\,dk = cb\,m_{K_0\cap P}(K_0\cap P)\int_{K_0}\dot f(k)\,dk.\end{aligned}$$

By adjusting the normalizations of Haar measures we can arrange that $I_{P\backslash G}(\dot f) = \int_{K_0}\dot f(k)\,dk$ for all $f \in C_c^\infty(G)$. $\square$

Corollary. *If Haar measures on G, K_0 and P are normalized so that $m_G(K_0) = m_{K_0}(K_0) = m_{\ell,P}(P\cap K_0) = m_{r,P}(P\cap K_0) = 1$, then*

$$\int_G f(g)\,dg = \int_{K_0}\int_P f(pk)\,d_\ell p\,dk = \int_{K_0}\int_P f(kp)\,dk\,d_r p, \qquad f \in C_c^\infty(G).$$

Proof. According to the preceding lemma, there exists a positive real number c such that

$$\int_G f(g)\,dg = I_{P\backslash G}(f) = c\int_{K_0} \dot{f}(k)\,dk = c\int_{K_0}\int_P f(pk)\,d_\ell p\,dk, \qquad f\in C_c^\infty(G).$$

Evaluating both sides for f equal to the characteristic function of K_0, we find that $c=1$ when the measures are normalized as indicated. For the second equality in the statement of the corollary, observe that

$$\begin{aligned}\int_G f(g)\,dg = \int_G f(g^{-1})\,dg &= \int_{K_0}\int_P f(k^{-1}p^{-1})\,d_\ell p\,dk\\ &= \int_{K_0}\int_P f(kp)\,d_r p\,dk\end{aligned}$$

since $d_\ell p^{-1} = d_r p$ when measures are normalized as indicated. □

Let $P=MN$ be a parabolic subgroup of G. Let (σ,W) be a smooth representation of M. We have the exact sequence $1\to N\to P\to M\to 1$, and we can regard (σ,W) as a representation of P by extending trivially across N. Since $P\backslash G$ is compact, $\mathrm{Ind}_P^G(\sigma)$ is admissible whenever (σ,W) is an admissible representation of M.

Let $\mathfrak{n}$ be the Lie algebra of $N:\ \mathfrak{n}=\{\,u-1 \mid u\in N\,\}$. Given $m\in M$, $\mathrm{Ad}_{\mathfrak{n}}(m):\mathfrak{n}\to\mathfrak{n}$ is given by $\mathrm{Ad}_{\mathfrak{n}}(m)(X)=mXm^{-1}$, $X\in\mathfrak{n}$. Recall that $\delta_P=\Delta_P^{-1}$ It is left as an exercise to show that $\delta_P(mn)=|\det\mathrm{Ad}_{\mathfrak{n}}(m)|_F$ for $m\in M$ and $n\in N$. Define $i_P^G\sigma=\mathrm{Ind}_P^G(\sigma\otimes\delta_P^{1/2})$ (this is called normalized induction).

Lemma. *The contragredient of $i_P^G\sigma$ is equivalent to $i_P^G\widetilde{\sigma}$.*

Proof. Note that the contragredient of δ_P is δ_P^{-1} (because δ_P is one-dimensional and real-valued). According to a proposition from Section 5,

$$(i_P^G\sigma)\widetilde{\ } = (\mathrm{Ind}_P^G(\sigma\otimes\delta_P^{1/2}))\widetilde{\ } \simeq \mathrm{Ind}_P^G((\sigma\otimes\delta_P^{1/2})\widetilde{\ }\otimes\delta_P) = \mathrm{Ind}_P^G(\widetilde{\sigma}\otimes\delta_P^{1/2}) = i_P^G\widetilde{\sigma}.$$

□

Let $\langle\cdot,\cdot\rangle_W$ denote the usual pairing of with $\widetilde{W}$ with W.

Lemma. *Let $f_1\in i_P^G\sigma$ and $f_2\in i_P^G\widetilde{\sigma}$. Then, assuming that Haar measures are normalized suitably, $\langle f_1,f_2\rangle=\int_{K_0}\langle f_1(k),f_2(k)\rangle_W\,dk$.*

Proof. If $f_1\in i_P^G\sigma$ and $f_2\in i_P^G\widetilde{\sigma}$, define $\langle f_1,f_2\rangle_*(g)=\langle f_1(g),f_2(g)\rangle_*$. Then $\langle f_1,f_2\rangle_*\in\mathrm{Ind}_P^G\delta_P$. As seen in the proof of the relevant proposition in Section 5,

$$\langle f_1,f_2\rangle=\langle f_1,f_2\rangle_{P\backslash G}=I_{P\backslash G}(\langle f_1,f_2\rangle_*)$$

defines a pairing that gives the equivalence of the preceding lemma. Applying one of the lemmas above, we see that, if measures are suitably normalized,

$$\langle f_1, f_2\rangle = \int_{K_0} \langle f_1, f_2\rangle_* \, dk = \int_{K_0} \langle f_1(k), f_2(k)\rangle_W \, dk.$$

$\square$

Lemma. *Let (σ, W) be a smooth unitary representation of M. Let $(\cdot,\cdot)_W$ be an M-invariant inner product on W. Then $i_P^G\sigma$ is unitary, with G-invariant inner product given by*

$$(f_1, f_2)_V = \int_{K_0} (f_1(k), f_2(k))_W \, dk, \qquad f_1,\ f_2 \in i_P^G(W).$$

Proof. If $f_1,\ f_2 \in i_P^G(W)$, let $(f_1, f_2)_*(g) = (f_1(g), f_2(g))_W$, $g \in G$. Applying a result from Section 5, we have that $(f_1, f_2)_V = I_{P\backslash G}((f_1, f_2)_*)$ defines a G-invariant inner product on $i_P^G(W)$. According to a lemma above, if measures are suitably normalized, then

$$I_{P\backslash G}((f_1, f_2)_*) = \int_{K_0} (f_1, f_2)_*(k) \, dk = \int_{K_0} (f_1(k), f_2(k))_W \, dk.$$

$\square$

Because normalized induction takes unitary representations of Levi subgroups to unitary representations of G, it is standard to use normalized induction. (Many authors use the notation Ind_P^G for normalized induction.)

Let (π, V) be a smooth representation of G. Let N be the unipotent radical of a parabolic subgroup P of G. Set

$$V(N) = \mathrm{Span}(\{\pi(n)v - v \mid n \in N,\ v \in V\}).$$

Observe that $V(N)$ is an N-invariant subspace of V. Let $V_N = V/V(N)$. If W is an N-invariant subspace of V and N acts trivially on the quotient V/W, then $V(N) \subset W$. Therefore V_N is the largest N-module quotient of V on which N acts trivially. Because P normalizes N, the subspace $V(N)$ is P-invariant. Hence we have a representation of P on V_N, with N acting trivially.

Definition. The representation (π_N, V_N) is called the *Jacquet module* of V (with respect to P).

Lemma. *Let M be a Levi factor of P. Let (σ, W) be a smooth representation of M. Then* $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) = \mathrm{Hom}_M(\pi_N, \sigma)$.

Proof. As shown in the previous section, $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\sigma) \simeq \mathrm{Hom}_P(\pi, \sigma)$ (Frobenius reciprocity). So it suffices to show that $\mathrm{Hom}_P(\pi, \sigma) \simeq \mathrm{Hom}_M(\pi_N, \sigma)$.

If $A \in \mathrm{Hom}_P(\pi, \sigma)$, then $A\pi(p) = \sigma(p)A$ for all $p \in P$. If $n \in N$ and $v \in V$, then $A(\pi(n)v - v) = Av - Av = 0$. This implies that $A(V(N)) = \{0\}$. So A factors to an element $\dot{A}$ of $\mathrm{Hom}_{\mathbb{C}}(V_N, W)$. Clearly, an element of $\mathrm{Hom}_{\mathbb{C}}(V_N, W)$ is equal to $\dot{A}$ for a unique $A \in \mathrm{Hom}_N(\pi, \sigma)$. Thus $\mathrm{Hom}_P(\pi, \sigma)$ is isomorphic to a subspace of $\mathrm{Hom}_{\mathbb{C}}(V_N, W)$. The image $\dot{A}$ of an element $A \in \mathrm{Hom}_P(\pi, \sigma)$ has the property

$$\dot{A}\pi_N(m)(v + V(N)) = \dot{A}(\pi(m)v + V(N)) = A\pi(m)v = \sigma(m)Av, \qquad m \in M,\ v \in V.$$

That is, $\dot{A} \in \mathrm{Hom}_M(\pi_N, \sigma)$. □

The following theorem is a deep result - see the end of this section for the proof. The theorem was originally proved by Jacquet for $GL_n(F)$ in [J].

Theorem. *Let P be a parabolic subgroup of G with unipotent radical N. If (π, V) is an admissible representation of G, then (π_N, V_N) is an admissible representation of P.*

Lemma. *If (π, V) is a smooth finitely generated representation of G, then (π_N, V_N) is a smooth finitely generated representation of P.*

Proof. Becauase N acts trivially on V_N and a compact open subgroup K of P satisfies $K = (K \cap M)(K \cap N)$, it suffices to show that (π_N, V_N) is a smooth finitely generated representation of a Levi factor M of P, For smoothness of π_N, note that if K is a compact open subgroup of G, then $v \in V^K$ implies $v + V(N)$ is fixed by $K \cap M$.

If $v \in V$, let $G\cdot v = \{\pi(g)v \mid g \in G\}$. Suppose that $V = \mathrm{Span}(\cup_{1 \le j \le \ell} G\cdot v_j)$ for vectors $v_1, \dots, v_\ell \in V$. Choose a compact open subgroup K such that $v_j \in V^K$ for $1 \le j \le \ell$. Now $P\backslash G$ is compact, so $P\backslash G/K = \cup_{1 \le j \le r} Pg_jK$ for some $g_1, \dots, g_r \in G$. It follows that $V = \mathrm{Span}(\cup_{i,j} P \cdot \pi(g_i)v_j)$, and hence the M-orbits of the images of the vectors $\pi(g_i)v_j$ span V_N. Hence V_N is finitely generated. □

Lemma. *N is the union of its compact open subgroups.*

Proof. For $N = N_{(1,\dots,1)}$, let $U = N \cap K_0$. Set $a = \mathrm{diag}(1, \varpi, \varpi^2, \dots, \varpi^{n-1})$. Then

$$aUa^{-1} = \begin{pmatrix} 1 & \varpi^{-1}\mathfrak{o} & \varpi^{-2}\mathfrak{o} & \cdots & \cdots & \varpi^{-(n-1)} \\ 0 & 1 & \varpi^{-1}\mathfrak{o} & \varpi^{-2}\mathfrak{o} & \cdots & \varpi^{-(n-2)}\mathfrak{o} \\ \vdots & 0 & \ddots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

We can easily see that $N = \bigcup_{m\geq 1} a^m U a^{-m}$. If $N = N_\alpha$, then $N = \bigcup_{m\geq 1} a^m(U\cap N)a^{-m}$.

$\square$

If U is a compact open subgroup of N, let e_U be $m_N(U)^{-1}$ times the characteristic function of U. If (π, V) is a smooth representation of G, define $\pi(e_U) : V \to V$ by

$$\pi(e_U)v = \int_N e_U(n)\pi(n)\, v\, dn, \qquad v \in V.$$

Recall that we made this kind of definition previously only for compact open subgroups of G. Here, U is not an open subgroup of G. Note that the restriction of $m_N(U)^{-1}dn$ to U is normalized Haar measure du on U, so we have $\pi(e_U)v = \int_U \pi(u)v\, du$, $v \in V$.

Lemma. $V(N) = \{\, v \in V \mid \pi(e_U)v = 0$ *for some compact open subgroup* U *of* $N\,\}$.

Proof. Let $v \in V$, $n \in N$. By the previous lemma, $n \in U$ for some compact open subgroup U of N. Because $n \in U$, we have $e_U(n'n^{-1}) = e_U(n')$ for all $n' \in N$, so

$$\begin{aligned}\pi(e_U)\pi(n)v &= m_N(U)^{-1}\int_N e_U(n_1)\pi(n_1 n)v\, dn_1 = m_N(U)^{-1}\int_N e_U(n_2 n^{-1})\pi(n_2)v\, dn_2\\ &= m_N(U)^{-1}\int_N e_U(n_2)\pi(n_2)v\, dn_2 = \pi(e_U)v.\end{aligned}$$

This implies $\pi(e_U)(v - \pi(n)v) = 0$. Thus $V(N) \subset \{\, v \in V \mid \pi(e_U)v = 0$ for some compact open subgroup U of $N\,\}$.

Suppose $v \in V$ and $\pi(e_U)v = 0$ for some compact open subgroup U of N. Let $U_v = \{\, u \in U \mid \pi(u)v = v\,\}$. Then

$$\begin{aligned}\pi(e_U)v &= [U : U_v]^{-1} \sum_{k\in U/U_v} \pi(k)v\\ \text{and} \quad v &= [U : U_v]^{-1} \sum_{k\in U/U_v} v\\ \Longrightarrow v &= v - \pi(e_U)v = [U : U_v]^{-1} \sum_{k\in U/U_v} v - \pi(k)\cdot v \in V(N)\end{aligned}$$

$\square$

Proposition. *Let* M *be a Levi factor of* P. *Let* (π, V), (π', V') *and* (π'', V'') *be smooth representations of* G *such that*

$$0 \to V \to V' \to V'' \to 0$$

is an exact sequence of G-morphisms. Then $0 \to V_N \to V'_N \to V''_N \to 0$ is an exact sequence of M-morphisms.

Proof. If $n \in N$, $v \in V$, and v has image v' in V', then $\pi(n)v - v$ has image $\pi'(n)v' - v'$. Thus the one-to-one G-morphism $V \to V'$ restricts to a one-to-one P-morphism $V(N) \to V'(N)$.

If $v'' \in V''$ and $n \in N$, then, because the map $V' \to V''$ is onto, there exists $v' \in V'$ that maps onto v''. It follows that $\pi'(n)v' - v'$ maps onto $\pi''(n)v'' - v''$. Thus the surjective G-morphism $V' \to V''$ restricts to a surjective P-morphism $V'(N) \to V''(N)$.

If $v' \in V'(N)$ has image 0 in $V''(N)$, then there exists $v \in V$ whose image in V' is equal to v'. According to the above lemma, there exists a compact open subgroup U of N such that $\pi'(e_U)v' = 0$. Because the map $V \to V'$ is an N-morphism, we see that $\pi(e_U)v$ maps to $\pi'(e_U)v' = 0$. Since the map $V \to V'$ is one-to-one, this gives $\pi(e_U)v = 0$. Applying the above lemma again, we find that $v \in V(N)$.

It follows that the sequence $0 \to V_N \to V'_N \to V''_N$ is exact. That the maps are M-morphisms follows from the definitions, together with the fact that the first sequence consists of G-morphisms. $\square$

Lemma. *Assume that K is a compact totally disconnected Hausdorff topological group (that is, K is profinite). Suppose that K_1 and K_2 are closed subgroups of K. Let dk_j be normalized Haar measure on K_j, $j = 1, 2$, and let e_{K_j} be the characteristic function of K_j, $j = 1, 2$. Let (π, V) be a smooth representation of K. Define $\pi(e_{K_j})v = \int_{K_j} \pi(k_j)v\, dk_j$, $j = 1, 2$, $v \in V$. If $K = K_1K_2$, then $\pi(e_K) = \pi(e_{K_1})\pi(e_{K_2})$.*

Proof. Recall that $\pi(e_K)$ is determined by the following properties. First, if $v \in V^K$, then $\pi(e_K)v = v$. Secondly, if $v \in V$ is such that $\pi(k)v = \tau(k)v$ for all $k \in K$, where τ is a nontrivial irreducible representation of K, then $\pi(e_K)v = 0$.

If $v \in V^K$, then $V^K \subset V^{K_j}$, $j = 1, 2$, implies $\pi(e_{K_1})\pi(e_{K_2})v = \pi(e_{K_1})v = v$.

Suppose that $v \in V$ is such that $\pi(k)v = \tau(k)v$ for all $k \in K$, where (τ, W) is a nontrivial irreducible representation of K. Then, because $K = K_2K_1$ implies $(W^{K_2})^{K_1} = W^K$, we have $\pi(e_{K_1})\pi(e_{K_2}) \cdot v \in (W^{K_2})^{K_1} \subset W^K = \{0\}$. $\square$

Given a partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n, let

$$N_\alpha^- = \begin{pmatrix} I_{\alpha_1} & 0 & \cdots & 0 \\ * & I_{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & I_{\alpha_r} \end{pmatrix}.$$

Then $P_\alpha^- := M_\alpha N_\alpha^-$ is a parabolic subgroup of $GL_n(F)$ such that $P_\alpha^- \cap P_\alpha = M_\alpha$. The Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(F)$ is the vector space $M_n(F)$, together with the Lie bracket $[X, Y] = XY - YX$, X, $Y \in M_n(F)$. The Lie algebra $\mathfrak{m}_\alpha$ of M_α is $\mathfrak{gl}_{\alpha_1}(F) \oplus \cdots \oplus \mathfrak{gl}_{\alpha_r}(F)$. The Lie algebras $\mathfrak{n}_\alpha$ and $\mathfrak{n}_\alpha^-$ of N_α and N_α^-, respectively, are

$$\mathfrak{n}_\alpha = \{\, X \in \mathfrak{g} \mid 1 + X \in N_\alpha \} \;\text{ and }\; \mathfrak{n}_\alpha^- = \{\, X \in \mathfrak{g} \mid 1 + X \in N_\alpha^- \}.$$

It is easy to see that $\mathfrak{g} = \mathfrak{n}_\alpha^- \oplus \mathfrak{m}_\alpha \oplus \mathfrak{n}_\alpha$ (Note that this is a direct sum of Lie subalgebras of $\mathfrak{g}$, but the summands are not ideals in $\mathfrak{g}$.)

At the group level, the set $\{\, n^- m n^+ \mid n^- \in N_\alpha^-,\, m \in M_\alpha,\, n^+ \in N_\alpha \,\}$ contains an open neighbourhood of 1, but is not equal to G.

Definition. If K is a compact open subgroup of G, and $P = MN$ is a parabolic subgroup of G, we say that K has an *Iwahori factorization relative to* P whenever $K = K^+ K^\circ K^- = K^- K^\circ K^+$, where $K^+ = K \cap N$, $K^- = K \cap M$, and $K^- = K \cap N^-$.

Example: If $G = GL_2(F)$, $K_0 = GL_n(\mathfrak{o})$, and B is the standard Borel subgroup of G, we have

$$K_0^- K_0^\circ K_0^+ = \{\, g \in G \mid g_{11},\, g_{22} \in \mathfrak{o}^\times,\; g_{12},\, g_{21} \in \mathfrak{o} \,\}.$$

Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K_0$ and this matrix does not belong to $K_0^- K_0^\circ K_0^+$, we see that K_0 does not have an Iwahori factorization relative to the standard Borel subgroup B.

Recall that if j is a positive integer, we defined

$$K_j = \{\, g \in GL_n(F) \mid g - 1 \in M_n(\mathfrak{p}^j) \,\}.$$

Lemma. *Let P be a standard parabolic subgroup of $GL_n(F)$. Then each of the compact open subgroups K_ℓ, $\ell \geq 1$, has an Iwahori factorization relative to P.*

Sketch of proof. First, let $P = B$. Let ℓ be a positive integer. Let $g = (g_{ij}) \in K_\ell$. Note that $g_{j1} g_{11}^{-1} \in \mathfrak{p}^\ell$ for $2 \leq j \leq n$. If we perform the elementary row operation that subtracts $g_{j1} g_{11}^{-1}$ times the first row of g from the jth row of g, we are left multiplying g by an elementary matrix that lies in K_ℓ^- and we get a matrix in K_ℓ whose $j1$ entry is 0. Therefore, there exists $k \in K_\ell^-$ such that the matrix $h = kg$ satisfies $h_{j1} = 0$, $2 \leq j \leq n$. If $3 \leq j \leq n$, subtracting $h_{j2} h_{22}^{-1}$ times row 2 of h from row j of h, we are left multiplying h by an elementary matrix in K_ℓ^- and getting a matrix whose first column is the same as the first column of h and whose $j2$ entry is 0. Continuing in this manner, we see that there exists $k^- \in K_\ell^-$ such that $k^- g$ is in $B \cap K_\ell$. It is clear that

$$B \cap K_\ell = (M_{(1,\dots,1)} \cap K_\ell)(N \cap K_\ell) = K_\ell^\circ K_\ell^+.$$

Hence $g \in (k^-)^{-1} K_\ell^\circ K_\ell^+ \subset K_\ell^- K_\ell^\circ K_\ell^+$.

Now let $\dot{N} = N_{(1,\dots,1)}$, $\dot{M} = M_{(1,\dots,1)}$ and $\dot{N}^- = N^-_{(1,\dots,1)}$. Suppose that $P = P_\alpha$ and $\dot{k} \in \dot{N}^-$. We can use elementary row operations (like some of the ones used above) whose matrices lie in $N_\alpha^- \cap K_\ell$ to show that there exists $k^- \in N_\alpha^- \cap K_\ell$ such that $k^- \dot{k} \in \dot{N}^- \cap M_\alpha \cap K_\ell$. This gives

$$\dot{N}^- \cap K_\ell = (N_\alpha^- \cap K_\ell)(\dot{N}^- \cap M_\alpha \cap K_\ell).$$

Similarly,

$$\dot{N} \cap K_\ell = (\dot{N} \cap M_\alpha \cap K_\ell)(N_\alpha \cap K_\ell).$$

Thus

$$\begin{aligned} K_\ell &= (\dot{N}^- \cap K_\ell)(\dot{M} \cap K_\ell)(\dot{N} \cap K_\ell) \\ &= (N_\alpha^- \cap K_\ell)(\dot{N}^- \cap M_\alpha \cap K_\ell)(\dot{M} \cap K_\ell)(\dot{N} \cap M_\alpha \cap K_\ell)(N_\alpha \cap K_\ell) \\ &\subset (N_\alpha^- \cap K_\ell)(M_\alpha \cap K_\ell)(N_\alpha \cap K_\ell). \end{aligned}$$

□

Remark: Suppose that G is an arbitrary connected reductive p-adic group. Let P be a parabolic subgroup of G. Then there exists a sequence $\{ K_j' \ | j \geq 1 \}$ of compact open subgroups of G such that each K_j' has an Iwahori factorization relative to P, and $\{ K_j' \}$ is a neighbourhood basis of the identity in G. For example, if $G = Sp_{2n}(F)$, and P is a parabolic subgroup of G, then there exists a standard parabolic subgroup P_α of $GL_{2n}(F)$ and an element $x \in GL_{2n}(F)$ such that $P = Sp_{2n}(F) \cap x P_\alpha x^{-1}$. In this case, we can take $K_j' = Sp_{2n}(F) \cap x K_j x^{-1}$, $j \geq 1$.

Theorem. *(Jacquet) Let $P = MN$ be a parabolic subgroup of G. Suppose that K is a compact open subgroup of G that has an Iwahori factorization relative to P. Let (π, V) be a smooth representation of G and let $\mathcal{Q} : V \mapsto V_N$ be the quotient map. Then*

(1) $\mathcal{Q}(V^K) = (V_N)^{K^0}$.

(2) If (π, V) is admissible, then (π_N, V_N) is admissible.

Proof. For convenience, we assume that P is a standard parabolic subgroup P_α. It is easy to see that $\mathcal{Q}(V^K) \subset V_N^{K^\circ}$.

Let $w \in V_N^{K^\circ}$. Choose $v \in V^{K^\circ}$ such that $\mathcal{Q}(v) = w$. Because

$$\mathcal{Q}(\pi(e_{K^\circ})v) = \pi_N(e_{K^\circ})\mathcal{Q}(v) = \mathcal{Q}(v) = w,$$

after replacing v by $\pi(e_{K^\circ})v$, we may assume that $v \in V^{K^\circ}$.

Let a be the element of the centre of M given by:

$$a = \begin{pmatrix} I_{\alpha_1} & 0 & \cdots & 0 \\ 0 & \varpi^{-1} I_{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi^{-(r-1)} I_{\alpha_r} \end{pmatrix}.$$

the subgroups $a^{-m}K^-a^m$, $m \geq 0$, form a neighbourhood basis of the identity in N^-. Choose ℓ such that $a^{-\ell}K^-a^\ell \subset \mathrm{Stab}_G(v)$. Set $v' = \pi(a^\ell)v$. Note that v' is fixed by K^-. Because a belongs to the centre of M, v' is also fixed by K°. Set $v'' = \pi(e_{K^+})v'$. Then, applying the lemma above,

$$\begin{aligned} \pi(e_K)v'' &= \pi(e_{K^-})\pi(e_{K^\circ})\pi(e_{K^+})v'' = \pi(e_{K^-})\pi(e_{K^\circ})\pi(e_{K^+})v' \\ &= \pi(e_{K^+})\pi(e_{K^\circ})\pi(e_{K^-})v' = \pi(e_{K^+})v' = v''. \end{aligned}$$

Thus $v'' \in V^K$. It follows that

$$\mathcal{Q}(v'') = \mathcal{Q}(\pi(e_{K^+})\pi(a^\ell)v) = \pi_N(e_{K^+})\pi_N(a^\ell)w = \pi_N(a^\ell)w,$$

the final equality holding since N acts trivially on V_N. We have shown that for any $w \in V_N^{K^\circ}$, there exists an integer ℓ_w such that $\pi_N(a^\ell)w \in \mathcal{Q}(V^K)$ for all $\ell \geq \ell_w$.

Let $w_1, \ldots, w_s \in V_N^{K^\circ}$. Let ℓ be the maximum of ℓ_{w_i}. Then $\pi_N(a^\ell)w_j \in \mathcal{Q}(V^K)$. Since $\pi_N(a^\ell) : V_N \to V_N$ is invertible,

$$\dim(\mathrm{Span}\{w_1, \ldots, w_s\}) = \dim(\mathrm{Span}\{\pi_N(a^\ell)w_j \mid 1 \leq j \leq s\}) \leq \dim(\mathcal{Q}(V^K)).$$

Thus $\dim(V_N^{K^\circ}) \leq \dim(\mathcal{Q}(V^K))$. Since we have already seen that $\mathcal{Q}(V^K) \subset V_N^{K^\circ}$, this implies $\mathcal{Q}(V^K) = V_N^{K^\circ}$.

For (2), assume that π is admissible. Let K' be a compact open subgroup of M. Then $K' \supset K_j \cap M = K_j^\circ$ for some $j \geq 1$, and so $V_N^{K'} \subset V_N^{K_j^\circ}$. According to an earlier lemma, since P is a standard parabolic subgroup of $GL_n(F)$, K_j has an Iwahori factorization relative to P. Applying part (1), we have that $V_N^{K_j^\circ}$ is finite dimensional. Therefore $\dim(V_N^{K'}) \leq \dim(V_N^{K_j^\circ}) < \infty$. $\square$

Recall that we defined normalized induction as $i_P^G\sigma = \mathrm{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$.

Definition. If (π, V) is a smooth representation of G, the *normalized Jacquet module* $r_P^G\pi$ or $r_P^G(V)$ is the representation $(\pi_N \otimes \delta_P^{-1/2}, V_N)$ of P.

With this normalization, we have

$$\begin{aligned}\mathrm{Hom}_G(\pi, i_P^G\sigma) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma \otimes \delta_P^{1/2})) = \mathrm{Hom}_M(\pi_N, \sigma \otimes \delta_P^{1/2})\\ \simeq \mathrm{Hom}_M(\pi_N \otimes \delta_P^{-1/2}, \sigma) = \mathrm{Hom}_M(r_P^G\pi, \sigma).\end{aligned}$$

7. Supercuspidal representations and Jacquet's subrepresentation theorem

Let (π, V) be a smooth representation of G.

Definition. A complex valued function on G of the form $g \mapsto \langle \widetilde{v}, \pi(g)v \rangle$, for a fixed $v \in V$ and $\widetilde{v} \in \widetilde{V}$, is called a *matrix coefficient* of π.

Note that smoothness of π implies that a matrix coefficient is invariant under left and right translation by some compact open subgroup K of G.

Definition. (π, V) is *supercuspidal* if every matrix coefficient of π is compactly supported modulo the centre of G.

If π is a smooth irreducible representation of G, Z is the centre of G, then (by Schur's Lemma), there exists a smooth one-dimensional representation $z \mapsto \chi_\pi(z)$ such that $\pi(z) = \chi_\pi(z) \cdot I$, $z \in Z$. This implies that if Z is not compact, then no nonzero matrix coefficient can be compactly supported on G.

Set $G_0 = \{\, g \in GL_n(F) \mid |\det g|_F = 1 \,\}$. The set G_0 is an open normal subgroup of G. In some situations, it is easier to work with G_0 because it has compact centre. Define $\nu : F^\times \to \mathbb{Z}$ by $|a|_F = q^{-\nu(a)}$. Then we have an exact sequence

$$1 \to G_0 \to G \to \mathbb{Z} \to 0,$$

where the map $G \to \mathbb{Z}$ is $g \mapsto \nu(\det g)$.

Lemma. *(π, V) is supercuspidal if and only if the restriction of π to G_0 is supercuspidal.*

Proof. Let Z be the centre of G. Use the fact that G is the semi-direct product of G_0 and the cyclic group generated by the matrix

$$A_\varpi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & 1 \\ \varpi & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and $\langle A_\varpi \rangle / (Z \cap \langle A_\varpi \rangle)$ is finite. Note that $Z \cap \langle A_\varpi \rangle$ consists of all matrices of the form $\varpi^j I$, where j is an integer and I is the identity. $\square$

Lemma. *If π is supercuspidal and $\pi \,|\, G_0$ is finitely generated then π is admissible.*

Proof. Let K be a compact open subgroup of G_0. Fix $v \in V$. Set $S_v = \{\, \pi(e_K)\pi(g)v \mid g \in G_0 \,\} \subset V^K$. Suppose that there exist infinitely many $g_j \in G_0$, $j \geq 1$ such that the set of vectors $\pi(e_K)\pi(g_j)v$, $j \geq 1$, is linearly independent. Observe that this forces the cosets Kg_j to be disjoint. As seen earlier, $\widetilde{V}^K = \mathrm{Hom}(V^K, \mathbb{C})$. Choose $\widetilde{v} \in \widetilde{V}^K$ such that $\langle \widetilde{v}, \pi(e_K)\pi(g_j)v\rangle = j$. We have

$$\langle \widetilde{v}, \pi(g_j)v\rangle = \langle \widetilde{\pi}(e_K)\widetilde{v}, \pi(g_j)v\rangle = \langle \widetilde{v}, \pi(e_K)\pi(g_j)v\rangle = j,$$

contradicting the fact that the matrix coefficient $g \mapsto \langle \widetilde{v}, \pi(g)v\rangle$ is compactly supported on G_0 (and thus takes only finitely many distinct values on G_0).

Therefore S_v spans a finite subspace of V^K. As V is spanned by a finite number of sets of the form $G_0 \cdot v_\ell$, it follows that V^K is spanned by a finite number of S_{v_ℓ}, hence is finite dimensional. Hence the restriction $\pi \,|\, G_0$ of π to G_0 is admissible. To see that π is admissible, note that if K is a compact open subgroup of G, then $K \subset G_0$, as $|\det(K)|_F$ is compact, so is trivial. $\square$

In the above lemma, it is not enough to assume that π is finitely generated. For example, suppose that (σ, W) is an irreducible supercuspidal representation of G_0. Let V be the countable direct sum of W, one copy W_j for each element A_ϖ^j of $\langle A_\varpi\rangle$, with a representation π_0 of G_0 defined by $\pi_0(g) \,|\, W_j = \sigma(g)$, $g \in G_0$. Define a representation (π, V) of G by $\pi \,|\, G_0 = \pi_0$, and for $w \in W_j$, set $\pi(A_\varpi)(w)$ equal to the corresponding vector $w \in W_{j+1}$. Then (π, V) is supercuspidal and finitely generated. However (π_0, V) is not finitely generated, and (π, V) is not admissible.

Lemma. *π is supercuspidal if and only if for every compact open subgroup K of G and every $v \in V$, the function $g \mapsto \pi(e_K)\pi(g)v$ is compactly supported modulo the centre of G.*

Proof. ($\Leftarrow$) Fix $v \in V$ and $\widetilde{v} \in \widetilde{V}$. Let Let K be a compact open subgroup of $\mathrm{Stab}_G(\widetilde{v})$. Then

$$\langle \widetilde{v}, \pi(g)v\rangle = \langle \widetilde{\pi}(e_K)\widetilde{v}, \pi(g)v\rangle = \langle \widetilde{v}, \pi(e_K)\pi(g)v\rangle, \qquad g \in G,$$

and so the support of $g \mapsto \langle \widetilde{v}, \pi(g)v\rangle$ is a subset of the support of $g \mapsto \pi(e_K)\pi(g)v$ for any $K \subset \mathrm{Stab}_G(\widetilde{v})$.

($\Rightarrow$) Let $\pi_0 = \pi \,|\, G_0$. Assume that π_0 is supercuspidal. It suffices to show that the functions $g \mapsto \pi_0(e_K)\pi_0(g)v$ are compactly supported on G_0. There is no loss of generality if V is replaced by $\mathrm{Span}(G_0 \cdot v)$ (every matrix coefficient of a subrepresentation is a matrix coefficient for V). So we can assume that π_0 is finitely generated. Hence π_0 is admissible (by the preceding lemma). Choose a basis $\widetilde{v}_\ell$ for the (finite dimensional) vector space $\widetilde{V}^K$. Then the support of $g \mapsto \pi_0(e_K)\pi_0(g)v$ is a subset of $\cup_\ell \mathrm{supp}(g \mapsto \langle \widetilde{v}_\ell, \pi_0(g)v\rangle)$. $\square$

Theorem. *(Harish-Chandra) A smooth representation π is supercuspidal if and only if $V_N = \{0\}$ for all proper parabolic subgroups $P = MN$ of G.*

Proof. We will show that $\pi \,|\, G_0$ is supercuspidal if and only if $V_N = \{0\}$ for all maximal proper parabolic subgroups P of G.

Let P and P_1 be parabolic subgroups such that $P_1 \subset P$, with unipotent radicals N and N_1, respectively. Then $N_1 \supset N$, and there exist Levi factors M_1 and M of P_1 and P, respectively, such that $M_1 \subset M$. (This is easy to see for standard parabolic subgroups). If $V_N = \{0\}$, then $V = V(N) \subset V(N_1) \subset V$ implies $V = V(N_1)$, that is, $V_{N_1} = \{0\}$. Therefore $V_N = \{0\}$ for all proper parabolic subgroups $P = MN$ if and only if $V_N = \{0\}$ for all maximal proper parabolic subgroups $P = MN$. A standard parabolic subgroup P_α is a maximal proper parabolic subgroup of G if and only if $\alpha = (\ell, n-\ell)$ for some ℓ, $1 \le \ell \le n-1$.

Suppose that $V_N = \{0\}$ for all proper maximal parabolic subgroups of G. Recall the Cartan decomposition of G: $G = K_0 A_+ K_0$, where $K_0 = GL_n(\mathfrak{o})$ and A_+ consists of diagonal matrices of the form $a_e = \mathrm{diag}(\varpi^{e_1}, \dots, \varpi^{e_n})$, where $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ is such that $e_1 \ge \cdots \ge e_n$. Let E be the set of e such that $\sum_{i=1}^n e_i = 0$. As $K_0 \subset G_0$, it is easy to see that the double coset $K_0 a_e K_0 \subset G_0$ if and only if $e \in E$.

Let $v \in V$ and $\widetilde{v} \in \widetilde{V}$. Choose a compact open subgroup $K \subset K_0$ such that $v \in V^K$ and $\widetilde{v} \in \widetilde{V}^K$. Let $\{g_i\}$ be a set of representatives for the (finitely many) left cosets of K in K_0. Then it follows from the Cartan decomposition for G_0 that $G_0 = \amalg_{e \in E,\, i,j}\, K g_j^{-1} a_e g_i K$. Fix k, $k' \in K$.

$$\langle \widetilde{v}, \pi(k g_j^{-1} a_e g_i k')v\rangle = \langle \widetilde{v}, \pi(g_j^{-1} a_e g_i)v\rangle = \langle \widetilde{\pi}(g_j)\widetilde{v}, \pi(a_e)\pi(g_i)v\rangle.$$

We may replace $\widetilde{v}$ by $\widetilde{\pi}(g_j)\widetilde{v}$ and v by $\pi(g_i)v$. To show that $g \mapsto \langle \widetilde{v}, \pi(g)v\rangle$ is compactly supported on G_0 for all choices of v and $\widetilde{v}$, it suffices to show that $a_e \mapsto \langle \widetilde{v}, \pi(a_e)v\rangle$ is zero for all but finitely many $e \in E$.

Let m be a positive integer. The set of $e \in E$ such that $e_\ell - e_{\ell+1} < m$ for $1 \le \ell \le n-1$, is finite. We will prove that $\langle \widetilde{v}, \pi(a_e)v\rangle = 0$ whenever e belongs to the complement of the above subset of E, for some integer m.

Fix ℓ, $1 \le \ell \le n-1$. Let $P = P_{\ell, n-\ell}$ and $e \in E$. Then

$$a_e \begin{pmatrix} I_\ell & (x_{ij}) \\ 0 & I_{n-\ell} \end{pmatrix} a_e^{-1} = \begin{pmatrix} I_\ell & (\varpi^{e_i - e_j} x_{ij}) \\ 0 & I_{n-\ell} \end{pmatrix}.$$

Above we have $1 \leq i \leq \ell$ and $\ell+1 \leq j \leq n$. For such i and j, $e_i - e_j \geq e_\ell - e_{\ell+1}$. By an earlier lemma, since $V = V(N)$, there exists a compact open subgroup U of N such that $e_U v = 0$. Assuming that $e_\ell - e_{\ell+1} \geq m_\ell$, we see that if m_ℓ is sufficiently large, $a_e U a_e^{-1} \subset \mathrm{Stab}_N(\widetilde{v})$. From $0 = \pi(a_e) e_U v = e_{a_e U a_e^{-1}} \pi(a_e) v$, it follows that

$$\langle \widetilde{v}, \pi(a_e) v\rangle = \langle e_{a_e U a_e^{-1}} \widetilde{v}, \pi(a_e) v\rangle = \langle \widetilde{v}, e_{a_e U a_e^{-1}} \pi(a_e) v\rangle = 0.$$

Take $m = \max(m_1, \ldots, m_{n-1})$.

Now assume that π is supercuspidal. Fix ℓ, $1 \leq \ell \leq n-1$ and set $P = P_{\ell, n-\ell}$. Let

$$a = \begin{pmatrix} \varpi^{n-\ell} I_\ell & 0 \\ 0 & \varpi^{-\ell} I_{n-\ell} \end{pmatrix}$$

Note that $a \in G_0 \cap Z_M$, where Z_M is the centre of M. Choose a compact open subgroup K of G such that $v \in V^K$ and $K = K^+ K^0 K^-$ has an Iwahori decomposition relative to P. By a previous lemma, π supercuspidal implies that $\pi(e_K)\pi(a^m)v = 0$ for m sufficiently large. Thus

$$0 = m(K)^{-1} \int_K \pi(k)\pi(a^m) v \, dk = m(K)^{-1} \pi(a^m) \int_K \pi(a^{-m} k a^m) v \, dk = \pi(a^m)\pi(e_{a^{-m} K a^m}) v,$$

and, as $\pi(a^m)$ is invertible, we have $\pi(e_{a^{-m} K a^m}) v = 0$ for m sufficiently large. Note that $x \mapsto a^{-1} x a$ contracts N^-, so if m is large, $a^{-m} K^- a^m \subset \mathrm{Stab}_{N^-}(v)$. From this, the fact that $a \in Z_M$, and $v \in V^K$, it follows that

$$0 = e_{a^{-m} K^+ a^m} e_{a^{-m} K^0 a^m} e_{a^{-m} K^- a^m} v = e_{a^{-m} K^+ a^m} e_{K^0} v = e_{a^{-m} K^+ a^m} v$$

for m large. As $a^{-m} K^+ a^m$ is a compact open subgroup of N, we have $v \in V(N)$. Since v was arbitrary, $V = V(N)$. $\square$

Remark. The proofs so far are valid for G equal to the direct product of finitely many general linear groups.

Suppose that $f : G \to \mathbb{C}$ is locally constant and compactly supported modulo the centre $Z = Z_G$ of G. Because $Z \subset M$ for every Levi subgroup M of G and $M \cap N = \{1\}$, for any fixed x in G, $n \mapsto f(xn)$ belongs to $C_c^\infty(N)$.

Definition. A function $f : G \to \mathbb{C}$ which is locally constant and compactly supported modulo Z is a *cusp form* if $\int_N f(xn)\, dn = 0$ for every $x \in G$ and every unipotent radical N of a proper parabolic subgroup of G. (Here dn denotes Haar measure on the unimodular group N).

Corollary. *If (π, V) is a smooth supercuspidal representation of G, then every matrix coefficient of π is a cusp form.*

Proof. Let $P = MN$ be a proper parabolic subgroup of G. By Harish-Chandra's theorem, $V = V(N)$. Therefore, for each $x \in G$, and matrix coefficient f of π, $\int_N f(xn)\, dn$ is a linear combination of integrals of the form

$$\int_N \langle \widetilde{v}, \pi(xn)(\pi(n_0)v - v)\rangle \, dn = \int_N \langle \widetilde{v}, \pi(xnn_0)v\rangle \, dn - \int_N \langle \widetilde{v}, \pi(xn)v\rangle \, dn, \ \ \widetilde{v} \in \widetilde{V}, \ v \in V, \ n_0 \in N.$$

Use $dn = d(nn_0)$ to see that the above expression equals zero. □

Some more consequences of Harish-Chandra's theorem.

Theorem. *(Jacquet) Let (π, V) be an irreducible smooth representation of G. Then there exists a parabolic subgroup $P = MN$ and an irreducible supercuspidal representation (σ, W) of M such that π is a subrepresentation of $\mathrm{Ind}_P^G \sigma$.*

Proof. Consider the set of standard parabolic subgroups P of G such that $V_N \neq \{0\}$. Fix a minimal element P of this set. Suppose that P_1 is a parabolic subgroup of G properly contained in P. Then the unipotent radical N of P is a subset of the unipotent radical N_1 of P_1. And there exist Levi factors M and M_1 of P and P_1, respectively, such that $M \supset M_1$. Note that $M \cap P_1 = M_1(M \cap N_1)$ is a parabolic subgroup of M, and $N_1 = (M \cap N_1)N$. The map $v + V(N) \mapsto v + V(N_1)$ takes V_N onto V_{N_1}. Suppose that $v \in V$, $n_1 = mn \in N_1$, $m \in M \cap N_1$, $n \in N$. Then

$$\begin{aligned} \pi(n_1)v - v + V(N) &= \pi(n_1)v - \pi(n)v + V(N) \\ &= \pi_N(m)(\pi(n)v + V(N)) - (\pi(n)v + V(N)) \in V_N(M \cap N_1). \end{aligned}$$

Recall that $V_N(M \cap N_1)$ is the span of the vectors of the form $\pi_N(n')v' - v'$, as v' and n' range over V_N and $M \cap N_1$. Therefore the above display shows that the kernel of the map $v + V(N) \mapsto v + V(N_1)$ is equal to $V_N(M \cap N_1)$. That is, $(V_N)_{M \cap N_1} = V_N / V_N(M \cap N_1) \simeq V_{N_1}$, and the process of taking Jacquet modules is transitive. (We remark that it is also the case that $\mathrm{Ind}_P^G \mathrm{Ind}_{M \cap P_1}^M \sigma \simeq \mathrm{Ind}_{P_1}^G \sigma$).

By minimality of P, $V_{N_1} = \{0\}$. As $V_{N_1} \simeq (V_N)_{M \cap N_1}$, and every proper parabolic subgroup of M is of the form $M \cap P_1$ for some $P_1 \subset P$, $P_1 \neq P$, we may apply Harish-Chandra's theorem to conclude that π_N is supercuspidal. Now π, being irreducible, is finitely generated. By an earlier result, π_N is finitely generated. By a still earlier result,

π_N has an irreducible quotient σ. It is a simple matter to check that σ is supercuspidal. By Frobenius reciprocity, $0 \neq \mathrm{Hom}_M(\pi_N, \sigma) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma)$. $\square$

Theorem. *If (π, V) is an irreducible smooth representation of $GL_n(F)$, then π is admissible.*

Proof. Fix a parabolic subgroup $P = MN$ and an irreducible supercuspidal representation σ of M such that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \neq 0$. Since σ is irreducible, σ is finitely generated. Also $Z_M M_0$ has finite index in M. Hence $\sigma \,|\, Z_M M_0$ is finitely generated. By Schur's lemma, as σ is irreducible, the centre Z_M of M acts by scalar multiples of the identity on the space of σ. Thus $\sigma \,|\, M_0$ is finitely generated. As seen earlier, this implies that σ is admissible. That in turn implies that $\mathrm{Ind}_P^G \sigma$ is admissible, and so is π, being a subrepresentation of an admissible representation. $\square$

Conjecture. *If π is irreducible and supercuspidal, there exists an open compact mod centre subgroup H of G and a representation σ of H such that $\pi \simeq c\text{-}\mathrm{Ind}_H^G \sigma$.*

Remark. The conjecture has been proved for $GL_n(F)$ ([BK1]), $SL_n(F)$, and classical groups ([S]). Constructions of some *tame* supercuspidal representations of classical groups appear in Morris ([Mo1-3]) and J.-L. Kim ([Ki1]). Yu ([Y]) has constructed large families of tame supercuspidal representations of groups that split over tamely ramified extensions of F. J.-L. Kim ([Ki2]) has shown that under certain tameness hypotheses on G, the supercuspidal representations in [Y] exhaust the irreducible supercuspidal representations of G.

Lemma. *Let (σ, W) be a smooth representation of an open compact mod centre subgroup H of G. Set $\pi =$c-$\mathrm{Ind}_H^G \sigma$. Given a matrix coefficient φ of σ, the function $\dot\varphi$ defined by $\dot\varphi(h) = \varphi(h)$ for $h \in H$, and $\dot\varphi(x) = 0$ if $x \notin H$ is a matrix coefficient of π.*

Proof. Fix $\widetilde{w} \in \widetilde{W}$ and $w \in W$ such that $\varphi(h) = \langle \widetilde{w}, \sigma(h) w \rangle_W$, $h \in H$. Define $f_w \in V =$c-$\mathrm{Ind}_H^G(W)$ by $f_w(h) = \sigma(h) w$, $h \in H$, and $f_w(x) = 0$ if $x \notin H$. Define $\widetilde{v} \in V^*$ by $\langle \widetilde{v}, f \rangle = \langle \widetilde{w}, f(1) \rangle_W$, $f \in V$. Suppose that $\pi^*(x)\widetilde{v} = \widetilde{v}$. Let K be a compact open subgroup of G such that $K \subset H$ and $\sigma \,|\, K \equiv 1_W$. Then, for every $f \in V$,

$$\langle \pi^*(xk)\widetilde{v}, f \rangle = \langle \widetilde{w}, f(k^{-1}x^{-1}) \rangle_W = \langle \widetilde{w}, f(x) \rangle_W = \langle \pi^*(x)\widetilde{v}, f \rangle = \langle \widetilde{v}, f \rangle.$$

That is $\pi^*(xk)\widetilde{v} = \widetilde{v}$ for every $k \in K$. Hence $\widetilde{v} \in V^*_{sm} = \widetilde{V}$. For $x \in G$,

$$\langle \widetilde{v}, \pi(x) f_w \rangle = \langle \widetilde{w}, f_w(x) \rangle_W = \begin{cases} \varphi(x), & \text{if } x \in H \\ 0, & \text{if } x \notin H \end{cases} = \dot\varphi(x).$$

$\square$

The matrix coefficients of $\pi = $c-Ind$_H^G\sigma$ that are of the form $\dot{\varphi}$ are compactly supported modulo Z_G. What can be said about the support of other matrix coefficients of π?

Lemma. *Suppose that (π, V) is an admissible representation of G. Then π is irreducible if and only if $\widetilde{\pi}$ is irreducible.*

Proof. First, show that if $0 \to X \to V \to Y \to 0$ is an exact sequence of G-morphisms. Then $0 \to \widetilde{Y} \to \widetilde{V} \to \widetilde{X} \to 0$ is also exact. It suffices to show that exactness holds for the subspaces K-fixed vectors for every compact open subgroup K. This is clear from $\widetilde{V}^K = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$.

Now suppose $0 \to X \to \widetilde{V} \to Y \to 0$ is exact. Then from $0 \to \widetilde{Y} \to \widetilde{\widetilde{V}} \to \widetilde{X} \to 0$ and $\widetilde{\widetilde{V}} \simeq V$, irreducibility of π forces $X = 0$ or $Y = 0$. $\square$

Corollary. *Suppose that (π, V) is an irreducible smooth representation such that some nonzero matrix coefficient of π is compactly supported modulo the centre of G. Then π is supercuspidal.*

Proof. Fix nonzero $v \in V$ and $\widetilde{v} \in \widetilde{V}$ such that $x \mapsto \langle \widetilde{v}, \pi(x)v \rangle$ is compactly supported modulo Z_G. By an earlier theorem, π is admissible. By above, $\widetilde{\pi}$ is irreducible. Therefore $V = \mathrm{span}(G \cdot v)$ and $\widetilde{V} = \mathrm{span}(G \cdot \widetilde{v})$. It follows that an arbitrary matrix coefficient of π has the form

$$x \mapsto \langle \sum_j a_j \widetilde{\pi}(g_j)\widetilde{v}, \pi(x) \sum_\ell b_\ell \pi(y_\ell)v \rangle = \sum_{j,\ell} a_j b_\ell \langle \widetilde{v}, \pi(g_j^{-1} x y_\ell)v \rangle$$

for a finite collection $\{g_j,\, y_\ell\}$ of elements of G, and complex numbers a_j, b_ℓ. The support of this matrix coefficient is contained in $\cup_{j,\ell} g_j S y_\ell^{-1}$, where S is the support of $x \mapsto \langle \widetilde{v}, \pi(x)v \rangle$. Hence all matrix coefficients of π are compactly supported modulo Z_G. $\square$

Corollary. *Suppose that $\pi = $c-Ind$_H^G\sigma$ for some smooth representation σ of an open compact modulo centre subgroup H. If π is irreducible, then π is supercuspidal (and admissible).*

Proof. Let φ be a nonzero matrix coefficient of σ. Let $\dot{\varphi}$ be the function on G that is equal to φ on H and vanishes on points in G that are not in H. According to an earlier lemma, $\dot{\varphi}$ is a matrix coefficient of π. Since $\dot{\varphi}$ is nonzero and the support of $\dot{\varphi}$ lies in H, we have a nonzero matrix coefficient of π that is compactly supported modulo Z_G. If π is irreducible, we may apply the preceding corollary to conclude that π is supercuspidal. $\square$

Remark. In the setting of the above corollary, because π =c-$\mathrm{Ind}_H^G\sigma$ is admissible, an earlier lemma tells us that $\pi = \mathrm{Ind}_H^G\sigma$.

In order to write down explicit inducing data for supercuspidal representations, we must use basic properties of characters (smooth one dimensional representations) of p-adic fields of characteristic zero. We can define a nontrivial character of $\mathbb{Q}_p$ (a smooth one-dimensional unitary representation) as follows. Given $a \in \mathbb{Q}_p$, write $a = \sum_{j=\nu(a)}^{\infty} b_j p^j$, $b_j \in \{0, 1, \ldots, p-1\}$.

$$\psi_0(a) = \begin{cases} e^{2\pi i \sum_{j=\nu(a)}^{-1} b_j p^j}, & \text{if } a \notin \mathbb{Z}_p, \\ 1, & \text{if } a \in \mathbb{Z}_p. \end{cases}$$

If F has characteristic zero, then F is a finite extension of $\mathbb{Q}_p$ for some prime p. Given $a \in F$, let $\mathrm{tr}_{F/\mathbb{Q}_p}(a) \in \mathbb{Q}_p$ be the trace of the endomorphism of the $\mathbb{Q}_p$-vector space F given by left multiplication by a. Then $\psi(a) = \psi_0(\mathrm{tr}_{F/\mathbb{Q}_p}(a))$ defines a nontrival character of F. The set of characters $\widehat{F}$ of F forms a group. Fix a nontrivial character ψ of F. For each $a \in F$, $\psi_a : b \mapsto \psi(ab)$ defines an element of $\widehat{F}$, and $a \mapsto \psi_a$ is an isomorphism $F \simeq \widehat{F}$ of locally compact groups ([T]). As ψ is smooth, there exists an integer j such that $\psi \,|\, \mathfrak{p}^j \equiv 1$ and $\psi \,|\, \mathfrak{p}^{j-1} \not\equiv 1$. Replacing ψ by $\psi_{\varpi^{j-1}}$, we see that we can choose ψ so that $\psi \,|\, \mathfrak{p} \equiv 1$ and $\psi \,|\, \mathfrak{o} \not\equiv 1$.

Example: Assume that the residual characteristic p is odd. Let $E = F(\sqrt{\varpi})$. Then $[E : F] = 2$ and $\sqrt{\varpi}$ is a uniformizer in E and $E^\times \simeq \langle\sqrt{\varpi}\rangle \times \mathfrak{o}_E^\times$. Let θ be a continuous quasi-character of $E^\times$ (that is, a smooth one dimensional representation). Assume that $\theta \,|\, \mathfrak{o}_E^\times \not\equiv 1$. Then, by smoothness of θ, there exists a smallest positive integer j such that $\theta \,|\, 1+\mathfrak{p}_E^j \equiv 1$. For convenience, we assume that j is even. We can view the restriction of θ to $1+\mathfrak{p}_E^{j/2}$ as a character of $(1+\mathfrak{p}_E^{j/2})/(1+\mathfrak{p}_E^j)$. The map $x \mapsto 1+x$ induces an isomorphism of the additive group $\mathfrak{p}_E^{j/2}/\mathfrak{p}_E^j$ and the multiplicative group $(1+\mathfrak{p}_E^{j/2})/(1+\mathfrak{p}_E^j)$. Fix a nontrivial character $\psi : F \to \mathbb{C}^\times$ of F such that $\psi \,|\, \mathfrak{p}_F \equiv 1$ and $\psi \,|\, \mathfrak{o}_F \not\equiv 1$. Then $\psi \circ \mathrm{tr}_{E/F}$ is a nontrivial character of E which is trivial on $\mathfrak{p}_E$ and nontrivial on $\mathfrak{o}_E$. A character of $\mathfrak{p}_E^{j/2}$ which is trivial on $\mathfrak{p}_E^j$ and nontrivial on $\mathfrak{p}_E^{j-1}$ has the form $x \mapsto \psi(\mathrm{tr}_{E/F}(c\varpi_E^{-j+1}x))$ for some $c \in \mathfrak{o}_E^\times$. The element c has the form $a + b\sqrt{\varpi}$ for some a, $b \in \mathfrak{o}_F$. As $c \in \mathfrak{o}_E^\times$, we must have $a \in \mathfrak{o}_F^\times$, Let $c = c_\theta$ be the c which works for θ: $\theta(1+x) = \psi(\mathrm{tr}_{E/F}(c_\theta\varpi_E^{-j+1}x))$, $x \in \mathfrak{p}_E^{j/2}$.

Set

$$\mathfrak{i}_\ell = \begin{cases} \begin{pmatrix} \mathfrak{p}^{\ell/2} & \mathfrak{p}^{\ell/2} \\ \mathfrak{p}^{\ell/2+1} & \mathfrak{p}^{\ell/2} \end{pmatrix}, & \text{if } \ell \text{ is even,} \\ \begin{pmatrix} \mathfrak{p}^{\frac{\ell+1}{2}} & \mathfrak{p}^{\frac{\ell-1}{2}} \\ \mathfrak{p}^{\frac{\ell+1}{2}} & \mathfrak{p}^{\frac{\ell+1}{2}} \end{pmatrix}, & \text{if } \ell \text{ is odd.} \end{cases}$$

For $\ell \geq 1$, set $\mathcal{I}_\ell = 1 + \mathfrak{i}_\ell$. Let $\mathcal{I} = \mathcal{I}_0 = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}$. We embed $E^\times$ in $GL_2(F)$ via the basis $\{\sqrt{\varpi}, 1\}$. With this choice, conjugation by $E^\times$ fixes each of the sets $\mathfrak{i}_\ell$, $\ell \in \mathbb{Z}$. Therefore $E^\times$ normalizes $\mathcal{I}_\ell$, $\ell \geq 0$.

The inducing subgroup is $H = E^\times \mathcal{I}_{j/2}$, which is open as it contains the open compact subgroup $\mathcal{I}_{j/2}$, and compact modulo Z (note that $Z \simeq F^\times \subset E^\times$). To define σ, we take $\sigma \,|\, E^\times = \theta$ and $\sigma(1+X) = \psi(\mathrm{tr}(c_\theta \varpi_E^{-j+1} X))$, $X \in \mathfrak{i}_{j/2}$. (Here, tr is just the usual trace). It is clear that our choice of c_θ implies that the two definitions agree on $E^\times \cap \mathcal{I}_{j/2} = 1 + \mathfrak{p}_E^{j/2}$. We will show below that c-$\mathrm{Ind}_H^G \sigma$ is irreducible.

Other examples of irreducible supercuspidal representations of $GL_2(F)$ are defined using characters of the multiplicative group of an unramified quadratic extension E_{un} of F (an extension generated by the square root of a root of unity in $\mathfrak{o}_F^\times$ which is a nonsquare). In this case, the inducing subgroup will be of the form $E^\times K_{[(j-1)/2]}$, where j is as above (relative to E_{un}) and K_ℓ, $\ell \geq 1$, is as defined earlier. If j is odd, then σ will not be one dimensional - there is something called a Heisenberg construction which is required to get an extension σ of the one dimensional representation of $E^\times K_{[(j+1)/2]}$ to $H = E^\times K_{[(j-1)/2]}$ (in order that σ will induce to an irreducible representation of G).

More generally, if $p > n$, we can get all irreducible supercuspidal representations π of $GL_n(F)$ from "admissible" quasi-characters of $E^\times$, as E varies over all degree n extensions of F ([H],[My1]). To each such character θ, there is a procedure for getting an H and σ such that $\pi \simeq$c-$\mathrm{Ind}_H^G \sigma$. When n is prime, the construction is much like the one described above for $n = 2$ (it is much easier than for n composite because of the fact that an element of E which does not belong to F must generate E over F).

Lemma. *Let (σ, W) be a smooth representation of an open compact modulo centre subgroup H of G. Set $\pi =$ c-$\mathrm{Ind}_H^G \sigma$. Given $g \in G$, define a representation σ^g of $g^{-1}Hg$ by $\sigma^g(g^{-1}hg) = \sigma(h)$, $h \in H$. Then*

$$\pi \,|\, H = \bigoplus_g \mathrm{Ind}_{H \cap g^{-1}Hg}^H (\sigma^g \,|_{H \cap g^{-1}Hg}),$$

where g ranges over a set of representatives for the H-double cosets HgH in G.

Proof. Let V =c-$\mathrm{Ind}_H^G(W)$. For convenience, let $(\kappa_g, W_g) = \mathrm{Ind}_{H\cap g^{-1}Hg}(\sigma^g\,|_{H\cap g^{-1}Hg})$. Given $f \in V$, set $Af = \sum_g \varphi_g$, where $\varphi_g \in W_g$ is defined by $\varphi_g(h) = f(gh)$, $h \in H$. Note that because π is compactly induced, the sum defining Af is finite. Let $h_1 \in H \cap g^{-1}Hg$, $h_2 \in H$. Then

$$\varphi_g(h_1h_2) = f(gh_1h_2) = f((gh_1g^{-1})gh_2) = \sigma(gh_1g^{-1})f(gh_2) = \sigma^g(h_1)\varphi_g(h_2),$$

so $\varphi \in W_g$. As H acts by right translation on V and on each W_g, it follows that A intertwines $\pi\,|\,H$ and $\oplus_g\kappa_g$. Given $\sum_g \varphi \in \oplus_g W_g$, define $f \in V$ by $f(h_1gh_2) = \sigma(h_1)\varphi_g(h_2)$, h_1, $h_2 \in H$. Check that this is an inverse of $f \mapsto Af$. $\square$

Definition: Let (σ, W) be an irreducible smooth representation of an open and closed subgroup H of G. The *Hecke algebra of* $\widetilde{\sigma}$*-spherical functions* $\mathcal{H}(G/\!\!/H, \sigma)$ is the set of functions $f : G \to \mathrm{End}_{\mathbb{C}}(\widetilde{W})$ which are compactly supported modulo H and satisfy $f(h_1gh_2) = \widetilde{\sigma}(h_1)f(g)\widetilde{\sigma}(h_2)$, $g \in G$, h_1, $h_2 \in H$.

Proposition. *Let (σ, W) be an irreducible smooth representation of an open and closed subgroup H of G. Let π =c-$\mathrm{Ind}_H^G\sigma$. Then $\mathrm{Hom}_G(\pi) \simeq \mathcal{H}(G/\!\!/H, \widetilde{\sigma})$ (as complex vector spaces).*

Proof. Let $A \in \mathrm{Hom}_G(\pi)$. Given $w \in W$, define $f_w \in V$ =c-$\mathrm{Ind}_H^G W$ by $f_w(h) = \sigma(h)w$ and $f_w(x) = 0$ if $x \notin H$. Define $\varphi_A : G \to \mathrm{End}_{\mathbb{C}}(W)$ by $\varphi_A(x)w = (Af_w)(x)$, $x \in G$. The support of φ_A is a subset of the union of the supports of the functions Af_{w_α}, as w_α ranges over a basis of W. Now σ is irreducible and $H/(Z_G \cap H)$ is compact, so W is finite dimensional. Therefore, as each Af_{w_α} is compactly supported modulo Z_G, φ_A is compactly supported modulo H. For h_1, $h_2 \in H$, $x \in G$, and $w \in W$,

$$\begin{aligned}\varphi_A(h_1xh_2)w &= (Af_w)(h_1xh_2) = \sigma(h_1)(Af_w)(xh_2) = \sigma(h_1)(A\pi(h_2)f_w)(x)\\ &= \sigma(h_1)Af_{\sigma(h_2)w}(x) = \sigma(h_1)\varphi_A(x)\sigma(h_2)w.\end{aligned}$$

Thus $\varphi_A \in \mathcal{H}(G/\!\!/H, \widetilde{\sigma})$.

For fixed $x \in G$, $\varphi \in \mathcal{H}(G/\!\!/H, \widetilde{\sigma})$ and $f \in V$. The function $g \mapsto \varphi(xg^{-1})f(g) \in W$ is constant on right H cosets in G. It is supported on a finite union of right H cosets (since φ is). Let $\mathcal{R}$ be a set of representatives for the right H cosets in G. Set $(A_\varphi f)(x) = \sum_{g\in\mathcal{R}} \varphi(xg^{-1})f(g)$ (this is really a finite sum). Now choose $S \subset G$ such that the support of φ is contained in S and the image of S in $H\backslash G$ is compact. There exists a finite subset $\mathcal{R}_0$ of $\mathcal{R}$ (depending on f) such that $f(g) = 0$ for $g \in \mathcal{R} - \mathcal{R}_0$. It follows that the support of $A_\varphi f$ is contained in $\cup_{g\in\mathcal{R}_0} Sg$. Also, it is immediate from the definition that $(A_\varphi f)(hx) = \sigma(h)(A_\varphi f)(x)$, for $h \in H$ and $x \in G$. Hence $A_\varphi f \in V$. Let x, $y \in G$. Then, since the definition of A_φ is independent of the choice of $\mathcal{R}$, we have

$$(\pi(y)A_\varphi f)(x) = (A_\varphi f)(xy) = \sum_{g\in\mathcal{R}} \varphi(xyg^{-1})f(g) = \sum_{g\in\mathcal{R}y^{-1}} \varphi(xg^{-1})f(gy)$$
$$= \sum_{g\in\mathcal{R}y^{-1}} \varphi(xg^{-1})(\pi(y)f)(g) = (A_\varphi \pi(y)f)(x).$$

Thus $A_\varphi \in \mathrm{Hom}_G(\pi)$.

To complete the proof check that $A \mapsto \varphi_A$ and $\varphi \mapsto A_\varphi$ are inverses. $\square$

Lemma. *Let $g \in G$. The subspace of $\mathcal{H}(G/\!\!/H, \widetilde{\sigma})$ consisting of functions supported on HgH is isomorphic to $\mathrm{Hom}_{H\cap g^{-1}Hg}(\sigma, \sigma^g)$.*

Proof. Suppose that φ is supported on HgH. Then for $h \in H \cap g^{-1}Hg$,

$$\varphi(g)\sigma(h) = \varphi(gh) = \varphi(ghg^{-1}g) = \sigma(ghg^{-1})\varphi(g) = \sigma^g(h)\varphi(g).$$

The map $\varphi \mapsto \varphi(g)$ is the isomorphism. $\square$

Corollary. *Let $\pi =$c-$\mathrm{Ind}_H^G\sigma$. Then π is irreducible (admissible, supercuspidal) if and only if $\mathrm{Hom}_{H\cap g^{-1}Hg}(\sigma, \sigma^g) = \{0\}$ whenever $g \notin H$.*

Proof. Suppose that $\varphi \in \mathcal{H}(G/\!\!/H, \widetilde{\sigma})$ is supported on H. Let $A_\varphi \in \mathrm{Hom}_G(\pi)$ be defined as in the proof of the above proposition. Let $f \in V =$c-$\mathrm{Ind}_H^G W$ and $x \in G$. There exists a unique $g_0 \in \mathcal{R}$ and $h \in H$ such that $x = hg_0$. Then

$$(A_\varphi f)(x) = \sum_{g\in\mathcal{R}} \varphi(xg^{-1})f(g) = \varphi(h)f(g_0) = \varphi(1)\sigma(h)f(h^{-1}x) = \varphi(1)f(x),$$

as $\varphi(xg^{-1}) = 0$ unless $xg^{-1} \in H$. Now $\varphi(1) \in \mathrm{Hom}_H(\sigma)$, so by Schur's lemma, $\varphi(1)$ is a scalar multiple c_φ of the identity on W. Thus $A_\varphi f = c_\varphi f$. So the subspace of $\mathrm{Hom}_G(\pi)$ consisting of the scalar operators on V is isomorphic to the subspace of functions in $\mathcal{H}(G/\!\!/H, \widetilde{\sigma})$ which are supported on H. Thus (using Schur's lemma for one direction), the corollary follows. $\square$

Let E, $\mathfrak{i}_\ell$, $\ell \in \mathbb{Z}$ and $\mathcal{I}_\ell$, $\ell \geq 0$ be defined as in the example. The following results are used in applying the above irreducibility criterion to the induced representation considered in the example. Let $E^\perp$ denote the orthogonal complement to E in $M_2(F)$ (relative to the trace map).

Lemma. $i_\ell = \mathfrak{p}_E^\ell \oplus (E^\perp \cap \mathfrak{i}_\ell)$.

Lemma. *Let m be an odd integer. The map $X \mapsto \varpi_E^m X - X\varpi_E^m$ takes $E^\perp \cap \mathfrak{i}_\ell$ onto $E^\perp \cap \mathfrak{i}_{\ell+m}$.*

Let Ad denote the adjoint representation of $GL_n(F)$ on $M_n(F)$: $\mathrm{Ad}\, g(X) = gXg^{-1}$, $g \in GL_n(F)$, $X \in M_n(F)$.

Proposition. *Let $a \in \mathfrak{o}_F^\times$. Let m be an odd integer. Assume that $\ell \in \mathbb{Z}$ is such that $\ell > m$. Then*

$$\varpi_E^m a + \mathfrak{i}_\ell = \mathrm{Ad}\,\mathcal{I}_{\ell-m}(\varpi_E^m a + \mathfrak{p}_E^\ell).$$

Proof. Take $X \in \varpi_E^m a + \mathfrak{i}_\ell$. Using one of the lemmas, write $X = \varpi_E^m a + \alpha + Y$, $\alpha \in \mathfrak{p}_E^\ell$, and $Y \in E^\perp \cap \mathfrak{i}_\ell$. Using another lemma, write $Y = \varpi_E^m Z - Z\varpi_E^m$ for some $Z \in E^\perp \cap \mathfrak{i}_{\ell-m}$. Set $X' = \mathrm{Ad}\,(1-Z)(X)$. Check (exercise) that $X' \in \varpi_E^m a + \alpha + \mathfrak{i}_{2\ell-m}$. Since $\ell > m$, we have $2\ell - m > \ell$, so $X' \in \varpi_E^m a + \mathfrak{p}_E^\ell + \mathfrak{i}_{2\ell-m}$ is an $\mathcal{I}_{\ell-m}$-conjugate of X which is closer to $\varpi_E^m a + \mathfrak{p}_E^\ell$ than X. Continue inductively- using completeness to produce a sequence of $\mathcal{I}_{\ell-m}$-conjugates of X which converges to an element of $\varpi_E^m a + \mathfrak{p}_E^\ell$. $\square$

Lemma. *Let $a \in \mathfrak{o}_F^\times$. Assume that m is an odd integer. Suppose that $(\varpi_E^m a + \mathfrak{p}_E^{m+1}) \cap \mathrm{Ad}\, g^{-1}(\varpi_E^m a + \mathfrak{p}_E^{m+1}) \neq \emptyset$. Then $g \in E^\times$.*

Proof. Let α, $\beta \in \mathfrak{p}_E$ and $g \in G$ be such that $\varpi_E^m(a+\alpha) = g^{-1}\varpi_E^m(a+\beta)g$. After multiplying both sides by a^{-1} we can assume that $a = 1$. Taking p^rth powers of both sides, and multiplying by $\varpi_F^{-m(p^r-1)/2}$, which commutes with g, we get, using $\varpi_F = \varpi_E^2$,

$$\varpi_E^m(1+\alpha)^{p^r} = g^{-1}\varpi_E(1+\beta)^{p^r} g.$$

As r tends to infinity $(1+\alpha)^{p^r}$ and $(1+\beta)^{p^r}$ approach 1. This forces g to commute with ϖ_E^m, which, as m is odd, generates E over F. Hence $g \in E^\times$. $\square$

Example: Returning to the example, keeping the same notation. Suppose that $X \in g^{-1}\mathfrak{i}_{j/2}g$. Then

$$\sigma^g(1+X) = \sigma(1+gXg^{-1}) = \psi(\mathrm{tr}(c_\theta\varpi_E^{-j+1}gXg^{-1})) = \psi(\mathrm{tr}(g^{-1}c_\theta\varpi_E^{-j+1}gX)).$$

As σ and σ^g are one dimensional, $\mathrm{Hom}_{H\cap g^{-1}Hg}(\sigma,\sigma^g) \neq 0$ if and only if σ and σ^g coincide on $H \cap g^{-1}Hg$. Suppose that σ and σ^g coincide on $\mathcal{I}_{j/2} \cap g^{-1}\mathcal{I}_{j/2}g$. Then

$$\psi(\mathrm{tr}((g^{-1}c_\theta\varpi_E^{-j+1}g - c_\theta\varpi_E^{-j+1})X)) = 1 \qquad \text{for all } X \in g^{-1}\mathfrak{i}_{j/2}g \cap \mathfrak{i}_{j/2}.$$

It is straightforward to check that $\mathfrak{i}_{-\ell+1} = \{\, Y \in M_2(F) \mid \mathrm{tr}(Y\mathfrak{i}_\ell) \in \mathfrak{p} \,\}$. Recall that ψ was chosen to have conductor $\mathfrak{p}$. Therefore we must have

$$g^{-1} c_\theta \varpi_E^{-j+1} g - c_\theta \varpi_E^{-j+1} \in \mathrm{Ad}\, g^{-1}(\mathfrak{i}_{-j/2+1}) + \mathfrak{i}_{-j/2+1}.$$

Now assume that $g \notin E^\times$. Then

$$\mathrm{Ad}\, g^{-1}(c_\theta \varpi_E^{-j+1} + \mathfrak{i}_{-j/2+1}) \cap (c_\theta \varpi_E^{-j+1} + \mathfrak{i}_{-j/2+1}) \neq \emptyset.$$

Applying the above proposition (ignoring the fact that $c_\theta = a + b\sqrt{\varpi}$, $a \in \mathfrak{o}_F^\times$ and $b \in \mathfrak{o}_F$ might be nonzero - minor adjustments can be made to deal with this), we have

$$\mathrm{Ad}\,(h_1 g h_2)^{-1}(c_\theta \varpi_E^{-j+1} + \mathfrak{p}_E^{-j/2+1}) \cap (c_\theta \varpi_E^{-j+1} + \mathfrak{p}_E^{-j/2+1}) \neq \emptyset$$

for some h_1, $h_2 \in \mathcal{I}_{j/2}$. Because $j \geq 2$, $-j/2+1 \geq -j+2$ and we can apply a lemma from above to conclude that $h_1 g h_2 \in E^\times$. Therefore $g \in E^\times \mathcal{I}_{j/2} = H$. $\square$

8. Depth zero supercuspidal representations

General results about depth zero supercuspidal representations of reductive p-adic groups may be found in [MP2], [Mo4] and [Mo5]. In the first part of the section, we discuss a basic result about interwining properties of representations of compact open subgroups of a locally profinite group which are contained in the same irreducible smooth representation of the locally profinite group. Then we describe the standard parahoric subgroups of general linear groups. After that, we make some comments about depth zero K-types and depth zero supercuspidal representations.

Let G be a locally profinite group. If H is a subgroup of G, $g \in G$, and σ is a representation of H, let $H^g = g^{-1}Hg$. and let σ^g be the representation of H^g defined by $\sigma^g(g^{-1}hg) = \sigma(h)$, $h \in H$.

Definition. Let τ_j be an irreducible smooth representation of a compact open subgroup K_j of G, $j = 1, 2$. Let $g \in G$. We say that g *intertwines* τ_1 *with* τ_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(\tau_1^g, \tau_2) \neq \{0\}.$$

If $K_1 = K_2$ and $\tau_1 = \tau_2$, we say that g *interwines* τ_1 whenever g intertwines τ_1 with itself.

Remark. Because the restrictions of τ_1^g and τ_2 to $K_1^g \cap K_2$ are semisimple, the dimensions of

$$\mathrm{Hom}_{K_1^g \cap K_2}(\tau_1^g, \tau_2) \quad \text{and} \quad \mathrm{Hom}_{K_1^g \cap K_2}(\tau_2, \tau_1^g) \simeq \mathrm{Hom}_{K_1 \cap K_2^{g^{-1}}}(\tau_2^{g^{-1}}, \tau_1)$$

are equal. It follows that g interwines τ_1 with τ_2 if and only if g^{-1} intertwines τ_2 with τ_1.

Definition. Let σ be a smooth irreducible representation of a closed subgroup H of G and let (π, V) be a smooth representation of G. We say that π *contains* σ, or σ *occurs in* π if σ is a subrepresentation of the restriction $\pi \,|\, H$, that is, $\mathrm{Hom}_H(\sigma, \pi) \neq \{0\}$. If H is a compact open subgroup of G and π contains σ, we say that σ is a *K-type* of π.

Lemma 1. *Let τ_j be an irreducible smooth representation of a compact open subgroup K_j of G, $j = 1, 2$. Let (π, V) be an irreducible smooth representation of G that contains τ_1 and τ_2. Then there exists $g \in G$ which intertwines τ_1 with τ_2.*

Outline of proof: The restriction of π to K_j is semisimple. Let V^{τ_j} be the sum of all of the irreducible K_j-subrepresentations of V that are equivalent to τ_j. (Often, V^{τ_j} is referred to as the *τ_j-isotypic component of V*.) Define $e_{\tau_j} \in C_c^\infty(G)$ by

$$e_{\tau_j}(g) = \begin{cases} m(K_j)^{-1}(\dim \tau_j)\,\mathrm{trace}\,\tau_j(k^{-1}), & \text{if } g = k \in K_j, \\ 0, & \text{if } g \notin K_j. \end{cases}$$

Check that $\pi(e_{\tau_j})$ is K_j-projection of V onto the subspace V^{τ_j}.

Because V^{τ_1} is nonzero and π is irreducible, we have

$$V = \mathrm{Span}\{\pi(g)V^{\tau_1} \mid g \in G\}.$$

This implies that

$$\pi(e_{\tau_2})V = V^{\tau_2} = \mathrm{Span}\{\pi(e_{\tau_2})\pi(g)V^{\tau_1} \mid g \in G\}.$$

Since V^{τ_2} is nonzero, there exists $g \in G$ such that $\pi(e_{\tau_2})\pi(g)V^{\tau_1}$ is nonzero. Check that this g intertwines τ_1 with τ_2. $\square$

For the rest of this section, we assume that G is a connected reductive p-adic group. The affine building $\mathcal{B}(G)$ is a polysimplicial complex on which the group G acts. (References: [Ti], [BT1], [BT2]). The building $\mathcal{B}(G)$ is obtained by pasting together copies (called *apartments*) of an affine Euclidean space $\mathcal{A}$. The affine root system partitions $\mathcal{A}$ into facets. The stabilizer of a facet contains a certain compact open subgroup called a *parahoric* subgroup of G. In some examples, a parahoric subgroup is the full stabilizer of a facet.

Let q be the cardinality of $\mathfrak{o}/\mathfrak{p}$. Let $K_0 = GL_n(\mathfrak{o})$ and let $K_1 = \{ g \in K_0 \mid g - 1 \in M_n(\mathfrak{p}) \}$. The quotient K_0/K_1 is isomorphic to $GL_n(\mathbb{F}_q)$. The group K_0 is a maximal parahoric subgroup of $GL_n(F)$.

Definition.

(1) If $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a partition of n, let $P_\alpha(\mathbb{F}_q)$ be the associated standard parabolic subgroup of $GL_n(\mathbb{F}_q)$. The *standard parahoric subgroup* K^α *of* $GL_n(F)$ *associated to the partition* α is defined to be the inverse image of $P_\alpha(\mathbb{F}_q)$ in K_0 (relative to the quotient map $K_0 \to K_0/K_1$).

(2) A *parahoric subgroup* of $GL_n(F)$ is a conjugate of a standard parahoric subgroup of $GL_n(F)$.

A profinite group K is a pro-p-group if K/K' is a finite p-group for every open normal subgroup K' of K. Equivalently, K is a projective limit of p-groups.

Definition. If K is a parahoric subgroup of G, the *pro-unipotent radical* (or *pro-p-radical*) K^u of K is the maximal normal pro-p-subgroup of K.

If α is a partition of n, then $K^{\alpha\, u}$ is the inverse image of the unipotent radical $N_\alpha(\mathbb{F}_q)$ of $P_\alpha(\mathbb{F}_q)$ in K_0. Note that $K^\alpha/K^{\alpha\, u} \simeq M_\alpha(\mathbb{F}_q)$.

Definition. An irreducible smooth representation (π, V) of G has *depth zero* if there exists a parahoric subgroup K of G such that $V^{K^u} \neq \{0\}$.

Definition. Let $\mathbb{H}$ be a connected reductive $\mathbb{F}_q$-group and let $H = \mathbb{H}(\mathbb{F}_q)$. An irreducible representation σ of H is said to be *cuspidal* if every matrix coefficient f of σ is a cusp form:

$$\sum_{n \in N(\mathbb{F}_q)} f(xn) = 0, \qquad \forall\, x \in P(\mathbb{F}_q),$$

whenever $P(\mathbb{F}_q)$ is a proper parabolic subgroup of H, with unipotent radical $N(\mathbb{F}_q)$.

Lemma 2. *Let notation be as in the above definition. Let V be the space of σ. Then*

(1) σ is cuspidal if and only if for every proper parabolic subgroup $P(\mathbb{F}_q)$ of H (with unipotent radical $N(\mathbb{F}_q)$) the space $V^{N(\mathbb{F}_q)} = \{0\}$. (Here $V^{N(\mathbb{F}_q)}$ is the space of of $N(\mathbb{F}_q)$-fixed vectors in V.)

(2) If σ is not cuspidal, then there exists a proper parabolic subgroup $P(\mathbb{F}_q)$ of H and an irreducible cuspidal representation ρ of a Levi factor $M(\mathbb{F}_q)$ of $P(\mathbb{F}_q)$ such that σ is a subrepresentation of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{H} \rho$.

Lemma 3. *For $j = 1, 2$, let $\dot{\tau}_j$ be an irreducible representation of $GL_n(\mathbb{F}_q)$. Let τ_j be the inflation of $\dot{\tau}_j$ to K_0: $\tau_j(k) = \dot{\tau}_j(kK_1)$. Assume that τ_1 is cuspidal. Then*

(1) There exists a $g \in G$ that intertwines τ_1 with τ_2 if and only if and $\dot{\tau}_1 \simeq \dot{\tau}_2$. In that case, $g \in ZK_0$.

(2) An element $g \in G$ interwines τ_1 if and only if $g \in ZK_0$.

Idea of proof in case $n = 2$: Let $g \in G$, k_1, $k_2 \in K_0$ and $z \in Z$. Note that g intertwines τ_1 with τ_2 if and only if $k_1 g z k_2$ intertwines τ_1 with τ_2. Therefore, taking into account the Cartan decomposition, there is no loss of generality if we assume that

$$g = \begin{pmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}$$

for some nonnegative integer ℓ. Assume that g^{-1} intertwines τ_1 with τ_2. Then g intertwines τ_2 with τ_1 (see remark above). If $\ell = 0$, then (1) clearly holds. Assume $\ell > 0$. Check that the group $K_1^g \cap K_0$ contains

$$N_0 = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathfrak{o} \}.$$

Note that τ_2^g is trivial on $K_1^g \cap K_0$. Because $\dot{\tau}_1$ is cuspidal, τ_1 cannot contain the trivial representation of N_0. Thus

$$\{0\} = \mathrm{Hom}_{N_0}(\tau_2^g, \tau_1) \supset \mathrm{Hom}_{K_0^g \cap K_0}(\tau_2^g, \tau_1).$$

For (2), apply part (1) with $\tau_1 = \tau_2$.

This approach generalizes to $n > 2$ quite easily. $\square$

Proposition 1. *Let $\dot{\tau}$ be an irreducible cuspidal representation of $GL_n(\mathbb{F}_q)$. Let τ be the inflation of $\dot{\tau}$ to K_0. Let W be the space of $\dot{\tau}$.*

(1) Let $c \in \mathbb{C}^\times$. There is a unique extension τ_c of τ to ZK_0 such that $\tau(\varpi \cdot I_n) = c \cdot Id_W$. Let $\pi =$c-$\mathrm{Ind}_{ZK_0}^{GL_n(F)} \tau_c$. Then π is a depth zero irreducible supercuspidal representation of $GL_n(F)$.

(2) Let π' be an irreducible smooth representation of $GL_n(F)$ that contains τ. Then there exists a $c \in \mathbb{C}^\times$ such that $\pi' \simeq$c-$\mathrm{Ind}_{ZK_0}^{GL_n(F)} \tau_c$.

Proof. For part (1), recall that a corollary from Section 7 (page 60) tells us that π_c is irreducible if and only if $g \notin ZK_0$ implies g does not intertwine τ_c. Because any element that intertwines τ_c also intertwines τ, we may apply Lemma 3 to see that any $g \notin ZK_0$ does not intertwine τ_c.

For part (2), assume that π' contains τ. Because the restriction of π' to ZK_0 is semisimple, there exists an irreducible representation τ' of ZK_0 that occurs in π' and such that the restriction of τ' to K_0 contains τ. Because Z commutes with K_0, it follows from irreducibility of τ' (together with Schur's lemma) that $\tau' = \tau_c$ for some $c \in \mathbb{C}^\times$. Thus $\mathrm{Hom}_{ZK_0}(\tau_c, \pi') \neq \{0\}$. According to the variant of Frobenius reciprocity mentioned in a remark from Section 5 (page 31), because ZK_0 is open, we have

$$\mathrm{Hom}_{ZK_0}(\tau_c, \pi') \simeq \mathrm{Hom}_G(\mathrm{c{-}Ind}_{ZK_0}^G \tau_c, \pi').$$

Since both of the representations c-$\mathrm{Ind}_{ZK_0}^G \tau_c$ and π' are irreducible, this implies that they are equivalent. $\square$

Proposition 2. *([MP2]) Suppose that K is a maximal parahoric subgroup of connected reductive p-adic group G, with pro-unipotent radical K^u. Let σ be a representation of K that is trivial when restricted to K^u and factors to an irreducible cuspidal representation of $K/K^u \simeq \mathbb{H}(\mathbb{F}_q)$. If τ is an irreducible representation of the normalizer $N_G(K)$ and* *the restriction $\tau \,|\, K$ contains σ, then $c\text{--}\mathrm{Ind}_{N_G(K)}^G \tau$ is a depth zero irreducible cuspidal representation of G.*

When $G = GL_n(F)$, there is only one conjugacy class of maximal parahoric subgroups, and $N_G(K) = ZK$ for all such subgroups. Most reductive p-adic groups have more than one conjugacy class of maximal parahoric subgroups. Also, if K is a maximal parahoric subgroup, $N_G(K)$ can be larger than ZK.

Lemma 4. *Let π be a depth zero irreducible smooth representation of a reductive p-adic group G. There exists a parahoric subgroup K of G such that $\pi \,|\, K$ contains an irreducible representation τ of K that is trivial on K^u and τ factors to an irreducible cuspidal representation of K/K^u.*

Sketch of proof: By definition of depth zero representation, we know that there exist K and τ as above, except that τ factors to an irreducible representation $\dot{\tau}$ of K/K^u that is not necessarily cuspidal. There exists a connected reductive $\overline{F}_q$-group $\mathbb{H}$ such that $K/K^u \simeq \mathbb{H}(\mathbb{F}_q)$. There exist a parabolic subgroup $P(\mathbb{F}_q)$ of $\mathbb{H}(\mathbb{F}_q)$ with Levi factor $M(\mathbb{F}_q)$ and an irreducible cuspidal representation $\dot{\sigma}$ of $M(\mathbb{F}_q)$ such that $\dot{\tau}$ is a subrepresentation of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{\mathbb{H}(\mathbb{F}_q)} \dot{\sigma}$. According to Frobenius reciprocity,

$$\mathrm{Hom}_{\mathbb{H}(\mathbb{F}_q)}(\dot{\tau}, \mathrm{Ind}_{P(\mathbb{F}_q)}^{\mathbb{H}(\mathbb{F}_q)} \dot{\sigma}) \simeq \mathrm{Hom}_{P(\mathbb{F}_q)}(\dot{\tau}, \dot{\sigma}).$$

Hence $\dot{\sigma}$ occurs as a quotient of $\dot{\tau} \,|\, P(\mathbb{F}_q)$. Because we are working with represntations of a finite group, it follows that $\dot{\sigma}$ is a subrepresentation of $\dot{\tau} \,|\, P(\mathbb{F}_q)$. Let K' be the inverse image of $P(\mathbb{F}_q)$ in K. Then K' is a parahoric subgroup of G and $K'/K'^{\,u} \simeq P(\mathbb{F}_q)/N(\mathbb{F}_q) \simeq M(\mathbb{F}_q)$, where $N(\mathbb{F}_q)$ is the unipotent radical of $P(\mathbb{F}_q)$. The restriction of π to K contains τ and the restriction of τ to K' contains the inflation σ of $\dot{\sigma}$ to K'. Replacing K by K' and τ by σ, we see that the conclusion of the lemma holds. $\square$

Definition. ([MP1]) An *unrefined minimal K-type of depth zero* is a pair (K, τ) where K is a parahoric subgroup of G and τ is a representation of K that is trivial on K^u and factors to an irreducible cuspidal representation of K/K^u.

Theorem 1. *([MP1]) Let (π, V) be an irreducible smooth representation of G. Let (K, τ) and (K', τ') be unrefined minimal K-types that occur in π. Then (K, τ) and (K', τ') are associates of each other. That is, there exists $g \in G$ such that the images of $K \cap K'^{\,g}$ in $K/K^u \simeq \mathbb{H}(\mathbb{F}_q)$ and in $K'^{\,g}/K'^{\,g\,u}$ are both equal to $\mathbb{H}(\mathbb{F}_q)$, and $\tau \simeq {}^g\tau'$.*

If K is a parahoric subgroup of G, we may attach a Levi subgroup of G to K as follows. Let S be a maximal F-split torus in G such that $S \cap K$ is the maximal compact subgroup of S and the image of $S \cap K$ in K/K^u is a maximal $\mathbb{F}_q$-split torus in the group $K/K^u \simeq \mathbb{H}(\mathbb{F}_q)$. Let C be the maximal $\mathbb{F}_q$-split torus in the centre of $\mathbb{H}(\mathbb{F}_q)$. Let C' be a subtorus of S such that the image of $C' \cap K$ in $\mathbb{H}(\mathbb{F}_q)$ is equal to C. The centralizer M of C' in G is a Levi subgroup of G. The intersection $K \cap M$ is a maximal parahoric subgroup of M. Moreover, $C' \subset Z$ if and only if $M = G$ if and only if K is a maximal parahoric subgroup of G.

Example: $G = GL_n(F)$, $\alpha = (\alpha_1, \ldots, \alpha_f)$ a partition of n. Let K be the standard parahoric subgroup of G associated to α. Then $K/K^u \simeq \prod_{j=1}^r GL_{\alpha_j}(\mathbb{F}_q)$, and the subgroup of matrices of the form $(a_1 I_{\alpha_1}, \ldots, a_r I_{\alpha_r})$, $a_j \in \mathbb{F}_q^\times$ is a maximal $\mathbb{F}_q$-split torus in the centre of K/K^u. We can take S to be the subgroup of diagonal matrices in G and C' equal to the group of matrices $(a_1 I_{\alpha_1}, \ldots, a_r I_{\alpha_r})$, $a_j \in F^\times$. Clearly the centralizer of C' in $GL_n(F)$ is the standard Levi subgroup $M_\alpha = \prod_{j=1}^r GL_{\alpha_j}(F)$, and $K \cap M \simeq \prod_{j=1}^r GL_{\alpha_j}(\mathfrak{o})$ is a maximal parahoric subgroup of M.

Theorem 2. *([MP2]) Let (π, V) be an irreducible smooth representation of G. Let K be a parahoric subgroup of G and let M be the associated Levi subgroup of G. Let $P = MN$ be a parabolic subgroup of P with Levi factor M. Let $\mathcal{Q} : V \to V_N = V/V(N)$ be the quotient map. Then $\mathcal{Q} \,|\, V^{K^u} \to V_N^{K^u \cap M}$ is an isomorphism.*

Remark. The subgroup K^u has an Iwahori factorization relative to P. So we may apply an earlier theorem (due to Jacquet) to see that $\mathcal{Q}(V^{K^u}) = V_N^{K^u \cap M}$.

Theorem 3. *Let (π, V) be a depth zero irreducible supercuspidal representation of G. Then there exist a maximal parahoric subgroup K of G, an irreducible representation τ of $N_G(K)$ such that $\tau \,|\, K$ contains a representation σ of K that factors to an irreducible cuspidal representation of K/K^u and $\pi \simeq$c-$\mathrm{Ind}_{N_G(K)}^K \tau$. Furthermore, if (K', τ') is an unrefined minimal K-type, and K' is not conjugate to K, then τ' does not occur in π.*

Proof. The first conclusion of the theorem can be proved using Theorem 2. According to Lemma 4, there exist a parahoric subgroup K of K^u and a representation σ of K that is trivial on K^u and factors to an irreducible cuspidal representation of K/K^u, such that $\mathrm{Hom}_K(\sigma, \pi) \neq 0$. Let M be the Levi subgroup of G associated to K. If $M \neq G$, let $P = MN$ be a parabolic subgroup of G with Levi factor M. Because V^{K^u} is nonzero by assumption, it follows from Theorem 2 that $V_N^{K^u \cap M} \neq 0$. Therefore V_N is nonzero. That is, π is not supercuspidal. Hence $M = G$, which is equivalent to K being maximal. Now consider $\pi \mid N_G(K)$. Because $\mathrm{Hom}_K(\sigma, \pi) \neq 0$, there exists an irreducible constituent τ of $\pi \mid N_G(K)$ such that $\mathrm{Hom}_K(\sigma, \tau) \neq 0$. Because $\mathrm{Hom}_{N_G(K)}(\tau, \pi)$ is nonzero, and the variant of Frobenius reciprocity from Section 5 (page 31) tells us that $\mathrm{Hom}_{N_G(K)}(\tau, \pi) \simeq \mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{N_G(K)}^G \tau, \pi)$, we find that $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{N_G(K)}^G \tau, \pi)$ is nonzero. Applying Proposition 2, we have that $\text{c-}\mathrm{Ind}_{N_G(K)}^G \tau$ is irreducible. By assumption, π is irreducible. Therefore $\pi \simeq \text{c-}\mathrm{Ind}_{N_G(K)}^G \tau$.

For the second conclusion of the theorem, suppose that $\mathrm{Hom}_{K'}(\tau', \pi) \neq 0$. Then, we may apply Theorem 1 to conclude that if σ is any irreducible constituent of $\pi \mid K$, then (K, σ) and (K', τ') are associate. If $g \in G$ and $K'^{\,g} \neq K$, then it is possible to show that $K \cap K'^{\,g}$ lies inside a nonmaximal parahoric subgroup of G, so cannot have image in K/K^u equal to $\mathbb{H}(\mathbb{F}_q)$. This means that (K, σ) and (K', τ') cannot be associate. For more details, refer to [MP1]. $\square$

9. Parahoric subgroups of symplectic groups

In this section, we realize $G = Sp_{2n}(F)$ as $\{\, g \in GL_)2n(F) \mid {}^t gJg = J \,\}$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The subgroup $A = \{\, \mathrm{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) \mid a_1, \ldots, a_n \in F^\times \,\}$ is an F-split maximal torus in G. Before discussing parahoric subgroups, we list some information about parabolic subgroups of G. For $1 \le j \le n-1$, let $A^{(j)}$ be the n by n elementary matrix obtained from I_n by interchanging the jth and $j+1$st rows. Let $W = N_G(A)/A$. Then W has order $2^n n!$ and $W = \langle\, s_1, \ldots, s_n \,\rangle$, where

$$s_j = \begin{pmatrix} A^{(j)} & 0 \\ 0 & A^{(j)} \end{pmatrix}, \qquad 1 \le j \le n-1.$$

and s_n is obtained from I_{2n} by first multiplying the $2n$th row by -1 and then interchanging the nth and $2n$th rows. The matrices s_j, $1 \le j \le n$, correspond to the reflections associated to the simple roots that determine the Borel subgroup

$$P_{\min} = \left\{ \begin{pmatrix} A & B \\ & {}^t A^{-1} \end{pmatrix} \in G \;\Big|\; A \text{ is upper triangular} \right\}.$$

The standard parabolic subgroups (the ones containing $P_{\min}$) are parametrized by subsets S' of $S = \{\, s_1, \ldots, s_n \,\}$. Given $S' \subset S$, the associated standard parabolic subgroup $P_{S'}$ is the subgroup $\langle\, P_{\min}, S' \,\rangle$ of G generated by $P_{\min}$ and S'.

Suppose that $\ell \ge 2$ and $m_1 + \cdots + m_\ell = n$ for some positive integers $m_1, \ldots, m_\ell$. Let

$$S' = S \setminus \{\, s_{m_1},\, s_{m_1+m_2},\, s_{m_1+m_2+\cdots m_{\ell-1}} \,\}.$$

Then $P_{S'} = M \ltimes N$, where M is the set of matrices of the form

$$\begin{pmatrix} A' & & & \\ & A & & B \\ & & {}^t A'^{-1} & \\ & C & & D \end{pmatrix}, \quad \text{where } A' = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{\ell-1} \end{pmatrix}, \quad A_j \in GL_{m_j}(F),$$

and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2m_\ell}(F)$.

When $\ell = 2$, $P_{S'}$ is a maximal proper parabolic subgroup.

If $S' = \{ s_1, \ldots, s_{n-1} \}$, then we obtain a maximal proper parabolic subgroup $P_{S'} = M \ltimes N$, with

$$M = \left\{ \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix} \mid A \in GL_n(F) \right\}.$$

The subgroup $K_0 = Sp_{2n}(\mathfrak{o}) = Sp_{2n}(F) \cap GL_{2n}(\mathfrak{o})$ is a maximal parahoric subgroup of $Sp_{2n}(F)$. The pro-p-radical K_0^u of K_0 is the set of matrices $k \in G$ such that every entry of $k - 1$ belongs to $\mathfrak{p}$. The quotient K_0/K_0^u is isomorphic to $Sp_{2n}(\mathbb{F}_q)$ (where q is such that $\mathfrak{o}/\mathfrak{p} \simeq \mathbb{F}_q$). Replacing F by $\mathbb{F}_q$ above, we obtain a parametrization of a set of standard parabolic subgroups of $Sp_{2n}(\mathbb{F}_q)$. If $S' \subset S$, we use the notation $P_{S'}(\mathbb{F}_q)$ for the corresponding standard parabolic subgroup of $Sp_{2n}(\mathbb{F}_q)$. Let $P_{\min}(\mathbb{F}_q) = P_\emptyset(\mathbb{F}_q)$. Let I be the inverse image in K_0 (relative to the canonical projection $K_0 \to K_0/K_0^u$) of $P_{\min}(\mathbb{F}_q)$. The group I is an Iwahori subgroup (a minimal parahoric subgroup) of G. Although there is more than one conjugacy class of maximal parahoric subgroups of G, it is always the case that any two Iwahori subgroups are conjugate in G. Fix a prime element $\varpi \in \mathfrak{p}$. Let

$$s_0 = \begin{pmatrix} 0 & & & & -\varpi^{-1} & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \varpi & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

The affine Weyl group W^{aff} of G is the group $\langle s_0, s_1, \ldots, s_n \rangle$. This group is isomorphic to $N_G(A)/(A \cap K_0)$. The standard parahoric subgroups correspond bijectively to proper subsets of $\{ s_0, \ldots, s_n \}$. If $S' \subsetneq \{ s_0, \ldots, s_n \}$, then the corresponding standard parahoric subgroup $K_{S'}$ is the group $\langle I, S' \rangle$. If $S' \subset S$, then $K_{S'}$ is the inverse image of $P_{S'}(\mathbb{F}_q)$ in K_0. The other standard parahoric subgroups of G, that is, the ones of the form $K_{S'}$ where $s_0 \in S'$, do not lie inside K_0.

Let $K_0' = K_{\{s_0, \ldots, s_{n-1}\}}$. Then

$$K_0' = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(F) \;\middle|\; A,\, D \in \mathfrak{gl}_n(\mathfrak{o}),\ B \in \mathfrak{gl}_n(\mathfrak{p}^{-1}),\ C \in \mathfrak{gl}_n(\mathfrak{p}) \right\},$$

and $K_0' = gK_0g^{-1}$ where $g_{1,2n} = g_{2,2n-1} = \cdots = g_{n,n+1} = 1$, $g_{n+1,n} = g_{n+2,n-1} = \cdots = g_{2n,1} = \varpi$ and all other entries of g equal zero. Note that $g \in GSp_{2n}(F) = \{ g \in GL_{2n}(F) \mid {}^tgJg = c\,J,\ c \in F^\times \}$. The quotient $K_0'/K_0'^u$ is isomorphic to $Sp_{2n}(\mathbb{F}_q)$.

If $1 \leq m \leq n-1$ and $S_m = \{ s_0, \ldots, s_{m-1}, s_{m+1}, \ldots, s_n \}$, then the maximal parahoric K_{S_m} has the property that $K_{S_m}/K_{S_m}^u$ is isomorphic to $Sp_{2m}(\mathbb{F}_1) \times Sp_{2(n-m)}(\mathbb{F}_q)$.

If $K_{S'}$ is a maximal parahoric subgroup, then the parahoric subgroups contained in $K_{S'}$ may be obtained as the inverse images in $K_{S'}$ of parabolic subgroups of $K_{S'}/K_{S'}^u$.

10. Cuspidal representations and the Deligne-Lusztig construction

In this section, q is a power of a prime p, $\mathbf{G}$ is a connected reductive $\mathbb{F}_q$-group, and $G = \mathbf{G}(\mathbb{F}_q)$. If $\mathbf{T}$ is a maximal $\mathbb{F}_q$-torus in $\mathbf{G}$ we will refer to $T := \mathbf{T}(\mathbb{F}_q)$. as a maximal torus in G. The so-called "Deligne-Lusztig construction" associates class functions on G to characters of maximal tori in G. Certain of these class functions are, up to sign, equal to characters of irreducible representations of G. The book of Carter ([Cart]) is a good basic reference for this (and for other information about the representation theory of reductive groups over finite fields).

Let $\ell \neq p$ be a prime. The ℓ-adic cohomology groups with compact support play a role in the Deligne-Lusztig construction. Suppose that X is an algebraic variety over the algebraic closure $\widehat{\mathbb{F}}_q$ of $\mathbb{F}_1$. Each automorphism of X induces a nonsingular linear map $H^i_c(X, \overline{\mathbb{Q}}_\ell) \to H^i_c(X, \overline{\mathbb{Q}}_\ell)$. This makes $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ a module for the group of automorphisms of X. If g is an automorphism of X of finite order, the "Lefschetz number of g on X" $\mathcal{L}(g, X)$ is defined to be $\sum_i (-1)^i \mathrm{trace}(g, H^i_c(X, \overline{\mathbb{Q}}_\ell))$. It is known that $\mathcal{L}(g, X)$ is an integer that is independent of ℓ.

Let $T = \mathbf{T}(\mathbb{F}_q)$ be a maximal torus in G and let Let $\mathbf{B}$ be a Borel subgroup (minimal parabolic subgroup) of $\mathbf{G}$ that contains $\mathbf{T}$. Note that $\mathbf{B}$ is not necessarily defined over $\mathbb{F}_q$. If $\mathbf{N}$ is the unipotent radical of $\mathbf{B}$, then $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$. Let Fr be the Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. (That is, $\mathrm{Fr}(x) = x^q$ for every $x \in \overline{\mathbb{F}}_q$.) The notation Fr will also be used for the action of Fr on $\mathbf{G}$, $\mathbf{T}$, etc. Define $L(g) = g^{-1}\mathrm{Fr}(g)$, $g \in \mathbf{G}$. This map is called *Lang's map*. Note that $G = \mathbf{G}^{\mathrm{Fr}} = L^{-1}(1)$. The set $\widetilde{X} = L^{-1}(\mathbf{N})$ is an affine algebraic variety. The group G acts on $\widetilde{X}$ by left multiplication: if $g \in G$ and $x \in \widetilde{X}$, then $L(gx) = x^{-1}g^{-1}\mathrm{Fr}(g)\mathrm{Fr}(x) = x^{-1}\mathrm{Fr}(x) = L(x) \in \mathbf{N}$, so $gx \in \widetilde{X}$. The group T acts on $\widetilde{X}$ by right mul tiplication: if $x \in \widetilde{X}$ and $t \in T$, then $L(xt) = t^{-1}x^{-1}\mathrm{Fr}(x)\, t \in t^{-1}\mathbf{N}t = \mathbf{N}$. These actions commute with each other. Thus $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ is a left G-module and a right T-module such that $(gv)t = g(vt)$ for $g \in G$, $t \in T$ and $v \in H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$.

Let $\widehat{T}$ be the set of characters (one-dimensional representations) of the maximal torus T. Let $\theta \in \widehat{T}$. If n is the order of T and $t \in T$, then $\theta(t)$ is an nth root of unity, so $\theta(t)$ is an algebraic integer. Because $\overline{\mathbb{Q}}_\ell$ contains the algebraic numbers, we can view θ as a homomorphism from T to $\overline{\mathbb{Q}}_\ell^\times$. Let $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)_\theta$ be the T-submodule of $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ on which T acts by the character θ. Define $R_{T,\theta} = R^G_{T,\theta} : G \to \overline{\mathbb{Q}}_\ell$ by

$$R_{T,\theta}(g) = \sum_{i \geq 0} (-1)^i \mathrm{trace}(g, H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_\ell)_\theta), \qquad g \in G.$$

Properties of $R_{T,\theta}$:

(1) $R_{T,\theta}$ is a "generalized character" (an integral linear combination of characters of irreducible representations of G).

(2) $R_{T,\theta}(g) = |T| \sum_{t \in T} \theta(t^{-1}) \, \mathcal{L}((g,t), \widetilde{X})$, $g \in G$.

(3) $R_{T,\theta}$ is independent of the choice of Borel subgroup $\mathbf{B}$ that has $\mathbf{T}$ as Levi factor.

More to be added....

11. Maximal tori over $\mathbb{F}_q$ and unramified tori in p-adic groups

12. Depth zero supercuspidal L-packets

Hecke algebras and K-types for smooth representations

Throughout this section, (σ, W) denotes an irreducible smooth representation of a compact open subgroup K of G. Recall that $\mathcal{H}(G/\!/K,\sigma)$ is the set of compactly supported $\mathrm{End}_{\mathbb{C}}(\widetilde{W})$-valued functions φ on G such that $\varphi(k_1xk_2)=\widetilde{\sigma}(k_1)\varphi(x)\widetilde{\sigma}(k_2)$ for k_1, $k_2\in K$ and $x\in G$. It is an algebra with respect to convolution:

$$(\varphi_1*\varphi_2)(x)=\int_G \varphi_1(y)\circ\varphi_2(y^{-1}x)\,dy,\qquad \varphi_1,\varphi_2\in\mathcal{H}(G/\!/K,\sigma).$$

The element

$$e_{\widetilde{\sigma}}(x)=\begin{cases} m(K)^{-1}\widetilde{\sigma}(x), & \text{if } x\in K\\ 0, & \text{if } x\notin K.\end{cases}$$

is an identity in $\mathcal{H}(G/\!/K,\sigma)$.

Proposition. *Let* $\pi=$c-$\mathrm{Ind}_K^G(\sigma)$. *Then the algebra* $\mathrm{Hom}_G(\pi)$ *of operators which intertwine* π *is isomorphic to the Hecke algebra* $\mathcal{H}(G/\!/K,\widetilde{\sigma})$.

Proof. In the previous section, we produced vector space isomorphism $\varphi\mapsto A_\varphi$ between $\mathcal{H}(G/\!/K,\widetilde{\sigma})$ and $\mathrm{Hom}_G(\pi)$. Let $\varphi_j\in\mathcal{H}(G/\!/K,\widetilde{\sigma})$, $j=1,2$. Choosing a set $\mathcal{R}$ of right coset representatives for K, we have, for $x\in G$ and $f\in V=$c-$\mathrm{Ind}_K^G(\sigma)$,

$$\begin{aligned}(A_{\varphi_1}A_{\varphi_2}f)(x)&=A_{\varphi_2}(A_{\varphi_1}f)(x)=\sum_{g\in\mathcal{R}}\varphi_1(xg^{-1})(A_{\varphi_2}f)(g)\\
&=\sum_{g,y\in\mathcal{R}}\varphi_1(xg^{-1})\varphi_2(gy^{-1})f(y)=\sum_{y\in\mathcal{R}}\left(\sum_{g\in\mathcal{R}}\varphi_1(g^{-1})\varphi_2(gxy^{-1})\right)f(y)\\
&=\sum_{y\in\mathcal{R}}\left(\sum_{g^{-1}\in\mathcal{R}}\varphi_1(g)\varphi_2(g^{-1}xy^{-1})\right)f(y)\\
&=m(K)^{-1}\sum_{y\in\mathcal{R}}\left(\int_G\varphi_1(g)\varphi_2(g^{-1}xy^{-1})\,dg\right)f(y)\\
&=m(K)^{-1}\sum_{y\in\mathcal{R}}(\varphi_1*\varphi_2)(xy^{-1})f(y)=m(K)^{-1}(A_{\varphi_1*\varphi_2}f)(x).\end{aligned}$$

The map $\varphi\mapsto m(K)^{1/2}A_\varphi$ is the desired isomorphism. $\square$

Given $\alpha\in\mathrm{End}_{\mathbb{C}}(W)$, define $\alpha^\vee\in\mathrm{End}_{\mathbb{C}}(\widetilde{W})$ by

$$\langle\alpha^\vee(\widetilde{w}),w\rangle=\langle\widetilde{w},\alpha(w)\rangle,\qquad \widetilde{w}\in\widetilde{W},\ w\in W.$$

It is easy to check the following:

Lemma. *The map $\varphi \mapsto \varphi'$ from $\mathcal{H}(G /\!/ K, \widetilde{\sigma})$ to $\mathcal{H}(G /\!/ K, \sigma)$ defined by $\varphi'(g) = \varphi(g^{-1})^\vee$, $g \in G$, is an anti-isomorphism.*

There is a one to one correspondence between smooth representations of G and non degenerate representations of $\mathcal{H}(G) = C_c^\infty(G)$. The correspondence takes irreducible representations to irreducible representations. Given a smooth representation (π, V) of G, we define $\pi(f) = \int_G f(x)\pi(x)\,dx$, $f \in \mathcal{H}(G)$. Starting with a non degenerate representation $f \mapsto \pi'(f)$ of $\mathcal{H}(G)$ on a complex vector space V, given a vector $v \in V$, choose a compact open subgroup K' of G such that $\pi'(e_{K'})v = v$. Then $\pi(g)v = \pi'(\lambda(g)e_{K'})v$, $g \in G$. Here, $\lambda(g)$ is left translation by g. (See [JL], pp.25-26 for a proof that π is well-defined.)

As we want to work with $\mathcal{H}(G /\!/ K, \widetilde{\sigma})$, it is convenient to work with representations of $\mathcal{C} = C_c^\infty(G, \mathrm{End}_{\mathbb{C}}(W))$. For the moment, we are only thinking of W as a finite dimensional complex vector space. For a fixed $e_{K'}$, the algebra $\mathcal{H}_{K'} = (e_{K'} \otimes 1_W) * \mathcal{C} * (e_{K'} \otimes 1_W)$ has an identity and its modules are equivalent to the modules of $e_{K'} * C_c^\infty(G) * e_{K'}$. (See Jacobson, Basic Algebra II, pp.30–31.) If $\mathcal{V}$ is a non degenerate $\mathcal{C}$-module, then $\mathcal{V}$ is a direct limit of of modules $\mathcal{V}_{K'}$ corresponding to the algebras $\mathcal{H}_{K'}$. Together with the module equivalence for each K', this results in an equivalence $V \mapsto V \otimes W$ between the category of non degenerate $\mathcal{H}(G)$-modules and the category of non degenerate $\mathcal{C}$-modules. Starting with a smooth representation (π, V) of G, we then have

$$\pi'(f)(v \otimes w) = \int_G (\pi(x)v \otimes f(x)w)\,dx, \qquad v \in V,\ w \in W,\ f \in \mathcal{C}.$$

Set $\mathcal{H}(\sigma) = \mathcal{H}(G /\!/ K, \widetilde{\sigma})$. The function e_σ defined above is an idempotent in $\mathcal{C}$: that is, $e_\sigma * e_\sigma = e_\sigma$. Also $\mathcal{H}(\sigma) = e_\sigma * \mathcal{C} * e_\sigma$. We get a representation of K on $V \otimes W$ from the tensor product of the restriction of π to K with σ. Given $v \in V$ and $w \in W$,

$$\pi'(e_\sigma)(v \otimes w) = m(K)^{-1} \int_K (\pi \otimes \sigma)(k)(v \otimes w)\,dk,$$

so $\pi'(e_\sigma)$ projects $V \otimes W$ onto $(V \otimes W)^K$ (recall lemma in the section on Haar measure and characters). Given $f \in \mathcal{H}(\sigma)$, it follows from $f = e_\sigma * f * e_\sigma$ that $(V \otimes W)^K$ is $\pi'(f)$-invariant. Thus $f \mapsto \pi'(f) \,|\, (V \otimes W)^K$ is a representation of $\mathcal{H}(\sigma)$ (which is finite dimensional if π is admissible). We will denote this representation of $\mathcal{H}(\sigma)$ by $(\pi', (V \otimes W)^K)$.

Proposition.

(1) If (π, V) is an irreducible smooth representation of G, then $(\pi', (V \otimes W)^K)$ is irreducible, or 0, as an $\mathcal{H}(\sigma)$-representation.

(2) Suppose that (π_j, V_j) are irreducible smooth representations such that $(\pi'_j, (V_j \otimes W)^K) \neq 0$, $j = 1, 2$. Then $\pi_1 \simeq \pi_2$ if and only if $(\pi'_j, (V_j \otimes W)^K)$ are equivalent $\mathcal{H}(\sigma)$-modules.

(3) For each irreducible representation (τ, U) of $\mathcal{H}(\sigma)$, there is an irreducible smooth representation (π, V) of G such that $(\pi', (V \otimes W)^K) \simeq (\tau, U)$.

Proof. By remarks above, we can work with non degenerate $\mathcal{C}$-modules instead of smooth representations of G. Let (π', V) be a $\mathcal{C}$-module. Suppose that U is an $\mathcal{H}(\sigma)$-submodule of $e_\sigma \cdot V$. Let $V' = \mathcal{C} \cdot U$. Then, using $U = e_\sigma \cdot U$, we have

$$e_\sigma \cdot V' = e_\sigma * \mathcal{C} \cdot U = e_\sigma * \mathcal{C} * e_\sigma \cdot U = \mathcal{H}(\sigma) \cdot U = U.$$

Suppose that (π', V) is irreducible. Let U be a non zero $\mathcal{H}(\sigma)$-submodule of $e_\sigma \cdot V$ such that $U \neq e_\sigma \cdot V$. By the argument above $e_\sigma \cdot V' = U$ implies that the $\mathcal{C}$-submodule V' is not all of V and is nonzero, contradicting irreducibility of (π', V).

Let (π'_j, V_j) be irreducible $\mathcal{C}$-modules such that the corresponding $\mathcal{H}(\sigma)$-modules are non zero. Let $A : e_\sigma \cdot V_1 \to e_\sigma \cdot V_2$ be an $\mathcal{H}(\sigma)$-intertwining operator. Set

$$\mathcal{W} = \{\, (v, Av) \in V_1 \oplus V_2 \mid v \in e_\sigma \cdot V_1 \,\}.$$

Then $\mathcal{W}$ is an $\mathcal{H}(\sigma)$-submodule of $e_\sigma \cdot V_1 \oplus e_\sigma \cdot V_2$. Set $\mathcal{V} = \mathcal{C} \cdot \mathcal{W}$. By the argument above, $e_\sigma \cdot \mathcal{V} = \mathcal{W}$. This implies that the projection of $\mathcal{V}$ to each V_j is nonzero. As V_j is irreducible, the projection must then be onto. Let $L_j \subset \mathcal{V}$ be the kernel of projection of $\mathcal{V}$ onto V_j, $j = 1, 2$. If $e_\sigma \cdot L_1 \neq 0$, then there are vectors in $\mathcal{W}$ whose first component is zero, but whose second component is non-zero, which is impossible, by definition of $\mathcal{W}$. Thus $e_\sigma \cdot L_1 = 0$. Suppose that $e_\sigma \cdot L_2 \neq 0$. Then there is a non zero vector in $e_\sigma \cdot V_1$ whose image under A is zero. But A intertwines irreducible $\mathcal{H}(\sigma)$-modules, so is invertible. Thus $e_\sigma \cdot L_2 = 0$. Because $L_1 \subset V_2$, $L_1 = 0$ or V_2. As $e_\sigma \cdot L_1 = 0$, we must have $L_1 = 0$. Similarly, $L_2 = 0$. Thus both projections are bijective. Let $v_1 \in V_1$. By above, there exists a unique $v_2 \in V_2$ such that $(v_1, v_2) \in \mathcal{V}$. Since $\mathcal{V} = \mathcal{C} \cdot \mathcal{W}$, there exist $f \in \mathcal{C}$ and $v \in e_\sigma \cdot V_1$ such that $(v_1, v_2) = (\pi'(f)v, \pi'(f)Av)$. Using the above, check that the map $\mathcal{A} : V_1 \to V_2$ defined by $\mathcal{A}v_1 = \pi'(f)Av$ is a well-defined invertible linear transformation which intertwines π'_1 and π'_2.

Let (τ, U) be as in (3). Fix a non zero $u \in U$. Set $\mathcal{I} = \{\, f \in \mathcal{H}(\sigma) \mid \pi'(f)u = 0 \,\}$. Then $\mathcal{I}$ is a left ideal in $\mathcal{H}(\sigma)$ and $U \simeq \mathcal{H}(\sigma)/\mathcal{I}$. Letting $\mathcal{C}$ act on $\mathcal{H}(\sigma)$ on the left, set $M = \mathcal{C} \cdot \mathcal{H}(\sigma)$ and $N = \mathcal{C} \cdot \mathcal{I}$. Set $L = M/N$. Then $e_\sigma \cdot L = \mathcal{H}(\sigma)/\mathcal{I} = U$. Thus L is generated by U. Let V' be a submodule of L. Since U is irreducible, we have $e_\sigma \cdot V' = 0$ or $e_\sigma \cdot V' = U$. If $e_\sigma \cdot V' = 0$, then $e_\sigma \cdot (L/V') = U$. If $e_\sigma \cdot V' = U$, then L must equal V' (since L is generated by U). Hence, if L is not irreducible, there exists a proper irreducible quotient L/V of L (Zorn's lemma) and it follows that $e_\sigma \cdot (L/V) = U$ since $V \neq L$ forces $e_\sigma \cdot V = 0$. □

Corollary. *Fix K, σ, and W as above. Then there is a bijection between equivalence classes of irreducible smooth representations (π, V) of G such that $(\pi', (V \otimes W)^K)$ is not zero, and equivalence classes of irreducible $\mathcal{H}(G/\!\!/K, \widetilde{\sigma})$-modules.*

Let (π, V) be a smooth representation of G. Viewing V as the space of the restriction of π to K, we have $V = \bigoplus_{\tau \in \widehat{K}} V^\tau$, where V^τ is a direct sum of copies of the space W_τ of τ. The subspace V^τ is called the (τ, K)-*isotypic* or τ-isotypic subspace of V. If $V^\tau \neq \{0\}$, then (τ, K) is called a K-*type* of π. Suppose that $(W_\tau \otimes W) \cap (V \otimes W)^K$ is non zero. Then $\tau \otimes \sigma$ contains the trivial representation of K with multiplicity at least one.

Therefore
$$0 \neq \int_K \operatorname{tr}(\tau \otimes \sigma)(k)\, dk = \int_K \operatorname{tr} \tau(k) \operatorname{tr} \sigma(k)\, dk,$$
which, by the orthogonality relations for characters of irreducible representations of compact groups, forces $\tau \simeq \widetilde{\sigma}$. On the other hand, $W_{\widetilde{\sigma}} \otimes W = \widetilde{W} \otimes W$ affords exactly one copy of the trivial representation of K. It follows that $(V \otimes W)^K = (V^{\widetilde{\sigma}} \otimes W)^K$ has dimension equal to $\dim(V^{\widetilde{\sigma}})/\dim(W)$, that is, the number of copies of $\widetilde{\sigma}$ in $\pi \,|\, K$. Hence those π giving rise to nonzero $\mathcal{H}(G/\!\!/K, \widetilde{\sigma})$-modules are exactly the ones whose restriction to K contains $\widetilde{\sigma}$.

Example: $\mathcal{H}_{K_0} = \mathcal{H}(G/\!\!/K_0, 1) = \mathcal{H}(G/\!\!/K_0)$. This Hecke algebra is well understood. Let $B = AN$ be the standard Borel subgroup of $G = GL_n(F)$. Here, A is the group of diagonal matrices in G, and N is the group of upper triangular unipotent matrices in G. Given $f \in \mathcal{H}_{K_0}$, define $(Sf)(a) = \delta_B(a)^{1/2} \int_N f(an)\, dn$. Note that $Sf \in \mathcal{H}(A/\!\!/(A \cap K_0))$. As A is abelian, $\mathcal{H}(A/\!\!/(A \cap K_0)) = \mathbb{C}[A/(A \cap K_0)]$, the group algebra of $A/(A \cap K_0)$. The symmetric group $\mathcal{S}_n$ on n letters acts on A in the obvious way. As this action preserves $A \cap K_0$, we have an action of $\mathcal{S}_n$ on $\mathbb{C}[A/(A \cap K_0)]$. The map $f \mapsto Sf$ is an algebra isomorphism from $\mathcal{H}_{K_0}$ onto the $\mathcal{S}_n$ fixed points $\mathbb{C}[A/(A \cap K_0)]^{\mathcal{S}_n}$, called the *Satake isomorphism*. If time permits, we will discuss Satake isomorphisms for reductive p-adic groups. Characters of $A/(A \cap K_0)$ can be used to give integral formulas for certain functions in $\mathcal{H}_{K_0}$. Such functions can then be used to produce explicit realizations of irreducible admissible representations of G which have non zero K_0-invariant vectors.

Let $s_j \in \mathbb{C}$, $1 \leq j \leq n$. Then $\chi = \otimes_{1 \leq j \leq n} |\cdot|_F^{s_j}$ defines a quasi-character of A which is trivial on $A \cap K_0$. Such a character is said to be *unramified*. The set of representations of the form $\mathrm{Ind}_B^G \chi$, χ a a quasi-character of A is called the *principal series* of representations. The set of principal series representations with χ unramified is called the *unramified* principal series. Such representations have non zero K_0-fixed vectors. Unramified principal series will be studied later in the course. A few results are stated below.

A subgroup I of $GL_n(F)$ which is conjugate to the set of matrices $x = (x_{ij})$ such that $x_{ii} \in \mathfrak{o}_F^\times$, $x_{ij} \in \mathfrak{o}_F$ whenever $i < j$, and $x_{ij} \in \mathfrak{p}_F$ whenever $i > j$, is called an Iwahori subgroup. The Iwahori spherical Hecke algebra $\mathcal{H}_I = \mathcal{H}(G /\!\!/ I)$ contains $\mathcal{H}_{K_0}$ as a subalgebra. The structure of $\mathcal{H}_I$ (for general G) has been described by Iwahori and Matsumoto ([IM]). Note that any unramified principal series representation contains a nonzero Iwahori-fixed vector. Borel ([B]) has shown:

(1) Any subquotient of an unramified principal series representation contains a nonzero Iwahori fixed vector.
(2) Any irreducible admissible representation which has a nonzero Iwahori fixed vector occurs as a subquotient of an unramified principal series representation.
(3) Suppose that I is an Iwahori subgroup which has an Iwahori decomposition relative to a parabolic subgroup $P = MN$. If (π, V) is an irreducible admissible representation with a nonzero I-fixed vector, then the image of V^I in the Jacquet module V_N is equal to $V_N^{I \cap M}$.

Regarding part (1) above, if (σ, K) is a K-type of a smooth representation (π, V), it not usually the case that every irreducible subquotient of π will contain the K-type (σ, K). See the comments on types below.

Borel's results have been generalized by Moy and Prasad ([MP]) to *depth zero* minimal K-types of reductive p-adic groups. A parahoric subgroup of $GL_n(F)$ is conjugate to a group of the form

$$K_\alpha = \begin{pmatrix} GL_{\alpha_1}(\mathfrak{o}) & M_{\alpha_1 \times \alpha_2}(\mathfrak{o}) & \cdots & M_{\alpha_1 \times \alpha_r}(\mathfrak{o}) \\ M_{\alpha_2 \times \alpha_1}(\mathfrak{p}) & GL_{\alpha_2}(\mathfrak{o}) & \cdots & M_{\alpha_2 \times \alpha_r}(\mathfrak{o}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{\alpha_r \times \alpha_1}(\mathfrak{p}) & M_{\alpha_r \times \alpha_2}(\mathfrak{p}) & \cdots & GL_{\alpha_r}(\mathfrak{o}) \end{pmatrix}$$

for some partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n. The pro unipotent radical K_α^u of K_α is the subgroup of K_α with the $GL_{\alpha_j}(\mathfrak{o})$ blocks replaced by $1 + M_{\alpha_j}(\mathfrak{p})$. The quotient K_α / K_α^u

is isomorphic to $\prod_{j=1}^{r} GL_{\alpha_j}(\mathfrak{o}/\mathfrak{p})$. An irreducible representation σ of the quotient can be pulled back to K_α. Suppose that σ is a *cuspidal* representation of the quotient. That is, the matrix coefficients of σ must be cusp forms on the finite group K_α/K_α^u: given a matrix coefficient f, for any $x \in K_\alpha/K_\alpha^u$ and any unipotent radical N of a proper parabolic subgroup of K_α/K_α^u, $\sum_{n\in N} f(xn) = 0$. The representation (σ, K_α) is an unrefined minimal K-type of depth zero (in the sense of [MP1]). Let $P_\alpha = M_\alpha N_\alpha$ be the standard parabolic subgroup of $GL_n(F)$ corresponding to the partition α. The group $\mathrm{Norm}_{M_\alpha}(K_\alpha \cap M_\alpha)$ is open and compact modulo the centre of M_α. Let τ be any irreducible representation of $\mathrm{Norm}_{M_\alpha}(K_\alpha \cap M_\alpha)$ which contains σ upon restriction to $K_\alpha \cap M_\alpha$. Then $\rho_\tau =$ c-$\mathrm{Ind}_{\mathrm{Norm}_{M_\alpha}(K_\alpha\cap M_\alpha)}^{M_\alpha}\tau$ is a supercuspidal representation of M_α. In this setting, the result of Moy and Prasad can be stated as follows:

(1) Any subqotient of $i_{P_\alpha}^G(\rho_\tau)$ is generated by its (σ, K_α)-isotypic subspace.

(2) Suppose that (π, V) is an irreducible smooth representation of $GL_n(F)$ whose (σ, K_α)-isotypic subspace is nonzero (that is, which contains σ upon restriction to K_α). Then π is a subquotient of $i_{P_\alpha}^G \rho_\tau$ for some τ as above.

Howe and Moy ([HM]) have proved the following: Suppose that the characteristic p of $\mathfrak{o}/\mathfrak{p}$ is greater than n. To each smooth irreducible representation (π, V) of $G = GL_n(F)$, there is associated a triple (G', K, σ), where G' is a reductive group, K is a parahoric subgroup of G and σ is an irreducible representation of K. The group G' is a product $\prod_\ell GL_{m_\ell}(E_\ell)$ where each E_ℓ is an extension of F, with $n = \sum_\ell m_\ell[E_\ell : F]$. The intersection $I' = G' \cap K$ is an Iwahori subgroup of G' and the Hecke algebra $\mathcal{H}(G/\!\!/K, \widetilde{\sigma})$ is isomorphic to the Hecke algebra $\mathcal{H}(G'/\!\!/I')$ in a natural way. Also, the $(\widetilde{\sigma}, K)$-isotypic subspace of V is nonzero. These isomorphisms can be used to reduce the study of irreducible smooth representations of G to the study of irreducible smooth representations of products of general linear groups over extensions of F, representations which have nonzero Iwahori fixed vectors. Starting with (π, V), we can get a representation (π', V') of G'. Take the representation π of $\mathcal{H}(G/\!\!/K, \widetilde{\sigma})$ on $(V \otimes W)^K$ and transfer to a representation π' of $\mathcal{H}(G'/\!\!/I')$ via the isomorphism: if $f \leftrightarrow f'$, set $\pi'(f') = \pi(f)$. Then there is a unique irreducible admissible representation (π', V') of G' corresponding to $f' \mapsto \pi'(f')$.

This sort of approach has been used for other groups also. But it is *not* necessarily the case that the G' Hecke algebra is an Iwahori spherical Hecke algebra. Further, G' might not be a subgroup of G. See papers of Moy ([My2-4]) where this is done for 3×3

unramified unitary groups and for $GSp_4(F)$. Ju-Lee Kim has constructed many K-types and associated Hecke algebra isomorphisms for classical groups, subject to constraints on the residual characteristic of F. There is a paper of Lusztig ([L]) on classification of unipotent representations of p-adic groups which is related, but more general types of Hecke algebras appear there.

Bushnell and Kutzko have defined a notion of *type*. A type is a K-type having particularly nice properties. Suppose that M is a Levi subgroup of G. Let $X(M)$ be the group of *unramified quasi-characters* of M: the quasi-characters of M which are trivial on all compact subgroups of M. (For example, if $G = GL_n(F)$, then $X(G)$ consists of all quasi-characters of G which are trivial on the subgroup $G_0 = \{\, g \in G \mid \det g \in \mathfrak{o}^\times \,\}$). If σ is an irreducible supercuspidal representation of M, let $\mathfrak{R}^{(M,\sigma)}(G)$ be the set of smooth representations (π, V) of G having the property that every irreducible subquotient of π appears as a compositon factor of $\mathfrak{i}_P^G(\sigma \otimes \chi)$ for some parabolic subgroup P of G with Levi factor M, and some $\chi \in X(M)$. It is known that $\mathfrak{R}^{(M,\sigma)}(G) = \mathfrak{R}^{(M',\sigma')}(G)$ if and only if there is a $\chi \in X(M')$ such that (M, σ) and $(M', \sigma' \otimes \chi)$ are G-conjugate. Let $\mathfrak{s}$ be the G-orbit of all pairs $(M, \sigma \otimes \chi)$, $\chi \in X(M)$. Set $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}^{(M,\sigma)}(G)$. Every irreducible smooth representation (π, V) lies in some $\mathfrak{R}^{\mathfrak{s}}(G)$.

Now suppose that (κ, W) is an irreducible representation of a compact open subgroup K of G. If (π, V) is a smooth representation of G, let $V_\kappa = \mathrm{Hom}_K(W, V)$. Let $A \in V_\kappa$ and $f \in \mathcal{H}(G/\!\!/K, \kappa)$. Set

$$(f \cdot A)(w) = \int_G \pi(g) A(f(g)^\vee(w))\, dg, \qquad w \in W.$$

Note that this makes V_κ into an $\mathcal{H}(G/\!\!/K, \kappa)$-module.

Let $\mathfrak{R}_\kappa(G)$ be the set of smooth representations (π, V) of G such that V is generated (as a representation of G) by the (κ, W)-isotypic subspace $V^\kappa = \sum_{A \in V_\kappa} A(W)$. The map $(\pi, V) \mapsto V_\kappa$ is a functor M_κ from $\mathfrak{R}_\kappa(G)$ to $\mathcal{H}(G/\!\!/K, \kappa)$-modules.

If $\mathfrak{s}$ and κ are as above, then (K, κ) is said to be an $\mathfrak{s}$-type if $\mathfrak{R}^{\mathfrak{s}}(G) = \mathfrak{R}_\kappa(G)$. It is known that if the categories $\mathfrak{R}^{\mathfrak{s}}(G)$ and $\mathfrak{R}_\kappa(G)$ have the same irreducible objects, then (K, κ) is an $\mathfrak{s}$-type. For a more detailed discussion of this, see section 4 of [BK2]. For a general discussion of types and covers, see pp54-55 of [BK3].

If M is a Levi subgroup of G, K_M is a compact open subgroup of M, and κ_M is an irreducible smooth representation of K_M, then a *G-cover* of (K_M, κ_M) is a compact open subgroup K of G and an irreducible smooth representation κ of K satisfying the following, for each parabolic subroup P of G with Levi factor M, and unipotent radical N,

(i) $K_M = K \cap M$ and $K = (K \cap N^-)K_M(K \cap N)$ (Iwahori factorization)

(ii) $K \cap N$ and $K \cap N^-$ belong to the kernel of κ, and $\kappa \,|\, K_M \simeq \kappa_M$.

(iii) There exists a (K, P)-positive element z in the centre of M and an invertible element $f_z \in \mathcal{H}(G/\!\!/K, \kappa)$ with support KzK. (The element z is (K, P)-positive if the sequences $z^i(K \cap N^-)z^{-i}$ and $z^{-i}(K \cap N^+)z^i$ tend monotonically to 1 as i tends to ∞).

One of the main results of [BK2] relates types and covers to properties of parabolic induction i_P^G.

Theorem. *([BK2]) Suppose that $\mathfrak{s}$ is as above, and suppose that M' is a Levi subgroup of G which contains M. Let $\mathfrak{s}_{M'}$ be defined relative to M', σ and M as $\mathfrak{s}$ is relative to G, σ and M. If $(K_{M'}, \kappa_{M'})$ is an $\mathfrak{s}_{M'}$-type, and (K, κ) is a G-cover of $(K_{M'}, \kappa_{M'})$, then (K, κ) is an $\mathfrak{s}$-type. If P' is a parabolic subgroup of G with Levi factor M', there exists a unique injective algebra homomorphism*

$$j_{P'} : \mathcal{H}(M'/\!\!/K_{M'}, \kappa_{M'}) \to \mathcal{H}(G/\!\!/K, \kappa)$$

such that

$$(j_{P'})^* \circ M_{\kappa_{M'}} = M_\kappa \circ i_{P'}^G.$$

Here, $j_{P'}^*$ is the map from $\mathcal{H}(M'/\!\!/K_{M'}, \kappa_{M'})$-modules to $\mathcal{H}(G/\!\!/K, \kappa)$-modules induced by $j_{P'}$.

Let $\mathcal{H} = \mathcal{H}(G/\!\!/K, \kappa)$ and $\mathcal{H}' = \mathcal{H}(M'/\!\!/K_{M'}, \kappa_{M'})$. Let V' be an $\mathcal{H}'$-module. Then

$$j_{P'}^*(V') = \mathrm{Hom}_{\mathcal{H}'}(\mathcal{H}, V'),$$

and if $B \in j_{P'}^*(V')$ and $f_1,\ f_2 \in \mathcal{H}$, $(f_1 \cdot B)(f_2) = B(f_2 * f_1)$ makes $j_{P'}^*(V')$ into an $\mathcal{H}$-module.

In [BK1], for $G = GL_n(F)$, Bushnell and Kutzko constructed $\mathfrak{s}$-types for $M = M_{(r,\dots,r)}$ and $\sigma = \sigma_0 \otimes \cdots \otimes \sigma_0$, where σ_0 is an irreducible supercuspidal representation of $GL_r(F)$. The corresponding Hecke algebras are of affine type (5.4.6 [BK1]). The paper [BK3] is concerned with construction of more general types, and covers, for $G = GL_n(F)$.

If m is a positive integer and $z \in \mathbb{C}^\times$, the affine Hecke algebra $\mathcal{H}(m, z)$ is an associative $\mathbb{C}$-algebra with identity. It is generated by elements $[s_i]$, $1 \le i \le m-1$, and $[\zeta]$, $[\zeta']$, subject to relations:

(i) $[\zeta][\zeta'] = [\zeta'][\zeta] = 1$

(ii) $([s_i]+1)([s_i]-z)=0, \quad 1 \le i \le m-1$

(iii) $[\zeta]^2[s_1]=[s_{m-1}][\zeta]^2$

(iv) $[\zeta][s_i]=[s_{i-1}][\zeta], \quad 1 \le i \le m-2$

(v) $[s_i][s_{i+1}][s_i]=[s_{i+1}][s_i][s_{i+1}], \quad 1 \le i \le m-2$

(vi) $[s_i][s_j]=[s_j][s_i], \quad 1 \le i,\, j \le m-1,\, |i-j| \ge 2.$

The Iwahori-spherical Hecke algebra of $G=GL_n(F)$ is isomorphic to the affine Hecke algebra $\mathcal{H}(n,q)$, where q is the cardinality of the residue class field $\mathfrak{o}/\mathfrak{p}$ of F. Let A_ϖ be as in the section on supercuspidal representations, and let D be the subgroup of G consisting of diagonal matrices, all of whose eigenvalues are powers of ϖ. For $1 \le i \le n-1$, let s_i be the permutation matrix which interchanges the ith and $i+1$st standard basis vectors, and leaves all others fixed. Then Let W be the group generated by D and the $\{\, s_i \mid 1 \le i \le n-1 \,\}$. Then A_ϖ normalizes W. The group $\widetilde{W}=\langle A_\varpi\rangle W$ is called the affine Weyl group of G, and

$$G=\coprod_{w\in\widetilde{W}} \mathcal{I}w\mathcal{I}.$$

If $w \in \widetilde{W}$, let f_w be the characteristic function of the double coset $\mathcal{I}w\mathcal{I}$. The map $[\zeta] \mapsto f_{A_\varpi}$, $[s_i] \mapsto f_{s_i}$, $1 \le i \le n-1$, extends uniquely to an isomorphism of algebras $\mathcal{H}(n,q) \simeq \mathcal{H}(G/\!\!/I,1)$.

9. Discrete series representations

Lemma. *Let (π_j, V_j) be irreducible smooth representations of G_j, $j = 1, 2$. Then $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ is a smooth irreducible representation of $G_1 \times G_2$.*

Proof. Smoothness of $\pi_1 \otimes \pi_2$ is immediate.

Let (π, V) be an irreducible smooth representation of G. Let I be some indexing set. Then $\oplus_I \mathrm{Hom}_G(V, V) \simeq \mathrm{Hom}_G(V, \oplus_I V)$. The isomorphism is given by $\sum_{\alpha \in I} A_\alpha \mapsto B$, where $B(v) = \sum_{\alpha \in I} A_\alpha(v)$. By Schur's lemma, $\mathrm{Hom}_G(V, V) \simeq \mathbb{C}$. Irreducibility of π guarantees that given any nonzero $v \in V$, $V = \mathrm{Span}(G \cdot v)$, which implies surjectivity.

Now suppose that W is a complex vector space. As W is a direct sum of copies of $\mathbb{C} \simeq \mathrm{Hom}_G(V, V)$, by above, we have

$$(i) \qquad W \simeq \mathrm{Hom}_G(V, V \otimes W),$$

with the G-action on $V \otimes W$ given by $g \cdot (v \otimes w) = \pi(g)v \otimes w$, $v \in V$, $w \in W$.

Let X be a direct sum of copies of V. Then

$$(ii) \qquad \begin{aligned} V \otimes \mathrm{Hom}_G(V, X) &\to X \\ v \otimes f \quad &\mapsto \quad f(v) \end{aligned}$$

is an isomorphism.

Next, we can use (i) and (ii) to show that

$$\begin{array}{ccc} \{\, G\text{−subspaces of } V \otimes W \,\} & \longleftrightarrow & \{\, \mathbb{C}\text{−subspaces of } W \,\} \\ V \otimes U & \longleftarrow & U \\ X & \longrightarrow & \mathrm{Hom}_G(V, X) \subset \mathrm{Hom}_G(V, V \otimes W) = W \end{array}$$

Now apply the above with $G = G_1$, $(\pi, V) = (\pi_1, V_1)$ and $W = V_2$. As any $(G_1 \times G_2)$-invariant subspace X of $V_1 \otimes V_2$ is also a G_1-invariant subspace, we have $X = V_1 \otimes U$ for some complex subspace U of V_2. If $X \neq \{0\}$, then letting G_2 act, we get

$$\mathrm{Span}(G_2 \cdot X) = V_1 \otimes \mathrm{Span}(G_2 \cdot U) = V_1 \otimes V_2$$

by irreducibility of π_2. $\square$

Recall that a smooth representation (π, V) of G is unitary if there exists a positive definite G-invariant Hermitian form on V. The representation $(\bar{\pi}, \bar{V})$ conjugate to (π, V) is defined as follows. As a set $\bar{V} = V$, but $c \cdot v = \bar{c}v$, for $c \in \mathbb{C}$ and $v \in \bar{V}$. For $g \in G$ and $v \in V$, $\bar{\pi}(g)v = \pi(g)v$.

Lemma. *If (π, V) is admissible and unitary, then $(\bar{\pi}, \bar{V})$ is equivalent to $(\widetilde{\pi}, \widetilde{V})$.*

Proof. Let $(\cdot,\cdot)$ be a G-invariant inner product on V. Define $A : \bar{V} \to \widetilde{V}$ by $Av = (\cdot, v)$. Check that A is linear and intertwines $\bar{\pi}$ with $\widetilde{\pi}$. Earlier, we saw that for any compact open subgroup K, $\widetilde{V}^K = \mathrm{Hom}(V^K, \mathbb{C})$. To see that $A(\bar{V}^K) = \mathrm{Hom}(V^K, \mathbb{C})$, use the fact that $(\cdot,\cdot)\,|_{V^K}$ is nondegenerate. $\square$

Lemma. *Suppose that (π, V) is irreducible and smooth. Then a G-invariant positive definite inner product on V is unique up to scalar multiples.*

Proof. Let $(\cdot,\cdot)_j$, $j = 1, 2$, be two such inner products. Define $A_j : \bar{V} \to \widetilde{V}$ by $A_j(v) = (\cdot, v)_j$, $v{\in}V_j$. As $A_2^{-1} \circ A_1$ intertwines $\bar{V}$, by Schur's lemma, $A_2^{-1} \circ A_1$ is scalar. $\square$

Suppose that (π, V) is smooth and irreducible. By Schur's lemma, if $z \in Z = Z_G$ (the centre of G), then $\pi(z) = \chi_\pi(z) 1_V$, for some $\chi_\pi(z) \in \mathbb{C}^\times$. The map $z \mapsto \chi_\pi(z)$ is a quasi-character of Z, called the central character of π. Let $v \in V$ and $\widetilde{v} \in \widetilde{V}$. Then

$$\langle \widetilde{v}, \pi(gz)v \rangle = \chi_\pi(z) \langle \widetilde{v}, \pi(g)v \rangle, \qquad g \in G,\ z \in Z.$$

If χ_π is unitary, that is, $|\chi_\pi(z)|_\infty = 1$ for all $z \in Z$, then $g \mapsto |\langle \widetilde{v}, \pi(g)v \rangle|_\infty$ is constant on cosets of Z in G, so can be viewed as a function on G/Z.

Definition. π is *square integrable (modulo the centre)* if $\pi\,|\,Z$ is a (unitary) character of Z and if the (complex) absolute value of every matrix coefficient of π belongs to $L^2(G/Z)$. That is,

$$\int_{G/Z} |\langle \widetilde{v}, \pi(g)v \rangle|_\infty^2 \, dg^* < \infty, \qquad \widetilde{v} \in \widetilde{V},\ v \in V.$$

Here dg^* is a Haar measure on the group G/Z.

Definition. The *discrete series* of representations of G is the set of all irreducible smooth representations which are square integrable modulo the centre of G.

Lemma. *Suppose that π is irreducible and smooth with unitary central character. Then π is square integrable modulo Z if and only if there is some nonzero matrix coefficient of π which belongs to $L^2(G/Z)$.*

Proof. Fix $v_0 \in V$ and $\widetilde{v}_0 \in \widetilde{V}$ such that $gZ \mapsto |\langle \widetilde{v}_0, \pi(g)v_0 \rangle|_\infty$ is square integrable on G/Z, and nonzero. Let

$$V_0 = \{\, v \in V \mid g \mapsto |\langle \widetilde{v}_0, \pi(g)v \rangle|_\infty \in L^2(G/Z) \,\}.$$

Since V_0 is nonzero and G-invariant, $V_0 = V$. To finish, argue with v_0 fixed and use irreducibility of $\widetilde{V}$. $\square$

Lemma. *Suppose that π is an irreducible smooth representation of $GL_n(F)$. Then there exists a quasi character χ of G such that $\pi \otimes \chi$ has unitary central character.*

Proof. Note that $\varpi^{-\nu(z)} z \in \mathfrak{o}^\times$ for all $z \in Z \simeq F^\times$. There exists an $s \in \mathbb{C}$ such that $\chi_\pi(z) = |z|_F^s \chi_\pi(\varpi^{-\nu(z)} z)$, $z \in Z$. Set $\chi(g) = |\det g|_F^{-s/n}$. Then $|\chi(z)\chi_\pi(z)|_\infty = |\chi_\pi(\varpi^{-\nu(z)} z)|_\infty = 1$, and $\pi \otimes \chi$ has unitary central character. $\square$

Proposition. *Suppose that (π, V) is irreducible, smooth, and square integrable modulo the centre. Then (π, V) is unitarizable and there exists a unique $d(\pi) > 0$ such that*

$$\int_{G/Z} \langle \widetilde{u}, \pi(g^{-1})u \rangle \langle \widetilde{v}, \pi(g)v \rangle \, dg^* = d(\pi)^{-1} \langle \widetilde{u}, v \rangle \langle \widetilde{v}, u \rangle, \qquad u, v \in V, \ \widetilde{u}, \widetilde{v} \in \widetilde{V}.$$

If $(\cdot,\cdot)$ is a G-invariant positive definite inner product on V, then

$$\int_{G/Z} (\pi(g^{-1})u, u')(\pi(g)v', v) \, dg^* = d(\pi)^{-1} (u, v)(v', u'), \qquad u, u', v, v' \in V.$$

Proof. Let $L^2(G, \bar{\chi}_\pi)$ be the set of complex valued locally constant functions on G such that $f(zg) = \bar{\chi}_\pi(z) f(g)$, for $g \in G$ and $z \in Z$ and such that $\int_{G/Z} |f(g)|_\infty^2 \, dg^* < \infty$. The group G acts on $L^2(G, \bar{\chi}_\pi)$ via the left regular representation and the usual inner product given by $\int_{G/Z} f_1(g) \overline{f_2(g)} \, dg^*$ is G-invariant and positive definite. Fix a nonzero $\widetilde{v} \in \widetilde{V}$, Define a map $V \to L^2(G, \bar{\chi}_\pi)$ by mapping $v \in V$ to the function $g \mapsto \langle \widetilde{v}, \pi(g^{-1})v \rangle$. To see that the map is injective, note that if the function is zero and $v \neq 0$, then irreducibility of π implies $V = \mathrm{Span}(G \cdot v)$, which forces $\widetilde{v} = 0$, a contradiction. We now have a G-invariant inner product on V obtained by restriction of the inner product on $L^2(G, \bar{\chi}_\pi)$ to V. As the function $g \mapsto |\langle \widetilde{v}, \pi(g^{-1})v \rangle|_\infty^2$ is nonzero and locally constant (when $v \neq 0$), this inner product is positive definite. Hence π is unitarizable.

Fix any G-invariant positive definite inner product $(\cdot,\cdot)$ on V. Define a map from $V \otimes \bar{V}$ by mapping $v \otimes v'$ to the function $g \mapsto (\pi(g^{-1})v, v') \in L^2(G, \bar{\chi}_\pi)$. By an earlier lemma, $V \otimes \bar{V}$ is an irreducible $G \times G$-module. It follows that this map is injective. Hence we can define an inner product on $V \otimes \bar{V}$ by restricting the inner product on $L^2(G, \bar{\chi}_\pi)$. The group $G \times G$ acts on $L^2(G, \bar{\chi}_\pi)$ by $f^{(g_1, g_2)}(x) = f(g_1^{-1} x g_2)$, and the inner product is $G \times G$-invariant. As the above map is a $G \times G$-map, we now have a $G \times G$-invariant inner product on $V \otimes \bar{V}$.

We get another $G\times G$-invariant inner product on $V\otimes\bar V$ by using $(\cdot,\cdot)$ and $\overline{(\cdot,\cdot)}$:

$$(u\otimes u', v\otimes v') = (u,v)\overline{(u',v')} = (u,v)(v',u').$$

As $(\cdot,\cdot)$ is G-invariant, this inner product on $V\otimes\bar V$ is $G\times G$-invariant.

As $V\otimes\bar V$ is irreducible, these two inner products are related by a scalar $d(\pi)$. Use positive definiteness to see that $d(\pi)>0$. Thus

$$\int_{G/Z}(\pi(g^{-1})u,u')\overline{(\pi(g^{-1})v,v')}\,dg^* = \int_{G/Z}(\pi(g^{-1})u,u')(\pi(g)v',v)\,dg^* = d(\pi)^{-1}(u,v)(v',u').$$

Let $\widetilde u$, $\widetilde v\in\widetilde V$. Then there exist unique u_0, $v_0\in V$ such that $\langle\widetilde u,w\rangle=(w,u_0)$ and $\langle\widetilde v,w\rangle=(w,v_0)$ for all $w\in W$ (see lemma above). Substituting into the above orthogonality relations then results in the first set of orthogonality relations given in the statement of the proposition. $\square$

Definition. The constant $d(\pi)$ of the proposition is called the *formal degree* of π. (Note that $d(\pi)$ depends on a choice of Haar measure on G/Z.)

If π is such that $\pi\otimes\chi$ is square integrable modulo Z for some quasi character χ of G, then π is said to be *quasi square integrable modulo* Z. If we set $d(\pi)=d(\pi\otimes\chi)$, then it follows from

$$\langle\widetilde u,(\pi\otimes\chi)(g^{-1})u\rangle\langle\widetilde v,(\pi\otimes\chi)(g)v\rangle = \langle\widetilde u,\pi(g^{-1})u\rangle\langle\widetilde v,\pi(g)v\rangle,\qquad \widetilde u,\widetilde v\in\widetilde V,\ u,v\in V,$$

that the first type of relation stated in the proposition holds for π.

Lemma.

(1) If π is an irreducible supercuspidal representation of G, then π is quasi square integrable modulo the centre.

(2) Let (σ,W) be an irreducible smooth representation of an open compact modulo centre subgroup H of G. Suppose that $\pi=$c-$\mathrm{Ind}_H^G\sigma$ is irreducible. Then π is quasi square integrable modulo the centre and $d(\pi)=m(H/Z)^{-1}\deg\sigma$.

Proof. (1) As the matrix coefficients of π are compactly supported modulo Z, the same is true of the matrix coefficients of $\pi\otimes\chi$ for any quasi character χ of G. Choosing χ so that $\pi\otimes\chi$ has unitary central character, we get a discrete series representation.

For part (2), first note that irreducibility of π implies that π is supercuspidal and we may apply (1) to conclude that π is quasi square integrable modulo the centre.

Let $w \in W$ and $\widetilde{w} \in \widetilde{W}$ be such that $\langle \widetilde{w}, w \rangle \neq 0$. Define $f_w \in V = \text{c-Ind}_H^G(W)$ by $f_w(h) = \sigma(h)w$, for $h \in H$ and $f_w(x) = 0$ for $x \in G \backslash H$. Denote the linear functional $f \mapsto \langle \widetilde{w}, f(1) \rangle$, $f \in V$, by $\widetilde{v}$. Then

$$\begin{aligned}\int_{G/Z} \langle \widetilde{v}, \pi(g^{-1}) f_w \rangle \langle \widetilde{v}, \pi(g) f_w \rangle \, dg^* &= d(\pi)^{-1} \langle \widetilde{v}, f_w \rangle^2 \\ &= d(\pi)^{-1} \langle \widetilde{w}, f_w(1) \rangle^2 = d(\pi)^{-1} \langle \widetilde{w}, w \rangle^2.\end{aligned}$$

In an earlier lemma, we saw that $\langle \widetilde{v}, \pi(g) f_w \rangle = \langle \widetilde{w}, \sigma(g) w \rangle$, if $g \in H$, and zero otherwise. Thus the above inner product is also equal to

$$\int_{H/Z} \langle \widetilde{w}, \sigma(h^{-1}) w \rangle \langle \widetilde{w}, \sigma(h) w \rangle \, dh^* = m(H/Z)(\deg \sigma)^{-1} \langle \widetilde{w}, w \rangle^2$$

by orthogonality relations for irreducible representations of compact groups. □

Let $P = MN$ be a standard parabolic subgroup of $G = GL_n(F)$. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be the partition of n associated to P. Set

$$A_M = \left\{ \begin{pmatrix} \varpi^{j_1} I_{\alpha_1} & & & \\ & \varpi^{j_2} I_{\alpha_2} & & \\ & & \ddots & \\ & & & \varpi^{j_r} I_{\alpha_r} \end{pmatrix} \;\middle|\; j_1, \ldots, j_r \in \mathbb{Z} \right\}.$$

If $P = M = G$, then A_G is just the set of all matrices of the form $\varpi^j I_n$, $j \in \mathbb{Z}$. Let A_M^+ be the subset of A_M such that $j_1 \geq j_2 \geq \cdots \geq j_r$. Suppose that K_M is a compact open subgroup of M. Let (π, V) be an irreducible smooth representation of G. Then $(V_N)^{K_M}$ is $\pi_N(A_M)$-stable and finite dimensional, so $(V_N)^{K_M}$ decomposes into a direct sum of generalized eigenspaces:

$$(V_N)^{K_M} = \bigoplus_{\chi \in \mathrm{Hom}(A_M, \mathbb{C}^\times)} (V_N)_\chi^{K_M},$$

where

$$(V_N)_\chi^{K_M} = \{ \, v \in (V_N)^{K_M} \mid \exists \, \ell \geq 0 \text{ such that } (\pi_N(a) - \chi(a))^\ell v = 0 \; \forall \; a \in A_M \, \}$$

As every $v \in V_N$ belongs to $(V_N)^{K_M}$ for some compact open subgroup K_M of M, v belongs to a sum of generalized eigenspaces. Hence $V_N = \oplus_{\chi \in \mathrm{Hom}(A_M, \mathbb{C}^\times)} (V_N)_\chi$.

Definition. The set $\{ \, \chi \mid (V_N)_\chi \neq \{0\} \, \}$ is the set of *exponents* of π with respect to P.

Square integrability modulo the centre is reflected in certain properties of the exponents of π. The following theorem will be obtained as a consequence of results of Casselman on asymptotics of matrix coefficients (see next section).

Theorem. *Let (π, V) be an irreducible smooth representation of $G = GL_n(F)$ such that χ_π is unitary. Then π is square integrable modulo the centre if and only if for every standard parabolic subgroup $P = MN$ and every exponent χ of π with respect to P,*

$$|\chi(a)\delta_P^{-1/2}(a)|_\infty < 1, \qquad \forall a \in A_M^+ \backslash A_G.$$

Any reductive p-adic group which is not compact modulo its centre has some discrete series representations which are not supercuspidal. Let B be a Borel subgroup (minimal parabolic subgroup) of G. Then $i_B^G(\delta_B^{-1/2})$ has a unique square integrable irreducible subquotient, called the Steinberg representation of G. This representation has a nonzero Iwahori-fixed vector, and hence corresponds to an irreducible representation of $\mathcal{H}(G/\!\!/I)$ (I an Iwahori subgroup). For a description of this representation of $\mathcal{H}(G/\!\!/I)$, see [B]. The Steinberg representation is also discussed in [Ca].

Let σ be an irreducible supercuspidal representation of $GL_d(F)$, where d is a positive integral divisor of n. For $1 \leq j \leq n/d$, set $\sigma_j = \sigma \otimes |\det(\cdot)|_F^{j-1}$. Let ρ be the representation of the standard Levi subgroup $M_{(d,\ldots,d)}$ given by $\rho = \otimes_{1 \leq j \leq n/d} \sigma_j$. Then the induced representation $i_{P_{(d,\ldots,d)}}^G(\rho)$ has a unique irreducible quotient $\pi_{ds}(\sigma)$, . This quotient $\pi_{ds}(\sigma)$ is quasi square integrable, and is square integrable if and only if $\sigma \otimes |\det(\cdot)|_F^{(n/d-1)/2}$ is unitary. Every quasi square integrable irreducible smooth representation of $GL_n(F)$ is equivalent to $\pi_{ds}(\sigma)$ for some d and σ. (See [Z] and [JS], p. 109).

When $d = 1$, σ is a quasi character of $F^\times$ and $\pi_{ds}(\sigma)$ is a twist of the Steinberg representation by some quasi character of G. Thus when n is prime (in which case $d = 1$ or n), the discrete series consists of those irreducible supercuspidal representations which have unitary central character, together with unitary twists of the Steinberg representation.

Results on square integrable representations of classical groups appear in [Ja1-2], [MT], and [Ta1-2]. Many irreducible supercuspidal (hence quasi square integrable) representations are constructed in [Y].

Asymptotic behaviour of matrix coefficients of admissible representations

Let $P = P_\alpha$ be the standard parabolic subgroup of $GL_n(F)$ associated to a partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n. Let $M = M_\alpha$, $N = N_\alpha$, and $N^- = N_\alpha^-$. Define A_M and A_M^+ as in the section on discrete series representations.

Theorem. *(Casselman) Let (π, V) be an admissible representation of $G = GL_n(F)$. Then there exists a unique pairing $\langle\ ,\ \rangle_N$ between $\widetilde{V}_{N^-}$ and V_N such that for all $v \in V$ and $\widetilde{v} \in \widetilde{V}$, there exists $a \in A_M^+$ satisfying:*

$$(*) \qquad \langle \widetilde{v}, \pi(ab)v \rangle = \langle \bar{\widetilde{v}}, \pi_N(ab)\bar{v} \rangle_N \qquad \forall\ b \in A_M^+,$$

where $\bar{\widetilde{v}}$, resp. $\bar{v}$, denotes the image of $\widetilde{v}$, resp. v, in $\widetilde{V}_{N^-}$, resp. V_N. The pairing $\langle\ ,\ \rangle_N$ is non degenerate and M-invariant and hence induces an isomorphism $\widetilde{V}_{N^-} \simeq (V_N)\breve{}$ (this last being contragredient with respect to M).

Let $\mathcal{V}$ be the vector space of complex valued functions on the set of non negative integers. Let $T \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{V})$ be given by $(Tf)(n) = f(n+1)$. Each eigenspace of T is one dimensional: $f(n+1) = af(n) = a^n f(0)$. For each $a \in \mathbb{C}$, we have a generalized eigenspace of functions such that $(T-a)^r f = 0$ for some $r \geq 1$. A function $f \in \mathcal{V}$ such that $(T-a)^r f = 0$ has the form $f(n) = a^n p(n)$, where p is a polynomial of degree at most $r-1$ (exercise). A function $f \in \mathcal{V}$ is said to be *T-finite* if $\mathrm{Span}_{\mathbb{C}}\{f, Tf, T^2 f, \ldots\}$ is finite dimensional. Viewing $\mathcal{V}$ as a $\mathbb{C}[T]$-module, f is T-finite if and only if f has $\mathbb{C}[T]$-torsion. The subspace of $\mathcal{V}$ made up of the T-finite functions decomposes as the direct sum of generalized eigenspaces:

$$\bigoplus_{a \in \mathbb{C}} \{\, f \in \mathcal{V} \mid (T-a)^r f = 0 \text{ for some } r \geq 1 \,\}.$$

Given a T-finite f, we let f_0 denote the component of f in the generalized eigenspace for $a = 0$: so $f_0(n) = 0$ for n sufficiently large. Set $f^0 = f - f_0$. Then $f^0(n) = \sum_i a_i^n p_i(n)$ (finite sum) with each $a_i \neq 0$, and $f(n) = f^0(n)$ for n sufficiently large.

Proof of uniqueness. Suppose there exists a pairing $\langle\ ,\ \rangle_N$ satisfying $(*)$. Set

$$a_0 = \begin{pmatrix} I_{\alpha_1} & & & \\ & \varpi^{-1} I_{\alpha_2} & & \\ & & \ddots & \\ & & & \varpi^{-(r-1)} I_{\alpha_r} \end{pmatrix} \in A_M^+.$$

Fix $v \in V$ and $\widetilde{v} \in \widetilde{V}$. Set

$$f(n) = \langle \widetilde{v}, \pi(a_0^n)v \rangle \quad \text{and} \quad g(n) = \langle \bar{\widetilde{v}}, \pi_N(a_0^n)\bar{v} \rangle_N, \quad n \in \mathbb{Z}_{\geq 0}.$$

Note that $(*)$ implies that $f(n) = g(n)$ for n sufficiently large. Choose a compact open subgroup $K_M \subset M$ such that $\bar{v} \in (V_N)^{K_M}$. Then, since $a_0^n \in Z_M$, $\pi_N(a_0^n)\bar{v} \in (V_N)^{K_M}$ for all $n \geq 0$. Let $T = \pi_N(a_0) \,|\, (V_N)^{K_M}$. Define a map $(V_N)^{K_M} \to \mathcal{V}$ by

$$w \mapsto (n \mapsto g_w(n) = \langle \bar{\widetilde{v}}, \pi_N(a_0^n)w \rangle_N).$$

So $\bar{v} \mapsto g = g_{\bar{v}}$. Since

$$g_w(n+1) = (Tg_w)(n) = \langle \bar{\widetilde{v}}, T^n(T(w)) \rangle = g_{T_w}(n),$$

$w \mapsto g_w$ is a $\mathbb{C}[T]$-module map, and so takes torsion elements to torsion elements. Because V_N is admissible, $(V_N)^{K_M}$ is finite dimensional, and so $\bar{v}$ is a torsion element. Hence g is T-finite. Note also that, because $\pi_N(a_0)$ is an invertible operator on the finite dimensional vector space $(V_N)^{K_M}$, $\pi_N(a_0)$ does not have zero as an eigenvalue. This implies that $g_0 = 0$. Because $f(n) = g(n)$ for sufficiently large n, f is also T-finite, and furthermore, $f - g = (f - g)_0 = f_0 - g_0 = f_0$. Hence $g = f^0$. We have shown that any pairing satisfying $(*)$ is given by $\langle \bar{\widetilde{v}}, \bar{v} \rangle_N = f^0(0)$.

Lemma. *Let $P = MN$ be a standard parabolic subgroup and $K = K_\ell$, $\ell \geq 1$. If $a \in A_M^+$, then $m(KaK) = \delta_P(a)^{-1}m(K)$.*

Proof. First assume that $B = M_0N_0$ and $A^+ = A_{M_0}^+$. Fix $a \in A^+$. Then KaK is the union of some left K cosets of the form kaK, $k \in K$. From

$$kaK = k'aK \iff a^{-1}k'^{-1}ka \in K \iff k \in k'(aKa^{-1} \cap K)$$

it follows that $m(KaK) = [K : aKa^{-1} \cap K]\, m(K)$. Recall that K has an Iwahori factorization relative to B: $K = K^+K^oK^-$, where $K^+ = N_0 \cap K$, $K^o = K \cap M_0$, and $K^- = N_0^- \cap K$. Because $a \in A^+$, $aK^+a^{-1} \subset K^+$, and $aK^-a^{-1} \supset K^-$. Therefore

$$aKa^{-1} \cap K = (aK^+a^{-1})K^o(aK^-a^{-1}) \cap K^+K^oK^- = (aK^+a^{-1})K^oK^-,$$

and so $[K : aKa^{-1} \cap K] = [K^+ : aK^+a^{-1}]$. It is easy to check that if $a = \mathrm{diag}(\varpi^{j_1}, \ldots, \varpi^{j_n})$, then

$$[K^+ : aK^+a^{-1}] = \prod_{r<s} [\mathfrak{o} : \mathfrak{p}^{j_r - j_s}] = \prod_{r<s} q^{j_r - j_s} = |\det(\mathrm{Ad}\, a)_{\mathfrak{n}_0}|_F^{-1} = \delta_B(a)^{-1}.$$

Now take $P = MN$ standard. Note that $A_M^+ \subset A^+$. Let $a \in A_M^+$. Also $\mathfrak{n}_0 = \mathfrak{n} \oplus \mathfrak{n}'$, where $1 + \mathfrak{n}' = M \cap N_0$, and since $a \in Z_M$, $\operatorname{Ad} a \,|\, \mathfrak{n}'$ is the identity. If follows that

$$\delta_B(a) = |\det(\operatorname{Ad} a)_{\mathfrak{n}_0}|_F = |\det(\operatorname{Ad} a)_{\mathfrak{n}}|_F = \delta_P(a), \qquad a \in A_M^+.$$

□

Lemma. *Let $K = K_\ell$ for some $\ell \geq 1$. Let a, $b \in A_M^+$. Denote the characteristic function of the double coset KaK by $ch(KaK)$. Then*

(1) $KaK \cdot KbK = KabK$

(2) $\frac{ch(KaK)}{m(KaK)} * \frac{ch(KbK)}{m(KbK)} = \frac{ch(KabK)}{m(KabK)}$

(3) If π is a smooth representation of G, then $\pi(e_K)\pi(a)\pi(e_K) = \pi(\frac{ch(KaK)}{m(KaK)})$.

Proof. Use the Iwahori factorization of K with respect to P: $K = K^+K^oK^-$, $K^+ = K \cap N$, $K^o = K \cap M$, $K^- = K \cap N^-$. With the obvious notation,

$$ak^+k^ok^-b = (ak^+a^{-1})(ak^ob)(b^{-1}k^-b) = (ak^+a^{-1}k^o)ab(b^{-1}k^-b)$$

the second equality following from the fact that $a \in Z_M$, so must commute with k^o. The set A_M^+ has the property that $aK^+a^{-1} \subset K^+$ and $a^{-1}K^-a \subset K^-$ for all $a \in A_M^+$. Therefore

$$(ak^+a^{-1}k^o)ab(b^{-1}k^-b) \subset K^+k^oabK^- \subset KabK$$

Next, for (2),

$$(ch_{KaK} * ch_{KbK})(x) = \int_G ch_{KaK}(y)ch_{KbK}(y^{-1}x)\, dy = \int_{KaK} ch_{KbK}(y^{-1}x)\, dy$$

If the value is non zero, then $x \in yKbK$ for some $y \in KaK$. By (1), $x \in KabK$. Suppose that $x = k_1abk_2$ and $y = k_3ak_4$, for some $k_j \in K$, $1 \leq j \leq 4$. Then

$$y^{-1}x \in KbK \iff a^{-1}k_3^{-1}k_1a \in K \iff y \in k_1(aKa^{-1} \cap K)aK = k_1aK.$$

In this case, the above convolution is equal to $\int_{k_1aK} dy = m(k_1aK) = m(K)$. Hence $ch_{KaK} * ch_{KbK} = m(K)\, ch_{KabK}$. Statement (2) now follows from the previous lemma (using the fact that δ_P is a homomorphism).

Fix $a \in A_M^+$ and $v \in V$. Note that $k_1 \mapsto \int_K \pi(k_1 a k_2) v \, dk_2$ is constant on left cosets of $aKa^{-1} \cap K$ in K. Therefore

$$\begin{aligned}
\pi(e_K)\pi(a)\pi(e_K)v &= m(K)^{-2} \int_K \int_K \pi(k_1 a k_2) v \, dk_2 \, dk_1 \\
&= m(K)^{-2} m(aKa^{-1} \cap K) \sum_{k_1 \in K/(aKa^{-1} \cap K)} \int_K \pi(k_1 a k_2) v \, dk_2 \\
&= [K : aKa^{-1} \cap K]^{-1} m(K)^{-1} \int_{KaK} \pi(x) v \, dx \\
&= m(KaK)^{-1} \int_G ch_{KaK}(x) \pi(x) v \, dx = \pi\left(\frac{ch(KaK)}{m(KaK)}\right) v
\end{aligned}$$

□

Proof of existence. Given $v \in V$ and $\widetilde{v} \in \widetilde{V}$, set $f_{\widetilde{v},v}(n) = \langle \widetilde{v}, \pi(a_0^n) v \rangle$. First, we show that $f_{\widetilde{v},v}$ is T-finite. Choose $\ell \geq 1$ such that $v \in V^K$ and $\widetilde{v} \in \widetilde{V}^K$ for $K = K_\ell$. Define a $\mathbb{C}[T]$-module structure on V^K by letting T act via $\pi(e_K a_0 e_K)$. Then, for $w \in V^K$,

$$\begin{aligned}
\langle \widetilde{v}, T^n w \rangle &= \langle \widetilde{v}, (e_K a_0 e_K)^n w \rangle = \langle \widetilde{v}, e_K a_0^n e_K w \rangle \\
&= \langle e_K \widetilde{v}, a_0^n e_K w \rangle = \langle \widetilde{v}, a_0^n w \rangle = f_{\widetilde{v},w}(n)
\end{aligned}$$

Map $V^K \to \mathcal{V}$ by $w \mapsto f_{\widetilde{v},w}$. This is a $\mathbb{C}[T]$-module map, and admissibility of π guarantees that V^K is finite dimensional, so v is T-finite. Hence $f_{\widetilde{v},v}$ is T-finite.

Set $\langle \widetilde{v}, v \rangle_1 = {f_{\widetilde{v},v}}^0(0)$. $\langle \ , \ \rangle_1$ is bilinear and M-invariant (since $f_{m\widetilde{v},mv} = f_{\widetilde{v},v}$ by invariance of $\langle \ , \ \rangle$). Next we will show that $\langle \ , \ \rangle_1$ is really a pairing between $\widetilde{V}_{N^-}$ and V_N. Let $v \in V$. Then

$$\begin{aligned}
\langle \widetilde{v}, v \rangle_1 = 0 \ \forall \ \widetilde{v} \in \widetilde{V} &\Longleftrightarrow \forall \ \widetilde{v} \in \widetilde{V}, \ \forall \ n \geq 0, \ 0 = \langle a_0^{-n} \widetilde{v}, v \rangle_1 = \langle \widetilde{v}, a_0^n v \rangle_1 = {f_{\widetilde{v},v}}^0(n) \\
&\Longleftrightarrow \forall \ \widetilde{v} \in \widetilde{V}, \ f_{\widetilde{v},v}(n) = 0 \text{ for } n >> 0 \\
&\Longleftrightarrow \forall \, \ell' \geq \ell, \ K' = K_{\ell'}, \ e_{K'} a_0^n v = 0 \text{ for } n >> 0 \qquad (**)
\end{aligned}$$

Proof of final equivalence: ($\Longleftarrow$) Given $\widetilde{v} \in \widetilde{V}$, choose $\ell' \geq \ell$ such that $\widetilde{v}$ is K'-invariant. Then

$$f_{\widetilde{v},v}(n) = \langle \widetilde{v}, a_0^n v \rangle = \langle e_{K'} \widetilde{v}, a_0^n v \rangle = \langle \widetilde{v}, e_{K'} a_0^n v \rangle = 0, \qquad \text{for } n >> 0.$$

($\Longrightarrow$) Given $\ell' \geq \ell$. Then, as $\widetilde{V}^{K'}$ is finite dimensional, the span of the functions $f_{\widetilde{v},v}$, $\widetilde{v} \in \widetilde{V}^{K'}$, is finite dimensional. As each such function vanishes for sufficiently large n, it follows from the finite dimensionality that there exists $N > 0$ such that for all $\widetilde{v} \in \widetilde{V}^{K'}$, $f_{\widetilde{v},v}(n) = 0$ whenever $n \geq N$. That is,

$$\forall \widetilde{v} \in \widetilde{V}^{K'} = \mathrm{Hom}(V^{K'}, \mathbb{C}),\ \langle \widetilde{v}, a_0^n v\rangle = \langle \widetilde{v}, e_{K'} a_0^n v\rangle = 0, \qquad \text{for } n \geq N.$$

Thus $e_{K'} a_0^n v = 0$ for $n >> 0$ and we have the desired equivalence.

Suppose that $\ell' \geq \ell$. Note that

$$e_{K'} a_0^n v = 0 \Longleftrightarrow e_{a_0^{-n} K' a_0^n} v = 0$$
$$\text{and } a_0^{-n} K' a_0^n = (a_0^{-n} K'^{+} a_0^n)(a_0^{-n} K'^{o} a_0^n)(a_0^{-n} K'^{-} a_0^n) = (a_0^{-n} K'^{+} a_0^n) K'^{o} (a_0^{-n} K'^{-} a_0^n).$$

As $a_0 \in A_M^+$, $a_0^{-n} K'^{-} a_0^n \subset K'^{-}$. Since $\ell' \geq \ell$, $v \in V^{K'}$. So

$$e_{K'} a_0^n v = 0 \Longleftrightarrow e_{a_0^{-n} K'^{+} a_0^n} v = 0.$$

For $n \geq 0$, $a_0^{-n} K'^{+} a_0^n$ is a compact open subgroup of N. Using the equivalence $(**)$, we conclude that $\langle \widetilde{v}, v\rangle_1 = 0$ for all $\widetilde{v} \in \widetilde{V}$ if and only if $e_U v = 0$ for some compact open subgroup U of N, that is, if and only if $v \in V(N)$.

By a similar argument, $\widetilde{V}(N^-) = \{\, \widetilde{v} \in \widetilde{V} \mid \langle \widetilde{v}, v\rangle_1 = 0 \ \forall\ v \in V \,\}$. We can now set $\langle \bar{\widetilde{v}}, \bar{v}\rangle_N = \langle \widetilde{v}, v\rangle_1$.

It remains to prove that $\langle\ ,\ \rangle_N$ satisfies $(*)$. Fix $K = K_\ell$, $\ell \geq 1$. It suffices to show there exists $n \geq 0$ such that for all $v \in V^K$ and $\widetilde{v} \in \widetilde{V}^K$,

$$(i) \qquad \langle \widetilde{v}, a_0^n b v\rangle = \langle \widetilde{v}, a_0^n b v\rangle_1, \qquad \forall\ b \in A_M^+.$$

Because V^K and $\widetilde{V}^K$ are finite dimensional, there exists $n \geq 0$ such that

$$f_{\widetilde{v},v}(j) = f_{\widetilde{v},v}{}^{0}(j), \qquad \forall\ j \geq n,\ v \in V^K,\ \widetilde{v} \in \widetilde{V}^K.$$

Thus

$$(ii) \qquad \langle \widetilde{v}, a_0^n v\rangle = \langle \widetilde{v}, a_0^n v\rangle_1, \qquad \forall v \in V,\ \widetilde{v} \in \widetilde{V}^K.$$

Next, observe that, for $b \in A_M^+$,

$$\begin{aligned} f_{\widetilde{v}, e_K b e_K v}(n) &= \langle \widetilde{v}, a_0^n e_K b e_K v\rangle = \langle \widetilde{v}, e_K a_0^n e_K \cdot e_K b e_K v\rangle \\ &= \langle \widetilde{v}, e_K a_0^n b e_K v\rangle = \langle \widetilde{v}, a_0^n b v\rangle = f_{\widetilde{v}, bv}(n). \end{aligned}$$

By definition of $\langle\ ,\ \rangle_1$, this implies

$$(iii) \qquad \langle \widetilde{v}, bv\rangle_1 = \langle \widetilde{v}, e_K b e_K v\rangle_1, \qquad \forall\, b \in A_M^+.$$

For $b \in A_M^+$, $v \in V^K$ and $\widetilde{v} \in \widetilde{V}^K$,

$$\begin{aligned}
\langle \widetilde{v}, a_0^n bv\rangle &= \langle \widetilde{v}, a_0^n e_K b e_K v\rangle \ \text{ argue as above for } (iii)\\
&= \langle \widetilde{v}, a_0^n e_K b e_K v\rangle_1 \ \text{ by } (ii)\\
&= \langle \widetilde{v}, e_K a_0^n e_K \cdot e_K b e_K v\rangle_1 \ \text{ by } (iii)\\
&= \langle \widetilde{v}, e_K a_0^n b e_K v\rangle_1 = \langle \widetilde{v}, a_0^n bv\rangle_1 \text{ by } (iii)
\end{aligned}$$

Hence (i) holds. This completes the proof of the theorem. $\square$

Theorem. *Let (π, V) be an irreducible smooth representation of $G = GL_n(F)$ such that (the central character) χ_π is unitary. Then π is square integrable modulo the centre if and only if for every standard parabolic subgroup $P = MN$ and every exponent χ of π with respect to P,*

$$|\chi(a)\delta_P^{-1/2}(a)|_\infty < 1, \qquad \forall\, a \in A_M^+ \backslash A_G.$$

Sketch of proof. We want to work in $G_{ad} = G/Z$. Given $S \subset G$, we denote the image of S in G/Z by S_{ad}. Let $g_1, \ldots, g_t$ be coset representatives for $K_{ad} \lhd K_{0,ad}$, where $K = K_m$. Then

$$\begin{aligned}
&G_{ad} = \coprod_{a \in A_{ad}^+} K_{0,ad} a' K_{0,ad} \qquad \text{(Cartan decomposition)}\\
&[K_{ad} : a' K_{ad} a'^{-1} \cap K_{ad}] = [K : aKa^{-1} \cap K] = \delta_B(a)^{-1}\\
&\frac{m(K_{ad} g_i a' g_j K_{ad})}{m(K_{ad})} = \delta_B(a)^{-1}
\end{aligned}$$

Let $\mathbb{Z}^t$ act via right translations on functions $f : \mathbb{Z}^t \to \mathbb{C}$. Eigenfunctions: $(y \cdot f)(x) = f(x+y) = \chi(y)f(x)$, where $y \mapsto \chi(y)$ is a quasi character. Let $z_1 = \chi(1, 0, \ldots, 0), \ldots, z_t = \chi(0, 0, \ldots, 1)$. Then $\chi(k_1, \ldots, k_t) = z_1^{k_1} \cdots z_t^{k_t}$. Generalized eigenfunctions have the form $p(k_1, \ldots, k_t) z_1^{k_1} \cdots z_t^{k_t}$, for a polynomial $p(k_1, \ldots, k_t)$.

Suppose that π is square integrable modulo Z. Given $P = MN$ a standard parabolic subgroup and χ an exponent of π with respect to P, choose $v \in V$ such that $\bar{v} \in V_N$ is a non zero eigenvector for χ: $\pi_N(a)\bar{v} = \chi(a)\bar{v}$, for $a \in A_M$. Pick $\widetilde{v} \in \widetilde{V}$ such that $\langle \bar{\widetilde{v}}, \bar{v}\rangle_N \neq 0$. By Casselman's theorem, there exists $a \in A_M^+$ such that for all $b \in A_M^+$,

$$\langle \widetilde{v}, abv\rangle = \langle \bar{\widetilde{v}}, ab\bar{v}\rangle_N = \chi(ab)\langle \bar{\widetilde{v}}, \bar{v}\rangle_N.$$

Let $c = \langle \bar{\widetilde{v}}, \bar{v} \rangle_N$. Then

$$\infty > \int_{G/Z} |\langle \widetilde{v}, \pi(g)v \rangle|_\infty^2 \, dg^* \geq \sum_{b \in (A_M^+)_{ad}} |\langle \widetilde{v}, abv \rangle|_\infty^2 \, m(K_{ad} ab K_{ad})$$
$$= |\chi(a)c|_\infty^2 \delta_P(a)^{-1} \sum_{b \in (A_M^+)_{ad}} |\chi(b)|_\infty^2 \delta_P(b)^{-1}$$

Thus $\sum_{b \in (A_M^+)_{ad}} |\chi(b)\delta_P^{-1/2}(b)|_\infty^2 < \infty$. Suppose that $M = M_\alpha$, $\alpha = (\alpha_1, \dots, \alpha_r)$. Then $(A_M^+)_{ad} \simeq \mathbb{Z}_{\geq 0}^{r-1}$ is generated by $(1, 0, \dots, 0)$, $(1, 1, 0, \dots, 0), \dots, (1, 1, \dots, 1, 0)$. The above sum has the form

$$\sum_{1 \leq j_i \leq \infty} w_i^{j_i} = \prod_{1 \leq i \leq r-1} \left(\sum_{j_i \geq 0} w_i^{j_i} \right),$$

where $w_i = \chi^2 \delta_P^{-1}(1, 1, \dots, 1, 0, \dots, 0)$, with i ones. Applying the convergence criterion for geometric series, we find that $|\chi\delta_P^{-1/2}(a)|_\infty < 1$ for all $a \in A_M^+ \backslash A_G$.

For the other direction, to prove that π is square integrable modulo Z, it follows from the decomposition $G_{ad} = \cup_{i,j,a \in A_{ad}^+} K_{ad} g_i a g_j K_{ad}$ and remarks above concerning measures, that it suffices to show that

$$\sum_{a \in A_{ad}^+} |\langle \widetilde{v}, k_1 g_i a g_j k_2 v \rangle|_\infty^2 \delta_B(a)^{-1} < \infty \qquad \forall\, i, j \text{ and } k_1, k_2 \in K_{ad}.$$

Replacing $\widetilde{v}$ by $g_i^{-1} k_1^{-1} \widetilde{v}$ and v by $g_j k_2 v$, we must show that

$$\sum_{a \in A_{ad}^+} |\langle \widetilde{v}, av \rangle|_\infty^2 \delta_B(a)^{-1} < \infty \qquad \forall\, \widetilde{v} \in \widetilde{V},\ v \in V.$$

Suppose that $G = GL_3(F)$. Fix $a = (\ell_1, \ell_2, 0) \in A_{ad}^+$, $\ell_1 \geq \ell_2 \geq 0$. As $b = (m_1, m_2, 0)$, $m_1 \geq m_2 \geq 0$ varies over A_{ad}^+, the set $\{\, ab \mid b \in A_{ad}^+ \,\}$ has the following form:

Applying the first theorem with $P = B$, there exists $a \in A_{ad}^+$ such that

$$(*) \qquad \sum_{b \in A_{ad}^+} |\langle \widetilde{v}, abv \rangle|_\infty^2 \delta_B(ab)^{-1} = \sum_{b \in A_{ad}^+} |\langle \bar{\widetilde{v}}, ab\bar{v} \rangle_N|_\infty^2 \delta_B(ab)^{-1}.$$

Expressing $\bar{\tilde{v}}$ and $\bar{v}$ as sums of generalized eigenvectors for the action of A_{ad}, we can show that, for $b \in A_{ad}^{+}$,

$$\langle \bar{\tilde{v}}, ab\bar{v} \rangle_N = \sum_{\chi} p_\chi(coord(ab))\chi(ab),$$

where the sum is over finitely many exponents χ, p_χ is a polynomial in the coordinates $coord(ab)$ of the element ab. Then the assumptions on the exponents of π with respect to B guarantee that the sum in $(*)$ is convergent. The details are left as an exercise.

It remains to show that the following sum converges:

$$\sum_{\{b \in A_{ad}^{+} \mid b \notin aA_{ad}^{+}\}} |\langle \tilde{v}, bv \rangle|_\infty^2 \delta_B(b)^{-1}.$$

The sum is over the points in the unshaded part of the above diagram. This sum can be broken up into a finite sum of terms (each summand looking like the sum over lattice points on a half line in the diagram), the convergence of each term following from an application of the first theorem with P an intermediate parabolic subgroup: that is, P conjugate to $P_{(2,1)}$. $\square$

More about Jacquet modules

Assume that G is (the F-rational points of) a split group. For example, $G = GL_n(F)$, $SL_n(F)$, or $Sp_{2n}(F)$. Many of the descriptions and arguments will be given for the case $G = GL_n(F)$.

Before discussing Jacquet modules, we make a few comments about compositions series. Let (π, V) be a smooth representation of G. π (or V) has a *composition series* if there exists a descending chain of G-invariant subspaces of V:

$$V = V_0 \supset V_1 \supset \cdots \supset V_\ell \supset V_{\ell+1} = \{0\},$$

such that V_j/V_{j+1} is irreducible. The subquotients V_j/V_{j+1} are called the *(Jordan-Holder) factors* of the composition series, and the *length* of the composition series is the number of factors. Any two composition series have the same length. We say that a representation has finite length if it has a (finite) composition series. It is known that any finitely generated admissible representation has finite length.

Let $B = AN$ be a standard Borel subgroup of G. Let $W = N_G(A)/A$, where $N_G(A)$ denotes the normalizer of A in G. The group W is called the *Weyl group* of G. Then we have a disjoint union $G = \coprod_{w \in W} BwB$. If M is a standard Levi subgroup, let W_M denote the Weyl group of M - we can view it as a subgroup of W since A is also the Levi subgroup of a standard Borel subgroup of M.

Lemma. *Let $P = MN$ and $Q = LU$ be standard parabolic subgroups of G. Then $P\backslash G/Q \longleftrightarrow W_M\backslash W/W_L$.*

Sketch of proof. Since Q is standard, $B \subset Q$. Hence $wB \subset Q$ if and only if $w \in Q$. Suppose that $x = \ell u \in N_G(A) \cap Q$. Then, because $\ell \in L$ and $A \subset L$, it follows that $\ell u A u^{-1} \ell^{-1} = A$ implies that $uAu^{-1} \subset L$. Multiplying on the left by elements of A, we see that $a^{-1}uau^{-1} \subset L$ for all $a \in A$. But $A \subset L$ and $u \in U$ imply $a^{-1}ua \in U$. So we have $a^{-1}uau^{-1} \in L \cap U = \{1\}$, that is, u centralizes A. Since the only elements in G which centralize all of A lie in A, and $A \cap U = \{1\}$, we see that $u = 1$. Thus $N_G(A) \cap Q \subset L$. It follows that $N_G(A) \cap Q = N_L(A)$ (and also $N_G(A) \cap P = N_M(A)$). $\square$

Define a partial ordering on $W_M\backslash W/W_L$ by $w < w'$ if $PwQ \subset \overline{Pw'Q}$, the bar denoting closure. Set $G_w = \cup_{w' > w} Pw'Q$. Given $w \in W$, let $d(w)$ be the dimension of the F-variety $P\backslash PwQ$ (it is useful to look at examples to see what this looks like - computing $d(w)$ is counting the number of independent coordinates in matrices which are coset representatives).

Suppose that σ is an irreducible supercuspidal representation of M. We use normalized induction $i_P^G(\sigma) = \mathrm{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$, and normalized Jacquet modules $r_P^G(\pi) = \pi_N \otimes \delta_P^{-1/2}$. The goal is to describe the Jacquet module $r_Q^G i_P^G \sigma$. For each integer $j \geq 0$, let

$$I_j = \{\, f \in i_P^G(\sigma) \mid \operatorname{supp} f \subset \cup_{\{w \,\mid\, d(w) \geq j\}} G_w \,\}.$$

Note that each I_j is a Q-module. This allows us to define $(I_j)_U$ (even though I_j is not a G-module).

Theorem. *(Casselman) The filtration $0 \subset I_{j_\ell} \subset \cdots \subset I_0 = i_P^G(\sigma)$ has the property that*

$$(I_j/I_{j+1})_U \simeq (I_j)_U/(I_{j+1})_U \simeq \bigoplus_{d(w)=j} J_{w,U}$$

$$\text{where} \quad J_{w,U} \otimes \delta_Q^{-1/2} = \begin{cases} i^L_{w^{-1}Pw \cap L}(w^{-1}\sigma), & \textit{if } w^{-1}Mw \subset L \\ 0, & \textit{otherwise.} \end{cases}$$

In general $w^{-1}Pw \cap L$ is a parabolic subgroup in L with Levi component $w^{-1}Mw \cap L$ and unipotent radical $w^{-1}Nw \cap L$. When $w^{-1}Mw \subset L$, then its Levi component is all of $w^{-1}Mw$. The above theorem says something about decomposing $(i_P^G(\sigma))_U$ in terms of certain (generally reducible) subquotients which are themselves parabolically induced representations of L. There is a similar filtration of $i_P^G(\sigma)$ for σ irreducible admissible, and $(I_j/I_{j+1})_U$ decomposes according to certain parabolically induced representations, but there are more terms in the decomposition (arising from the fact that if σ is not supercuspidal, then σ will have non zero Jacquet modules).

We will not discuss the proof of the theorem. Later it will be applied in the special case $P = B = Q$ with $\sigma = \chi$ a quasi character of A. In this case the theorem says that $r_B^G i_B^G(\chi)$ has a composition series with Jordan-Holder factors equal to χ^w, $w \in W$.

Linear independence of characters

Let (π, V) be an admissible representation of G. Recall that the character Θ_π of π is the distribution defined by

$$\Theta_\pi(f) = \operatorname{trace} \pi(f) \quad \text{where} \quad \pi(f) = \int_G f(x)\pi(x)\, dx, \qquad f \in C_c^\infty(G).$$

Theorem. *Let (π_j, V_j), $1 \le j \le r$, be irreducible admissible representations of G which are pairwise inequivalent. Then the distributions Θ_{π_j}, $1 \le j \le r$, are linearly independent.*

Proof. Choose a compact open subgroup K of G such that $V_j^K \neq \{0\}$, $1 \le j \le r$. The irreducible representation of $\mathcal{H}(G/\!\!/K)$ corresponding to π_j will also be denoted by π_j. By earlier results the Hecke algebra representations are pairwise inequivalent. We have a map

$$\begin{aligned} \mathcal{H}(G/\!\!/K) &\longrightarrow \operatorname{End}_{\mathbb{C}}(V_1^K) \times \cdots \times \operatorname{End}_{\mathbb{C}}(V_r^K) \\ f &\longmapsto (\pi_1(f), \ldots, \pi_r(f)) \end{aligned}$$

The main part of the proof consists of showing that this map is onto. Set $\mathcal{H} = \mathcal{H}(G/\!\!/K)$.

By Schur's lemma, $\operatorname{Hom}_{\mathcal{H}}(V_j^K, V_j^K) \simeq \mathbb{C}$. We view V_j^K as a vector space over $\operatorname{Hom}_{\mathcal{H}}(V_j^K, V_j^K)$. We have a map $\mathcal{H} \to \operatorname{End}_{\mathbb{C}}(V_j^K)$: $f \mapsto \pi_j(f)$. In this case, the Jacobson density theorem (see for example Hungerford, p. 420) says that the endomorphisms $\{\, \pi_j(f) \mid f \in \mathcal{H} \,\}$ are isomorphic to a dense ring of endomorphisms of the $\operatorname{Hom}_{\mathcal{H}}(V_j^K, V_j^K)$-vector space V_j^K. That is for every positive integer ℓ and linearly independent set $\{u_1, \ldots, u_\ell\} \subset V_j^K$ and every subset $\{v_1, \ldots, v_\ell\} \subset V_j^K$, there exists $f \in \mathcal{H}$ such that $\pi_j(f)u_j = v_j$. Since V_j^K is finite dimensional, this is equivalent to $\{\, \pi_j(f) \mid f \in \mathcal{H} \,\} = \operatorname{End}_{\mathbb{C}}(V_j^K)$.

We need a more general version of the Jacobson density theorem. Set $\mathcal{M} = V_1^K \times \cdots \times V_r^K$, with the action of $\mathcal{H}$ as above. For each j, let $p_j : \mathcal{M} \to V_j^K$ be projection onto the jth component. Then $p_j \in \operatorname{Hom}_{\mathcal{H}}(\mathcal{M}, V_j^K)$. Let $i_\ell : V_\ell^K \to \mathcal{M}$ be the usual inclusion map. Then, given $\varphi \in \operatorname{End}_{\mathcal{H}}(\mathcal{M})$, the composition $p_j \circ \varphi \circ i_\ell : V_\ell^K \to V_j^K$ belongs to $\operatorname{Hom}_{\mathcal{H}}(V_\ell^K, V_j^K)$. If $\ell \neq j$, then $\operatorname{Hom}_{\mathcal{H}}(V_\ell^K, V_j^K) = \{0\}$ since π_ℓ and π_j are not equivalent. This forces $\varphi(i_\ell(V_\ell^K)) \subset i_\ell(V_\ell^K)$. By irreducibility of π_ℓ, if $\varphi(i_\ell(V_\ell^K)) \neq 0$, then $\varphi(i_\ell(V_\ell^K)) = i_\ell(V_\ell^K)$. We have $\operatorname{End}_{\mathcal{H}}(\mathcal{M}) = \operatorname{End}_{\mathcal{H}}(V_1^K) \times \cdots \times \operatorname{End}_{\mathcal{H}}(V_r^K)$, this last equaling $\mathbb{C} \times \cdots \times \mathbb{C}$ by Schur's lemma.

We view $\mathcal{M}$ as an $\operatorname{End}_{\mathcal{H}}(\mathcal{M})$-module in the usual way: given $v \in \mathcal{M}$ and $\varphi \in \operatorname{End}_{\mathbb{C}}(\mathcal{H})$, $\varphi \cdot v = \varphi(v)$. From the above description of $\operatorname{End}_{\mathcal{H}}(\mathcal{M})$, we see that $\operatorname{End}_{\operatorname{End}_{\mathcal{H}}(\mathcal{M})}(\mathcal{M})$ consists of maps from $\mathcal{M}$ to $\mathcal{M}$ commuting with all endomorphisms of $V_1^K \times \cdots \times V_r^K$ of the form $c_1 I_{V_1{}^K} \times \cdots \times c_r I_{V_r{}^K}$, $c_j \in \mathbb{C}$. Thus $\mathrm{End}_{\mathrm{End}_{\mathcal{H}}(\mathcal{M})}(\mathcal{M}) = \mathrm{End}_{\mathbb{C}}(V_1^K) \times \cdots \times \mathrm{End}_{\mathbb{C}}(V_r^K)$. By the Jacobson density theorem, given $T \in \mathrm{End}_{\mathrm{End}_{\mathcal{H}}(\mathcal{M})}(\mathcal{M})$ and $x_1, \ldots, x_s \in \mathcal{M}$, there exists $f \in \mathcal{H}$ such that $f \cdot x_i = T(x_i)$.

Having shown that $f \mapsto (\pi_1(f), \ldots, \pi_r(f))$ maps onto $\prod_{1 \leq j \leq r} \mathrm{End}_{\mathbb{C}}(V_j^K)$, we can choose $f_j \in \mathcal{H}$ such that $\pi_j(f_j) = 1_{V_j^K}$ and $\pi_\ell(f_j) = 0$ if $\ell \neq j$. Linear independence of the characters is now immediate from $\operatorname{tr} \pi_\ell(f_j) = \delta_{j\ell} \dim V_\ell^K$. □

To compare characters of π, π' and subquotients, it is enough (by the proof of the above proposition) to compare their restrictions to function which are K bi invariant for some compact open subgroup chosen such that the space of K fixed vectors is non zero for all of the subquotients. Since the representations of the group are admissible, the corresponding representations of the Hecke algebra are acting on finite dimensional subspaces. It is straightforward to check that Θ_π equals the sum of the characters of the irreducible subquotients of π.

Corollary. *Let (π, V) and (π', V') be admissible representations of finite length. Then π and π' have the same Jordan-Holder factors if and only if $\Theta_\pi = \Theta_{\pi'}$.*

The above corollary is a generalization of a well known result for compact groups. If G is compact, then $\Theta_\pi = \Theta_{\pi'}$ if and only if $\pi \simeq \pi'$. As the representations of compact groups are completely reducible, π and π' have the same Jordan-Holder factors if and only if $\pi \simeq \pi'$. For G non compact, two representations can have the same Jordan-Holder factors without being equivalent.

Principal series representations

Fix a standard Borel subgroup $B = AN$ of G. The *principal series* of representations is the set of representations of the form $i_B^G(\chi)$, where χ is a quasi character of A.

Proposition. *Let V be a nonzero subquotient of $i_B^G(\chi)$. Then $r_B^G(V) \neq 0$.*

The following theorem will be used in the proof of the proposition. We will not discuss the proof of the theorem.

Theorem. *Suppose that $G = GL_n(F)$. Set $G_0 = \{ g \in GL_n(F) \mid |\det(g)|_F = 1 \}$. Let (π, V) be a smooth representation of G_0. Then there exists a unique decomposition $V = V_{sc} \oplus V_{nc}$ where V_{sc} is a supercuspidal G_0-invariant subspace and V_{nc} is a G_0-invariant subspace having no supercuspidal subquotients.*

Lemma. *Suppose that (π, V) and (π', V') are representations of $G = GL_n(F)$ having the property that on the centre Z, π and π' act by scalars, and $\pi \mid Z = \pi' \mid Z$. Suppose also that $\mathrm{Hom}_{G_0}(\pi', \pi) \neq 0$. Then $\mathrm{Hom}_G(\pi', \pi \otimes \eta) \neq 0$ for some character η of G.*

Proof. Recall that $G_0 Z$ is normal in G, the quotient $G/G_0 Z$ cyclic of order n generated by cosets of by a matrix a_ϖ whose determinant equals ϖ. Consider the representation of G on $\mathrm{Hom}_G(\pi', \pi)$ given by $g \cdot A = gAg^{-1}$. As $G_0 Z$ acts trivially, it factors to a representation of $G/G_0 Z$, and hence is a direct sum of characters of $G/G_0 Z$. Choose a character η and a nonzero A such that $g \cdot A = \eta^{-1}(g) A$. Then it is a simple matter to check that $A \in \mathrm{Hom}_G(\pi', \pi \otimes \eta)$. $\square$

Proof of proposition. Consider the set of all standard parabolic subgroups P of G such that $r_P^G(V) \neq 0$. As $r_G^G(V) \neq 0$, there is a minimal such P (minimal with respect to inclusion). Fix such a minimal P. By transitivity of Jacquet modules, all proper Jacquet modules of $r_P^G(V)$ are zero. That is, $r_P^G(V)$ is a supercuspidal representation of M ($P = MN$). Since V is a subquotient of $i_B^G(V)$, $r_P^G(V)$ is a subquotient of $r_P^G i_B^G(\chi)$.

Any irreducible subquotient $\mathcal{W}$ of $r_P^G(V)$ must therefore be a subquotient of some $i^M_{w^{-1}Bw \cap M}(w^{-1}\chi)$, $w \in W$. The group $B' = w^{-1}Bw \cap M$ is a Borel subgroup of M with Levi component A. The centre Z_M of M is contained in B' and acts via $\chi^w \mid Z_M$. Applying the above theorem and lemma with M and M_0 in place of G and G_0, we see that some twist of $\mathcal{W}$ is a subrepresentation of $i_{B'}^M(\chi^w)$. By exactness of the Jacquet functor, we have

$$0 \to r_{B'}^M(\mathcal{W}) \to r_{B'}^M i_{B'}^M(\chi^w) \to r_{B'}^M(i_{B'}^M(\chi^w)/\mathcal{W}) \to 0.$$

Since $r^M_{B'}(i_{B'}(\chi^w) \neq 0$, it follows that $r^M_{B'}(\mathcal{W}) \neq 0$. As $\mathcal{W}$ is supercuspidal, this forces $B' = M$. That is, $A = M$ and $P = B$. Thus $r^G_B(V) \neq 0$. $\square$

Corollary. *$i^G_B(\chi)$ has finite length, the length being at most $|W|$.*

Proof. Suppose we have a chain $i^G_B(\chi) \supset V_1 \supset V_2 \supset \cdots$, with strict containment. Then at the level of Jacquet modules we have $r^G_B i^G_B(\chi) \supset r^G_B(V_1) \supset r^G_B(V_2) \supset \cdots$. By the proposition, $r^G_B(V_i)/r^G_B(V_{i+1}) = r^G_B(V_i/V_{i+1}) \neq 0$. By previous results, the length of $r^G_B i^G_B(\chi)$, which we have just seen is an upper bound for the length of $i^G_B(\chi)$, is equal to $|W|$. $\square$

Proposition. *If χ is unitary and the stabilizer of χ in W is trivial, then $i^G_B(\chi)$ is irreducible.*

Proof. Suppose that χ is unitary. Then $i^G_B(\chi)$ is unitary. By the corollary, $i^G_B(\chi)$ has finite length. Hence $i^G_B(\chi)$ is a finite direct sum of irreducible unitary representations, and $i^G_B(\chi)$ is irreducible if and only if $\mathrm{End}_G(i^G_B(\chi)) = \mathbb{C}$. By Frobenius reciprocity, $\mathrm{End}_G(i^G_B(\chi)) = \mathrm{Hom}_A(r^G_B i^G_B(\chi), \chi)$. The Jordan-Holder factors of $r^G_B i^G_B(\chi)$ are all of the form χ^w, $w \in W$ (each occurring once). Thus $\dim(\mathrm{End}_G(i^G_B(\chi)) \leq |\mathrm{Stab}_W(\chi)|$. $\square$

Now we assume that G is a split reductive group, for example $GL_n(F)$, $SL_n(F)$, $Sp_{2n}(F)$. In this case, $A \simeq (F^\times)^d$, where $d = \dim(A) = \mathrm{rank}(G)$. Fix an isomorphism $a \mapsto (a_1, \ldots, a_d) \in (F^\times)^d$. The subgroup $A_0 \simeq (O_F^\times)^d$ is the maximal compact subgroup of A. A quasi character of A which is trivial on A_0 is said to be *unramified*. If χ is unramified, then from properties of quasi characters of $F^\times$, it follows that there exist $s_1, \ldots, s_d \in \mathbb{C}$ such that $\chi(a_1, \ldots, a_d) = |a_1|_F^{s_1} \cdots |a_d|_F^{s_d}$. The d-tuple $z = (|\varpi|_F^{s_1}, \ldots, |\varpi|_F^{s_d}) = (q^{-s_1}, \ldots, q^{-s_d}) \in (\mathbb{C}^\times)^d$ determines χ. Conversely any element $z \in (\mathbb{C}^\times)^d$ gives rise to a unique unramified character, denoted by χ_z. Thus we can identify the set $\mathrm{Hom}(A/A_0, \mathbb{C}^\times)$ of unramified characters of A with $(\mathbb{C}^\times)^d$.

Fix a quasi character χ of A. As z varies over the complex variety $(\mathbb{C}^\times)^d$, we get a family $\chi\chi_z$ of quasi characters of A. The ring R of regular functions on $(\mathbb{C}^\times)^d$ is the group algebra $\mathbb{C}[A/A_0]$ of $A/A_0 \simeq \mathbb{Z}^d$. Let χ_{un} denote the inclusion map $A/A_0 \to R = \mathbb{C}[A/A_0]$. Given $z \in (\mathbb{C}^\times)^d$, the evaluation map $\alpha_z : \mathbb{C}[A/A_0]^\times \to \mathbb{C}^\times$ is a $\mathbb{C}$-algebra homomorphism. The composition

$$A \longrightarrow A/A_0 \xrightarrow{\chi_{un}} \mathbb{C}[A/A_0]^\times \xrightarrow{\alpha_z} \mathbb{C}^\times$$

is the map $a \mapsto \chi_z(a)$.

Define an R-module M by

$$M = i_B^G(\chi\chi_{un}) = \{\, f : G \to R \mid f(bg) = \delta_B^{-1/2}(b)\chi(b)\chi_{un}(b)f(g),$$
$$f \text{ right invariant wrt some compact open subgroup } \}.$$

Letting G act by right translation on the functions in M gives a map $G \to \mathrm{Aut}_R(M)$. Given $z \in (\mathbb{C}^\times)^d$, we define a complex representation M_z of G by $M_z = M \otimes_R \mathbb{C}$, using $\alpha_z : R^\times \to \mathbb{C}^\times$.

Lemma. *As G-modules, $M_z \simeq i_B^G(\chi\chi_z)$.*

Proof. Composition with α_z gives a natural map $M_z \to i_B^G(\chi\chi_z)$. To see that this map is an isomorphism, it suffices to show that $i_B^G(\chi\chi_z)^K \simeq M_z^K$ for every compact open subgroup K of G.

Let $f \in i_B^G(\chi\chi_z)^K$. Then f is right K-invariant, so can be viewed as a function on G/K. Given $g \in G$, the values of f on BgK are determined by $f(g)$. Suppose that $f(g) \neq 0$. For $k \in K$, $f(gk) = f(gkg^{-1} \cdot g)$. If $gkg^{-1} \in B$, then $f(g) = f(gk) = (\chi\chi_z)(gkg^{-1})f(g)$ implies $(\chi\chi_z)(gkg^{-1}) = 1$. Thus f can have nonzero values on BgK if and only if $(\chi\chi_z) \,|\, B \cap gKg^{-1} \equiv 1$. As gKg^{-1} is compact and χ_z is unramified, $\chi_z \,|\, B \cap gKg^{-1} \equiv 1$. For such g, $f(g) \in \mathbb{C}$ can be chosen freely. It follows that

$$i_B^G(\chi\chi_1)^K = \bigoplus_{\{g \in B\backslash G/K \;\mid\; \chi \,|\, B \cap gKg^{-1} \equiv 1\,\}} \mathbb{C}.$$

Similarly, $M^K = \oplus R$, with the same index set. Now $\otimes_R \mathbb{C}$ converts M^K to M_z^K. $\square$

We can choose n such that G is a subgroup of $GL_n(F)$. For an integer $j \geq 1$, let K_j be the intersection of G with the subgroup $\{\, x \in GL_n(F) \mid x - 1 \in M_n(\mathfrak{p}^j)\,\}$. We assume that the inclusion $G \hookrightarrow GL_n(F)$ is chosen in such a way that K_j has an Iwahori decomposition with respect to B.

Lemma. *Choose $K = K_j$ such that $\chi \,|\, A \cap K \equiv 1$. Then, for every $z \in (\mathbb{C}^\times)^d$, any subquotient of $i_B^G(\chi\chi_z)$ is generated by its K-fixed vectors.*

Proof. It suffices to prove the lemma for a subrepresentation V of $i_B^G(\chi\chi_z)$. Set $V_0 = \mathrm{Span}\{G \cdot V^K\}$ and $V_1 = V/V_0$. Then $0 \to V_0 \to V \to V_1 \to 0$ implies $0 \to V_0^K \to V^K \to V_1^K \to 0$. Thus $V_0^K = V^K$ implies $V_1^K = 0$.

The group K has an Iwahori decomposition with respect to B. Therefore (recall earlier results on admissibility of Jacquet modules) the map $V_1^K \to r_B^G(V_1)^{K\cap A}$ is onto. Hence

$V_1^K = 0$ implies $r_B^G(V_1)^{K\cap A} = 0$. The Jacquet module $r_B^G(V_1)$ is a subquotient of $r_B^G(V)$, which is itself a subrepresentation of $r_B^G(i_B^G(\chi\chi_z))$, the latter having Jordan-Holder factors equal to $(\chi\chi_z)^w$, $w \in W$. By choice of K, $(\chi\chi_z)^w \mid A \cap K \equiv 1$ (recall that w permutes the a_j's, so preserves $A \cap K$). As $r_B^G(V_1)^{A\cap K} = 0$, this forces $r_B^G(V_1) = 0$. Since V is a subrepresentation of $i_B^G(\chi\chi_z)$, V_1 is a subquotient of $i_B^G(\chi\chi_z)$. By the first proposition in this section, $r_B^G(V_1) = 0$ implies $V_1 = 0$. That is, $V_0 = V$. $\square$

Set $\mathcal{H}_K = \mathcal{H}(G/\!\!/K)$. Smooth G-modules generated by their K-fixed vectors correspond to $\mathcal{H}_K$-modules. This was discussed in an earlier section for irreducible smooth representations having non zero K-invariant vectors (being irreducible, such representations are generated by their K-invariant vectors). In the more general setting, the representation of $\mathcal{H}_K$ is defined as in the irreducible case, with $\mathcal{H}_K$ acting on the space of K-invariant vectors (see section 7). The above lemma allows us to reduce questions about subquotients of $i_B^G(\chi\chi_z)$ to questions about $\mathcal{H}(G/\!\!/K)$-modules.

As M^K is an $\mathcal{H}_K$-module and an R-module ($R = \mathbb{C}[A/A_0]$), so we can consider M^K as an $\mathcal{H}_K \otimes_{\mathbb{C}} R$-module. As an R-module, M^K is free and of finite rank (recall our earlier description of M_K).

Abstract Lemma. *Let H be a $\mathbb{C}$-algebra and R a finitely generated commutative $\mathbb{C}$-algebra. Let $Z = \operatorname{Spec} R$. Suppose that M is an $H \otimes_{\mathbb{C}} R$-module which is free of finite rank as an R-module. Set*

$$Z_0 = \{\, z \in Z \mid M_z = M \otimes_{R\nearrow\alpha_z} \mathbb{C} \text{ is an irreducible } H-\text{module}\,\}.$$

Then Z_0 is Zariski dense in Z.

Outline of proof. Fix an R-module isomorphism $M \simeq R^k$. Fix j, $0 < j < k$. Let $G(j,k)$ be the set of all j-dimensional subspaces of $\mathbb{C}^k$. Given $h \in H$, we have a $k \times k$ matrix A_h with entries in R, and also a corresponding $k \times k$ complex matrix $A_{h,z}$. For each z, we can identify M_z and $R^k \otimes_{R\nearrow\alpha_z} \mathbb{C} = \mathbb{C}^k$. The set $\{(z,W) \in Z \times G(j,k) \mid A_{h,z}W \subset W\}$ is Zariski closed in $Z \times G(j,k)$. Taking the intersection over all $h \in H$, we get the Zariski closed set

$$\{\,(z,W) \mid W \text{ is an } H-\text{submodule of } \mathbb{C}^k = M_z\,\}.$$

Project this variety onto Z and using properties of $G(j,k)$, show that the set of all $z \in Z$ having the property that there exists a j-dimensional H-submodule in M_z is Zariski closed. To finish, take the union over all j. $\square$

Applying the above lemma with $H = \mathcal{H}_K$, $R = \mathbb{C}[A/A_0]$, etc., we obtain:

Corollary. *Fix a quasi character $\chi : A \to \mathbb{C}^\times$. Then $\{ z \in (\mathbb{C}^\times)^d \mid i_B^G(\chi\chi_z)$ is irreducible $\}$ is Zariski open in $(\mathbb{C}^\times)^d$.*

Lemma. *Fix a quasi character $\chi : A \to \mathbb{C}^\times$ and $w \in W$. Set $B' = wBw^{-1}$. Then $i_B^G(\chi \circ w) \simeq i_{B'}^G(\chi)$.*

Proof. We remark that whenever necessary we choose representatives for elements of w in the compact open subgroup $K_0 = G(\mathfrak{o}_F) = G \cap GL_n(\mathfrak{o}_F)$. Given $f \in i_B^G(\chi \circ w)$, define $A_w f : G \to \mathbb{C}$ by $(A_w f)(g) = f(w^{-1}gw)$. Then, for $b' \in B'$ and $g \in G$,

$$(A_w f)(b'g) = ((\chi \circ w)\delta_B^{1/2})(w^{-1}b'w)\,(A_w f)(g).$$

To see that $A_w f \in i_{B'}^G(\chi)$, it necessary to verify that $\delta_B(w^{-1}b'w) = \delta_{B'}(b')$ for $b' \in B'$. This is left as an exercise in the general case. For $G = SL_2(F)$, if w is the non trivial element of the Weyl group, and $a \in A$ is the diagonal matrix with diagonal entries t and t^{-1}, then $w^{-1}aw$ has diagonal entries t^{-1} and t, that is, $w^{-1}aw = a^{-1}$. If B is the upper triangular Borel subgroup, then B' is the lower triangular Borel subgroup. An easy calculation shows that $\delta_{B'}(a) = |t|_F^{-2} = \delta_B(a^{-1})$. □

Theorem. *For all $\chi : A \to \mathbb{C}^\times$ and all $w \in W$, $i_B^G(\chi)$ and $i_B^G(\chi \circ w)$ have the same character.*

Proof. Choose K as above. Fix $f \in \mathcal{H}_K$. Set $\pi_z = i_B^G(\chi\chi_z)$, and $\pi'_z = i_{B'}^G(\chi\chi_z)$, $z \in (\mathbb{C}^\times)^d$. With M as above, M^K is a free R-module, and f acts on M^K. So $\operatorname{tr}(f, M^K) \in R$. Thus $z \mapsto \operatorname{tr}\pi_z(f)$ is a regular function on $(\mathbb{C}^\times)^d$ (that is, belongs to R).

Let B' be such that $i_B^G(\chi \circ w) \simeq i_{B'}^G(\chi)$. Set $\pi'_z = i_{B'}(\chi\chi_z)$. Replacing B by B' above, we have $z \mapsto \operatorname{tr}\pi'_z(f)$ is regular.

Let

$$S = \{ z \in (\mathbb{C}^\times)^d \mid \Theta_{\pi_z} = \Theta_{\pi'_z} \}$$

$$S_1 = \{ z \in (\mathbb{C}^\times)^d \mid \pi_z \text{ and } \pi_{z'} \text{ are irreducible} \}$$

$$S_2 = \{ z \in (\mathbb{C}^\times)^d \mid \operatorname{Stab}_W(\chi\chi_z) = \{1\} \}$$

By regularity of the characters, S is Zariski closed in $(\mathbb{C}^\times)^d$. By the above lemma, S_1 is Zariski dense in $(\mathbb{C}^\times)^d$. As the Weyl group W acts on quasi characters as permutations on the d-tuples in $(\mathbb{C}^\times)^d$, S_2 is also Zariski dense.

Suppose that $z \in S_2$. The abelian group A acts on $r_B^G i_B^G(\chi\chi_z)$, which is finite dimensional (see earlier description of the composition series of Jacquet modules), so $r_B^G i_B^G(\chi\chi_z)$ decomposes as a direct sum of generalized eigenspaces. For each $w \in W$, $(\chi\chi_z) \circ w$

occurs as a subquotient. By assumption on z, $\mathrm{Stab}_W(\chi\chi_z) = 1$. Therefore, there are $|W|$ distinct one dimensional subquotients of $r_B^G i_B^G(\chi\chi_z)$, which itself is $|W|$-dimensional. Hence each generalized eigenspace has dimension one, that is, $r_B^G i_B^G(\chi\chi_z)$ is a direct sum of $(\chi\chi_z) \circ w$'s. Thus $\mathrm{Hom}_A(r_B^G i_B^G(\chi\chi_z), (\chi\chi_z) \circ w) \neq 0$ for all $w \in W$. By Frobenius reciprocity, $\mathrm{Hom}_G(\pi_z, \pi'_z) \neq 0$ (for $z \in S_2$).

Suppose $z \in S_1 \cap S_2$. Then, as π_z and π'_z are irreducible, $\mathrm{Hom}_G(\pi_z, \pi'_z) \neq 0$ forces π_z and π'_z to be equivalent, and hence $\Theta_{\pi_z} = \Theta_{\pi'_z}$. We can now conclude that $S_1 \cap S_2 \subset S$. As S is Zariski closed and contains a Zariski dense set, $S = (\mathbb{C}^\times)^d$. $\square$

Corollary. *The representations $i_B^G(\chi)$ and $i_B^G(\chi \circ w)$ have the same Jordan-Holder factors.*

Intertwining maps

As in the previous section, B denotes a standard Borel subgroup of G, with Levi decomposition $B = AN$, χ denotes a quasi character of A and w is an element of the Weyl group $W = N_G(A)/A$. Since $i_B^G(\chi)$ and $i_B^G(\chi \circ w)$ have the same Jordan-Holder factors, it is natural to try to construct intertwining maps $i_B^G(\chi) \to i_B^G(\chi \circ w)$. As $i_B^G(\chi \circ w) \simeq i_{B'}^G(\chi)$ for $B' = wBw^{-1}$, such maps can be viewed as intertwining $i_B^G(\chi)$ and $i_{B'}^G(\chi)$. Let N' be the unipotent radical of B': $N' = wNw^{-1}$.

Given $f \in i_B^G(\chi)$, set $(I_{B',B}f)(g) = \int_{(N\cap N')\backslash N'} f(ng)\, dn$. Here, dn is the N' invariant measure on the quotient coming from Haar measures on the unipotent (hence unimodular) groups N' and $N \cap N'$. As $f \in i_B^G(\chi)$, the function $n \mapsto f(ng)$ is left $N \cap N'$ invariant. However, the function isn't compactly supported on the quotient, so the integral defining $I_{B',B}f$ might not converge.

Lemma. *Let $f \in i_B^G(\chi)$. If $(I_{B',B}f)(g)$ converges for all $g \in G$, then $I_{B',B}f \in i_{B'}^G(\chi)$.*

Proof. Since dn is right N' invariant, $I_{B',B}f$ is left N' invariant. Let $a \in A$. Then, since A normalizes N' and $N' \cap N$,

$$(I_{B',B}f)(ag) = \chi(a)\delta_B(a)^{1/2} \int_{(N\cap N')\backslash N'} f(a^{-1}nag)\, dn = \chi(a)\delta_B(a)^{1/2}\Delta_w(a)\,(I_{B',B}f)(g),$$

where $\Delta_w(a)$ relates the two N' invariant measures dn and $d(a^{-1}na)$ on the quotient: $d(a^{-1}na) = \Delta_w(a)^{-1}dn$. We omit the calculation of $\Delta_w(a)$ in the general case. For $G = SL_2(F)$, assume that B is upper triangular and w is non trivial. Then B' is lower triangular and $N \cap N' = \{1\}$. We find that $\Delta_w(a) = |t|_F^{-2} = \delta_{B'}(a)$ for a with diagonal entries t and t^{-1}, $t \in F^\times$. In the previous section, we saw that $\delta_B(a) = |t|_F^2$. Hence $\delta_B(a)^{1/2}\Delta_w(a) = \delta_{B'}(a)^{1/2}$, so $I_{B',B}f \in i_{B'}^G(\chi)$. $\square$

If the integral $(I_{B',B}f)(1)$ converges for all $f \in i_B^G(\chi)$, then, since $f(ng) = (g \cdot f)(n)$ and $g \cdot f \in i_B^G(\chi)$, we have $(I_{B',B}f)(g)$ converging for all $f \in i_B^G(\chi)$ and all $g \in G$. Thus for purposes of checking convergence, we need only look at $(I_{B',B}f)(1)$.

Using $|\cdot|_\infty$ to denote the usual absolute value on the complex numbers, we observe that, for $f \in i_B^G(\chi)$,

$$|f(bg)|_\infty = |\delta_B^{1/2}\chi(b)|_\infty |f(g)|_\infty = \delta_B^{1/2}(b)|\chi|_\infty(b)|f(g)|_\infty.$$

Note that, by compactness of A_0, $|\chi|_\infty \mid A_0 \equiv 1$. So $|\chi|_\infty$ is unramified and real valued (as $|\cdot|_\infty$ is real valued). We will replace χ by $|\chi|_\infty$. For now, assume that $G = SL_2(F)$. In this case, $A \simeq F^\times : \mathrm{diag}(t, t^{-1}) \mapsto t$, and $|\chi|_\infty$ is given by:

$$|\chi|_\infty(\mathrm{diag}(t, t^{-1})) = \chi_s(\mathrm{diag}(t, t^{-1})) = |t|_F^s, \qquad \text{for some } s \in \mathbb{R}.$$

Let $f \in i_B^G(\chi_s)$. We have $G = BK$, where $K = K_0 = SL_2(\mathfrak{o}_F)$. As f is locally constant and K is compact, f takes finitely many values on K and therefore $|f|$ is bounded on K. Set $f_{s,B}(bk) = \delta_B^{1/2}(b)\chi_s(b)$. As $\delta_B^{1/2}\chi_s \,|\, K \equiv 1$, it follows that $f_{s,B} \in i_B^G(\chi_s)$. There exists a constant C such that $|f|_\infty \le C|f_{s,B}|_\infty$. Hence in order to check absolute convergence, it suffices to consider

$$(I_{B',B} f_{s,B})(1) = \int_F f_{s,B} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} dx.$$

Case 1: $x \in \mathfrak{o}_F$. Here $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in K$, so $f_{s,B}$ takes the value 1.

Case 2: $x \notin \mathfrak{o}_F$. Here

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} K.$$

Multiplying the second column by $-x^{-1}$ (which belongs to $\mathfrak{o}_F$) and subtracting from the first column, we get

$$\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} = \begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \in \begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix} K.$$

Thus, when $x \notin \mathfrak{o}_F$,

$$f_{s,B} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = (\delta_B^{1/2}\chi_s) \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = |x|_F^{-1-s}.$$

We are assuming that the Haar measure dx is normalized in the usual way: so that $\mathfrak{o}_F$ has volume one. Therefore, with $q = |\mathfrak{o}_F/\mathfrak{p}_F|$,

$$\begin{aligned} (I_{B',B} f_{s,B})(1) &= 1 + \int_{F - \mathfrak{o}_F} |x|_F^{-1-s}\, dx = 1 + \sum_{j=1}^{\infty} \int_{|x|_F = q^j} q^{-j(s+1)}\, dx \\ &= 1 + \sum_{j=1}^{\infty} q^{-j(s+1)} q^j (1 - q^{-1}) = 1 + (1 - q^{-1}) \sum_{j=1}^{\infty} q^{-sj}. \end{aligned}$$

Above we have used $\{\, x \in F \mid |x|_F = q^j \,\} = \varpi_F^{-j}\mathfrak{o}_F^\times$, $d(\varpi_F^{-j}x) = q^j dx$, and the measure of $\mathfrak{o}_F^\times = \mathfrak{o}_F - \mathfrak{p}_F = \mathfrak{o}_F - \varpi_F\mathfrak{o}_F$ equals $1 - q^{-1}$. It follows that, for $s > 0$, $(I_{B',B} f_{s,B})(1)$ converges to $(1 - q^{-s-1})/(1 - q^{-s})$.

Now let $s \in \mathbb{C}$ and define $f_{s,B}$ as above. As $f_{s,B} \in i_B^G(\chi_s)^K$ is non zero and $i_B^G(\chi_s)^K$ is one dimensional any other element of $i_B^G(\chi_s)^K$ is a scalar multiple of $f_{s,B}$. Our calculation above shows that $I_{B',B} f_{s,B}$ converges whenever $\mathrm{Re}\,(s) > 0$. In that case, $I_{B',B} : i_B^G(\chi_s)^K \to i_{B'}^G(\chi_s)^K$, and so $I_{B',B} f_{s,B}$ must be a scalar multiple of $f_{s,B'}$. As we have computed $(I_{B',B} f_{s,B})(1)$ explicitly, and $f_{s,B'}(1) = 1$, it follows that

Lemma. *Suppose that* $G = SL_2(F)$. *For* $s \in \mathbb{C}$ *such that* $\mathrm{Re}\,(s) > 0$, $I_{B',B} f_{s,B} = \dfrac{1-q^{-1-s}}{1-q^{-s}} f_{s,B'}$.

Suppose now that G is a split reductive group with $B = AN$ and $B' = AN'$ standard Borel subgroups, and χ a quasi character of A. In order to study convergence of $I_{B',B} f$, the integral is expressed as an iterated integral, each piece of which looks like the intertwining integral for $SL_2(F)$. We consider the example of $G = GL_3(F)$, with B upper triangular and B' lower triangular. To begin we conjugate in stages, using transpositions in W, from N to N'. Let

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & z \\ x & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in F \right\} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} N \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$N_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & x \\ z & 0 & 1 \end{pmatrix} \mid x, y, z \in F \right\} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} N_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$$

$$N' = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ y & x & 1 \end{pmatrix} \mid x, y, z \in F \right\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} N_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

In the integrals defining I_{B',B_1}, I_{B_2,B_1}, $I_{B_1,B}$, we integrate over

$$(N' \cap N_2)\backslash N' = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix} \right\}$$

$$(N_1 \cap N_2)\backslash N_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix} \right\}$$

$$(N_1 \cap N)\backslash N_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

respectively. Therefore $I_{B',B} = I_{B',B_2} \circ I_{B_2,B_1} \circ I_{B_1,B}$, and if each of the three integrals on the right converge, then so does $I_{B',B}$. Fix an unramified quasicharacter χ_s of $A \simeq (F^\times)^3$: $s = (s_1, s_2, s_3) \in \mathbb{C}^3$, $\chi_s(\mathrm{diag}(a_1, a_2, a_3)) = |a_1|_F^{s_1} |a_2|_F^{s_2} |a_3|_F^{s_3}$. Using our earlier calculations for $SL_2(F)$, we see

$$\begin{aligned}(I_{B_1,B}f_{s,B})(1) &= \int_F f_{s,B}\begin{pmatrix}1&0&0\\x&1&0\\0&1&0\end{pmatrix}dx = 1+\int_{F-\mathfrak{o}_F}(\delta_B^{1/2}\chi_s)\begin{pmatrix}x^{-1}&0&0\\0&x&0\\0&0&1\end{pmatrix}dx\\ &= 1+\int_{F-\mathfrak{o}_F}|x|_F^{-1}|x|_F^{-s_1+s_2}\,dx = 1+(1-q^{-1})\sum_{j=1}^{\infty}q^{-(s_1-s_2)j}\\ &= \frac{1-q^{-(s_1-s_2)-1}}{1-q^{-(s_1-s_2)}},\qquad \text{if } \mathrm{Re}\,(s_1-s_2)>0\end{aligned}$$

Thus $(I_{B_1,B}f_{s,B}) = (1-q^{-(s_1-s_2)-1})/(1-q^{-s_1-s_2})f_{s,B_1}$. After carrying out the same type of calculations for $I_{B_2,B_1}f_{s,B_1}$ and $I_{B',B_2}f_{s,B_2}$, we get

$$(I_{B',B}f_{s,B}) = \prod_{i<j}\frac{1-q^{-(s_i-s_j)-1}}{1-q^{-(s_i-s_j)}}\,f_{s,B'},\qquad \text{if } \mathrm{Re}(s_i-s_j)>0,\ \text{for } i<j.$$

It is convenient to express the above result in terms of roots and co roots. Let $X^*(A)$ be the set of rational characters of the split torus A. As $A\simeq (F^\times)^d$, such characters correspond to elements of $\mathbb{Z}^d$: $a=(a_1,a_2,\dots,a_d)\mapsto a_1^{k_1}\cdots a_d^{k_d}$ corresponds to $(k_1,\dots,k_d)\in\mathbb{Z}^d$. The set of rational homormorphisms $F^\times\to A$ is denoted by $X_*(A)$. This set can also be identified with $\mathbb{Z}^d$: the homomorphism $t\mapsto(t^{\ell_1},\dots,t^{\ell_d})\in(F^\times)^d\simeq A$ corresponds to $(\ell_1,\dots,\ell_d)\in\mathbb{Z}^d$. We have a natural pairing between $X_*(A)$ and $X^*(A)$ denoted by $\langle\cdot,\cdot\rangle$: it is given by composition. The composition map is a rational homomorphism from $F^\times$ to itself, hence is of the form $t\mapsto t^r$ for some integer r. With the above identifications, $r=\langle(\ell_1,\dots,\ell_d),(k_1,\dots,k_d)\rangle=\sum_{j=1}^d\ell_jk_j$. Whenever convenient we extend the pairing $\langle\cdot,\cdot\rangle$ to $X_*(A)\otimes_{\mathbb{Z}}\mathbb{C}$ and $X^*(A)\otimes_{\mathbb{Z}}\mathbb{C}$ in the obvious way. An unramified quasi character of A has the form $\chi_s(a)=|a_1|_F^{s_1}\cdots|a_d|_F^{s_d}$ and so can be identified with the element $s=(s_1,\dots,s_d)$ of $X^*(A)\otimes_{\mathbb{Z}}\mathbb{C}=\mathbb{C}^d$. (Recall that in an earlier section we had identified an unramified quasi character with an element of $z=(z_1,\dots,z_d)\in(\mathbb{C}^\times)^d$ - the relation to the above is simply $z_j=q^{-s_j}$).

Consider the adjoint action of A on the Lie algebra $\mathfrak{g}$ (for subgroups of $GL_n(F)$ this is just conjugation by elements of A). The Lie algebra $\mathfrak{g}$ is a direct sum of the eigenspaces for this action. The non zero eigenfunctions are called *roots*, and they belong to $X^*(A)$. For example, if we consider $G=GL_3(F)$ we find the roots are $a\mapsto a_ia_j^{-1}$ for $i\neq j$, or $\pm(1,-1,0)$, $\pm(1,0,-1)$, $\pm(0,1,-1)\in\mathbb{Z}^3$. Suppose α is a root. Let $\mathfrak{g}_\alpha\subset\mathfrak{g}$ be the corresponding eigenspace. Given any standard Borel subgroup $B''=AN''$, either $\mathfrak{g}_\alpha\subset\mathfrak{n}''$

($\mathfrak{n}''$ the Lie algebra of N''), in which case we say that α is *positive for* B'', or $\mathfrak{g}_\alpha \subset \mathfrak{n}''^-$ (the Lie algebra of the opposite unipotent radical N''^-), in which case α is *negative for* B''.

Corresponding to each root α, we have an *co root* $\alpha^\vee \in X_*(A)$ which has the property that the image of $\alpha^\vee$ in $\mathbb{Z}^d$ is identified with a particular multiple of the image of α in $\mathbb{Z}^d$, the multiple being determined by $\langle \alpha^\vee, \alpha \rangle = 2$. For groups other than $GL_n(F)$, it can happen that α and $\alpha^\vee$ do not correspond to the same element of $\mathbb{Z}^d$.

Proposition. *Let B and B' be standard Borel subgroups. Let χ_s, $s \leftrightarrow (s_1, \ldots, s_d) \in X^*(A) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^d$ be an unramified quasi character of A. Then, with the product over all roots α which are positive for B and negative for B', if s is such that* $\mathrm{Re}\,\langle \alpha^\vee, s \rangle > 0$ *for all such α,*

$$I_{B',B} f_{s,B} = \prod \frac{1 - q^{-(1+\langle \alpha^\vee, s \rangle)}}{1 - q^{-\langle \alpha^\vee, s \rangle}} \, f_{s,B'}.$$

Remarks.

(1) If we set $z = (q^{-s_1}, \ldots, q^{-s_d})$, we see that $(1 - q^{-(1+\langle \alpha^\vee, s \rangle)})/(1 - q^{-\langle \alpha^\vee, s \rangle})$ is a rational function of z.

(2) The formula in the above proposition will be used to study the Satake isomorphism in a later section.

(3) For more information about interwining operators and the decomposition of unitary principal series representations, see the papers of Winarsky [Wi] and Keys [K].

The Satake isomorphism

We continue to assume that G is a split reductive group and $B = AN$ is a standard Borel subgroup of G. Returning to the notation of the section on principal series representations, we index unramified quasi characters of $A \simeq (F^\times)^d$ by elements $z \in (\mathbb{C}^\times)^d$: $z = (z_1, \dots, z_d)$ is given by $z_j = \chi_z(1, \dots, \varpi_F, 1, \dots, 1)$, where ϖ_F occurs in the jth component. Let $K = G(\mathfrak{o}_F)$. Fix $z \in (\mathbb{C}^\times)^d$. The function $f_{z,B}$ defined by $f_{z,B}(ank) = (\delta_B^{1/2}\chi_z)(a)$ is a basis for the one dimensional space $i_B^G(\chi_z)^K$. Let $\mathcal{H}_K = \mathcal{H}(G//K)$. Let $\pi_z = i_B^G(\chi_z)$. Given $f \in \mathcal{H}_K$, $\pi_z(f) : i_B^G(\chi_z)^K \to i_B^G(\chi_z)^K$, so there exists $f^\vee(z) \in \mathbb{C}$ such that $\pi_z(f)(f_{z,B}) = f^\vee(z) f_{z,B}$. As $f \mapsto \pi_z(f)$ is an algebra homomorphism, it follows that $f \mapsto f^\vee(z)$ is also an algebra homomorphism. Letting z vary over $(C^\times)^d$, we have a homomorphism $f \mapsto f^\vee$ from $\mathcal{H}_K$ to the algebra of complex valued functions on $(\mathbb{C}^\times)^d$.

Haar measures on G, K, A and N will be normalized so that K, $A \cap K$, and $N \cap K$ have volume one.

Lemma. *Let $f \in \mathcal{H}_K$ and $z \in (\mathbb{C}^\times)^d$. Then*
(1) $f^\vee(z) = \int_G f(g)\, E_z(g)\, dg$, where $E_z(g) = \int_K f_{z,B}(kg) dk$
(2) $f^\vee(z) = \int_A f^{(B)}(a)\, \chi_z(a)\, da$, where

$$f^{(B)}(a) = \delta_B(a)^{1/2} \int_N f(an)\, dn = \delta_B(a)^{-1/2} \int_N f(na)\, dn, \qquad a \in A.$$

Proof. $f_{z,B}(1) = 1 \implies f^\vee(z) = (\pi_z(f)(f_{z,B}))(1) = \int_G f(g) f_{z,B}(g)\, dg$. Since f is K invariant, and $d(kg) = dg$ for $k \in K$,

$$\int_G f(g) f_{z,B}(g)\, dg = \int_G f(g) f_{z,B}(kg)\, dg$$

for all $k \in K$. Integrating both sides of this equality over K, we obtain (1).

Recall the integral formula coming from the Iwasawa decomposition $G = BK = KB$ (section 5, p. 21),

$$\int_G h(g)\, dg = \int_K \int_B h(bk)\, dk\, d_\ell b = \int_K \int_B h(kb)\, dk\, d_r b, \qquad h \in C_c^\infty(G).$$

Therefore, using the fact that f and $f_{z,B}$ are right K invariant,

$$\begin{aligned} f^\vee(z) &= \int_B \int_K f(bk) f_{z,B}(bk)\, d_\ell b\, dk = \int_B f(b) f_{z,B}(b)\, db \\ &= \int_A \int_N f(an) \delta_B(a)^{1/2} \chi_z(a)\, dn\, da = \int_A f^{(B)}(a) \chi_z(a)\, da. \end{aligned}$$

$\square$

Recall that the ring of regular functions on $(\mathbb{C}^\times)^d$ is the group algebra $\mathbb{C}[A/A_0]$ where $A_0 = A \cap K$. As the Weyl group $W = N_G(A)/A$ permutes the entries $a_1, \dots, a_d$ of $a \in A$, W preserves A_0. Hence W acts on the unramified quasi characters of A and on $\mathbb{C}[A/A_0]$.

Theorem. *The map $f \mapsto f^\vee$ is an isomorphism of $\mathcal{H}_K$ with $\mathbb{C}[A/A_0]^W$.*

The map $f \mapsto f^\vee$ is called the *Satake isomorphism*. We will prove the theorem and derive Macdonald's formula for $f^\vee(z)$.

Let $f \in \mathcal{H}_K$. By definition of $f^\vee(z)$, $\operatorname{tr}\pi_z(f) = f^\vee(z)$. As we saw in the section on principal series representations (section 12), $z \mapsto \operatorname{tr}\pi_z(f)$ is a regular function of z. By a theorem of section 12, given any $w \in W$, $i_B^G(\chi_z)$ and $i_B^G(\chi_z \circ w)$ have the same character. Thus $f^\vee(z) = f^\vee(w(z))$. It remains to show that the image of $f \mapsto f^\vee$ is all of $\mathbb{C}[A/A_0]^W$ and the map is one to one.

Recall the Cartan decomposition $G = KA^+K$ (see the section on discrete series representations). For $G = GL_n(F)$, A^+ is the subset of $A \simeq (F^\times)^n$ which corresponds to elements $(\varpi_F^{j_1}, \dots, \varpi_F^{j_n})$ such that $j_1 \geq \cdots \geq j_n$. For general G, as A/A_0 can be identified with elements of the form $(\varpi_F^{j_1}, \dots, \varpi_F^{j_d})$ for arbitrary integers j_i, we can identify $X_*(A)$ (see the previous section) with A/A_0, simply by evaluating the homomorphism $t \mapsto (t^{j_1}, \dots, t^{j_d})$ at $t = \varpi_F$. The Weyl group W acts on $A/A_0 \simeq X_*(A)$ by permuting the components $\varpi_F^{j_i}$ and each W-orbit in $X_*(A)$ contains a unique element of A^+. That is, A^+ is a set of representatives for the W-orbits in $X_*(A)$. An element $\mu \in X_*(A)$ is said to by *dominant* with respect to B if $\langle \mu, \alpha \rangle \geq 0$ for every root α which is positive for B. The dominant elements in $X_*(A)$ correspond to the elements of A^+. We will denote the dominant elements by $X_*(A)^+$.

Given $\mu = (\mu_1, \dots, \mu_d) \in X_*(A)$, let $a_\mu = (\varpi_F^{\mu_1}, \dots, \varpi_F^{\mu_d})$ be the corresponding element of A/A_0, and let $f_\mu \in \mathcal{H}_K$ be the characteristic function of $Ka_\mu K$. As a consequence of the Cartan decomposition, we know that $\{ f_\mu \mid \mu \in X_*(A)^+ \}$ is a basis for $\mathcal{H}_K$. Given $\mu \in X_*(A)$, let $W \cdot \mu \subset X_*(A)$ be the W-orbit of μ. The elements $g_\mu = \sum_{\mu' \in W\cdot\mu} \mu'$, for $\mu \in X_*(A)^+$ form a $\mathbb{C}$-basis of $\mathbb{C}[A/A_0]^W = \mathbb{C}[X_*(A)]^W$.

Given f_μ, $\mu \in X_*(A)^+$, write $f_\mu^\vee$ as a linear combination of g_ν, $\nu \in X_*(A)^+$: $f_\mu^\vee = \sum_{\nu \in X_*(A)^+} c_{\mu,\nu} g_\nu$. We define a partial order on $X_*(A)$ as follows: $\nu \leq \mu$ if $\mu - \nu = \sum_{\alpha \text{ simple}} \ell_\alpha \alpha^\vee$ with every ℓ_α a non negative integer. Ths sum is over the positive simple roots (relative to B). If $G = GL_n(F)$ and B is upper triangular, the simple roots correspond to the following elements of $\mathbb{Z}^n$

$$(1,-1,0,\ldots,0),\ (0,1,-1,0,\ldots,0),\ldots,(0,\ldots,0,1,-1).$$

We will show that

(1) $c_{\mu,\mu} \neq 0$

(2) $c_{\mu,\nu} = 0$ unless $\nu \leq \mu$.

Lemma. *Let V be a vector space with a basis I having a partial order such that for every $\mu \in I$ the set $\{\nu \in I \mid \nu \leq \mu\}$ is finite. Suppose that A is a linear operator on V such that the matrix coefficients of A with respect to the basis I are $c_{\mu,\nu}$, and these coefficients satisfy (1) and (2). Then A is an isomorphism.*

Thus to complete the proof of the theorem, it suffices to prove (1) and (2).

Since f_μ and $\delta_B^{1/2}$ are right A_0 invariant, $f_\mu^{(B)}$ is right A_0 invariant. As χ_z is also A_0 invariant, and A_0 has volume one relative to Haar measure on A, the second formula for $f_\mu^\vee(z)$ (see earlier lemma) can be rewritten as

$$f_\mu^\vee(z) = \sum_{a\in A/A_0} f_\mu^{(B)}(a)\chi_z(a) = \sum_{\nu\in X_*(A)^+}\sum_{\nu'\in W\cdot\nu} f_\mu^{(B)}(a_{\nu'})\chi_z(a_{\nu'}).$$

Let $w \in W$ and $f \in \mathcal{H}_K$. From

$$0 = f^\vee(z) - f^\vee(w(z)) = \int_A f^{(B)}(a)(\chi_z(a) - \chi_z(w(a)))\,da = \int_A (f^{(B)}(a) - f^{(B)}(w^{-1}(a))\chi_z(a)\,da$$

which holds for all $z \in (\mathbb{C}^\times)^d$, it follows that $f^{(B)}(a) = f^{(B)}(w^{-1}(a))$ for all $a \in A$. That is, $f^{(B)}$ is constant on W-orbits in A. Hence the above expression for $f_\mu^\vee(z)$ equals

$$\begin{aligned} f_\mu^\vee(z) &= \sum_{\nu\in X_*(A)^+} f_\mu^{(B)}(a_\nu) \sum_{\nu'\in W\cdot\nu} \chi_z(a_{\nu'}) \\ &= \sum_{\nu\in X_*(A)^+} f_\mu^{(B)}(a_\nu) \sum_{\nu'\in W\cdot\nu} \langle \nu', z\rangle = \sum_{\nu\in X_*(A)^+} f_\mu^{(B)}(a_\nu) g_\nu(z). \end{aligned}$$

Here we are identifying $z = (z_1,\ldots,z_d) = (q^{-s_1},\cdots,q^{-s_d})$ with $s = (s_1,\ldots,s_d) \in X^*(A)\otimes_{\mathbb{Z}}\mathbb{C}$, and writing $\langle \nu', z\rangle$ in place of $\langle \nu', s\rangle$. We have shown

Lemma. *Let μ, $\nu \in X_*(A)^+$. Then $c_{\mu,\nu} = f_\mu^{(B)}(a_\nu) = \delta_B(a_\nu)^{-1/2}\int_N f_\mu(na_\nu)\,dn$.*

Lemma. *Let μ, $\nu \in X_*(A)^+$. Then*

(1') $N \cap Ka_\mu K a_\mu^{-1} \subset N \cap K$.

(2') $Na_\nu \cap Ka_\mu K \neq \emptyset \Longrightarrow \nu \leq \mu$.

Assuming the lemma for the moment, observe that

$$f_\mu(na_\mu) \neq 0 \Longleftrightarrow na_\mu = k_1 a_\mu k_2, \quad \text{for some } k_1, k_2 \in K \Longleftrightarrow n \in Ka_\mu K a_\mu^{-1}.$$

Thus (1') implies condition (1): $c_{\mu,\mu} \neq 0$. It is immediate that (2') implies (2). Hence to complete the proof of the theorem it suffices to prove the lemma.

For the moment, assume that $G = GL_n(F)$ and $K = GL_n(\mathfrak{o}_F)$. Given $X = (X_{ij}) \in M_n(F)$ (recall $M_n(F)$ is the set of $n \times n$ matrices with entries in F), set $\|X\| = \max_{i,j} |X_{ij}|_F$.

Lemma. *Let k_1, $k_2 \in K$ and $X \in M_n(F)$. Then $\|k_1 X k_2\| = \|X\|$.*

Proof. As the entries of k_1 and k_2 belong to $\mathfrak{o}_F$, $\|k_1 X k_2\| \leq \|X\|$. The entries of k_1^{-1} and k_2^{-1} are also in $\mathfrak{o}_F$. So $\|X\| = \|k_1^{-1}(k_1 X k_2)k_2^{-1}\| \leq \|k_1 X k_2\|$. □

Let $\mu = (\mu_1, \ldots, \mu_n) \in X_*(A)^+$. Note that $\|X\| = |\varpi_F^{\mu_n}|_F = q^{-\mu_n}$.

If we have a representation of a split reductive F-group in a finite dimensional F-vector space V, the eigenfunctions for the action of A on V belong to $X^*(A)$ and are called the *weights* of the representation. The weights can be related to each other through the roots. For details, see [Bo].

Example: Let $V = F^3$. Consider the usual (F-valued) representation of $GL_3(F)$ on V. Let $\{e_1, e_2, e_3\}$ be the standard basis of F^3. The weights of the representation are λ_j, $1 \leq j \leq 3$, where $\lambda_j(a_1, a_2, a_3) = a_j$, as elements of $X^*(A)$, they are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Recall that the simple roots are $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,1,-1)$. In $X^*(A)$, that is, writing things additively, we have $\lambda_1 - \alpha_1 = \lambda_2$, and $\lambda_1 - \alpha_1 - \alpha_2 = \lambda_2 - \alpha_2 = \alpha_3$. We can define a partial order in $X^*(A)$ as we did above for $X_*(A)$. The weight λ_1 is maximal and hence is called the *highest weight*, and λ_3 is minimal, so is the *lowest weight*. Observe that $-\lambda_3$ is dominant. Further, if μ is dominant, then $\|a_\mu\| = q^{-\mu_3} = |\lambda_3(a_\mu)|_F = q^{\langle -\lambda_3, \mu\rangle}$.

For general split G, $\lambda \in X^*(A)$ is dominant if $\langle \alpha^\vee, \lambda\rangle \geq 0$ for every root α which is positive for B. Given a dominant $\lambda \in X^*(A)$, there exists exactly one equivalence class of irreducible finite dimensional G-modules having lowest weight $-\lambda$ (and all other weights of the representation are of the form $-\lambda - \sum_{\alpha \text{ simple}} \ell_\alpha \alpha$). Let $(\rho_\lambda, V_\lambda)$ belong to the equivalence class corresponding to λ. Let n_λ be the dimension of V_λ. Then $\rho_\lambda : G \to GL_{n_\lambda}(F)$. It is possible to choose a basis of V_λ such that $\rho_\lambda(A)$ is a subgroup of the diagonal matrices in $GL_{n_\lambda}(F)$, $\rho_\lambda(N)$ is upper triangular and unipotent, and $\rho_\lambda(K) \subset GL_{n_\lambda}(\mathfrak{o}_F)$.

Now suppose that $Na_\nu \cap Ka_\mu K \neq \emptyset$. Let $n \in N$. Assume that we have chosen a basis of V_λ as above. Then, since every entry of $\rho_\lambda(a_\nu)$ is also an entry of $\rho_\lambda(na_\nu)$, we have $\|\rho_\lambda(na_\nu)\| \geq \|\rho_\lambda(a_\nu)\|$. Suppose that $na_\nu \in Ka_\mu K$ for some $n \in N$. Then $\|\rho_\lambda(a_\nu)\| \leq \|\rho_\lambda(na_\nu)\| = \|\rho_\lambda(Ka_\mu K)\| = \|\rho_\lambda(a_\mu)\|$. That is, $|\lambda^{-1}(a_\nu)|_F \leq |\lambda^{-1}(a_\mu)|_F$. Or, $q^{\langle -\mu+\nu,\lambda\rangle} \leq 1$.

As λ was an arbitrary dominant element of $X^*(A)$, we have $\langle -\mu+\nu,\lambda\rangle \leq 0$ for every dominant $\lambda \in X^*(A)$. We remark that $Na_\nu \cap Ka_\mu K \neq \emptyset$ implies that $|\det(\rho_\lambda(a_\mu))|_F = |\det(\rho_\lambda(a_\nu))|_F$ for every dominant λ. It is a simple matter to check that these conditions on μ and ν imply that $\mu \geq \nu$. Hence (2') holds. This completes the proof of the theorem.

Spherical representations and Macdonald's formula

We continue to use the notation from the sections on principal series representations and the Satake isomorphism. In this section, we discuss properties of spherical representations and we outline the proof of Macdonald's formula for $f_\mu^\vee(z)$, $z \in (\mathbb{C}^\times)^d$, $\mu \in X_*(A)^+$. The main reference for this section is [Cass]. See also [Mac] and [Mac'].

A (zonal) spherical function on G with respect to K is a function E on G which is bi-invariant under K, such that $E(1) = 1$, and satisfies the equivalent properties (i) and (ii):

(i) The map $f \mapsto \int_G f(g)\, E(g)\, dg$ from $\mathcal{H}_K$ to $\mathbb{C}$ is an algebra homomorphism.

(ii) $E(g)\, E(g') = \int_K E(gkg')\, dk$, g, $g' \in G$.

To see that (i) and (ii) are equivalent, note that

$$\begin{aligned}\int_G (f_1 * f_2)(g)\, E(g)\, dg &= \int_G \int_G E(g_1 g_2)\, f_1(g_1)\, f_2(g_2)\, dg_1\, dg_2 \\ &= \int_G \int_G f_1(g_1)\, f_2(g_2) \int_K E(g_1 k g_2)\, dk\, dg_1\, dg_2.\end{aligned}$$

Observe that the function $(g_1, g_2) \mapsto \int_K E(g_1 k g_2)\, dk$ on $G \times G$ is invariant under left and right translations by elements of $K \times K$. Hence (i) is equivalent to

$$\int_G \int_G f_1(g_1) f_2(g_2) \left(E(g_1) E(g_2) - \int_K E(g_1 k g_2)\, dk \right) dg_1 dg_2 = 0 \qquad f_1,\ f_2 \in \mathcal{H}_K,$$

which is then equivalent to (ii).

An irreducible smooth representation (π, V) is said to be *spherical* (or K-spherical) or *unramified* if V^K is nonzero. These representations are important in the theory of automorphic forms. Given an automorphic representation of the adele group $G(\mathbb{A}_F)$, all but finitely many of its local components are spherical.

Let E be a spherical function. Set $V_E = \{\, g \mapsto \sum_{i=1}^{\ell} c_i E(g g_i) \mid g_i \in G,\, c_i \in \mathbb{C} \,\}$. If $f \in V_E$ and $g \in G$ let $(\pi_E(g) f)(g') = f(g'g)$, $g' \in G$.

Lemma. *Let E be a spherical function. Then (π_E, V_E) is a spherical representation and V_E^K is spanned by E.*

Proof. It is clear that π_E is a smooth representation. Suppose that $f \in V_E$ and $f \neq 0$. Choose $g' \in G$ such that $f(g') \neq 0$. Let $ch_{Kg'}$ be the characteristic function of Kg'. Using property (ii) of spherical functions,

$$f(g')^{-1} \pi_E(ch_{Kg'}) f(g) = f(g')^{-1} \int_K f(gkg')\, dk = E(g).$$

Hence E belongs to any nonzero G-invariant subspace of V_E. By definition of V_E, $V_E = \mathrm{Span}(G \cdot E)$. It follows that π_E is irreducible.

Let $f \in V_E^K$. Then property (ii) of spherical functions, with $g' = 1$, takes the form

$$f(g) = \int_K f(gk)\,dk = E(g)f(1), \qquad g \in G.$$

Hence $f = f(1)E$. $\square$

Recall that for $z \in (\mathbb{C}^\times)^d$, $E_z(g) = \int_K f_{z,B}(kg)\,dk$, where $f_{z,B}(ank) = (\delta_B^{1/2}\chi_z(a))$, $a \in A$, $n \in N$, $k \in K$. Because $f \mapsto f^\vee(z)$ is an algebra homomorphism from $\mathcal{H}_K$ to $\mathbb{C}$, and (as seen in the previous section) $f^\vee(z) = \int_G f(g)E_z(g)\,dg$, it follows that E_z is a spherical function.

Theorem.

(1) If (π, V) is a spherical representation, there exists a unique spherical function E which occurs as a matrix coefficient of π. Furthermore, $\pi \simeq \pi_E$.

(2) If E is a spherical function, there exists $z \in (\mathbb{C}^\times)^d$ such that $E = E_z$.

(3) The function $g \mapsto E_z(g)$ is a matrix coefficient of $i_B^G(\chi_z)$.

We remark that, although the spherical function E_z of (1) and (2) is unique, z is not unique: $E_z = E_{w(z)}$ for all $w \in W$.

Proof of theorem: Since π is spherical and K contains an Iwahori subgroup $\mathcal{I}$, we have $V^{\mathcal{I}} \neq 0$. It is known (though not proved in these notes) that an irreducible smooth representation which has nonzero Iwahori fixed vectors is equivalent to a subrepresentation of some unramified principal series representation. So there exists a $z \in (\mathbb{C}^\times)^d$ such that π is equivalent to a subrepresentation of $i_B^G(\chi_z)$. Now we know that the subspace of K-invariant vectors in the space of $i_B^G(\chi_z)$ is one dimensional and is spanned by $f_{z,B}$. It follows that $\dim V^K = 1$. Because $\widetilde{V}^K = (V^K)^*$, we have $\dim \widetilde{V}^K = 1$. Choose nonzero vectors $v \in V^K$ and $\widetilde{v} \in \widetilde{V}^K$. Multiplying v be a nonzero scalar if necessary, we may assume that $\langle \tilde{v}, v\rangle = 1$. Note that the matrix coefficient $f_{\tilde{v},v}(g) = \langle \tilde{v}, \pi(g)v\rangle$ is K-bi-invariant. Because V^K and $\widetilde{V}^K$ are one dimensional, it is easy to see that $f_{\tilde{v},v}$ is the unique K-bi-invariant matrix coefficient of π which takes the value 1 at the identity.

Then the image of v in the space of $i_B^G(\chi_z)$ is a nonzero multiple of $f_{z,B}$. The contragredient of $i_B^G(\chi_z)$ is $i_B^G(\chi_{z^{-1}})$, and $f_{z^{-1},B} \in i_B^G(\chi_{z^{-1}})$. By earlier results on induced representations,

$$\langle f_{z^{-1},B}, i_B^G(\chi_z)(g)f_{z,B}\rangle = \int_K f_{z^{-1},B}(k)\, f_{z,B}(kg)\,dk = E_z(g),$$

the final equality holding because $f_{z^{-1},B}\,|\,K \equiv 1$. Hence (3) holds. Now, for each $g \in G$, $i_{z,B}(\chi_z)(g)f_{z,B}$ belongs to the image of V in the space of $i_{z,B}(\chi_z)$. And any smooth linear functional on the space of $i_B^G(\chi_z)$ restricts to a smooth linear functional on (the image of) the space V of π. Hence E_z is also a matrix coefficient of π. By the remarks above concerning uniqueness of K-bi-invariant matrix coefficients, $E_z(g) = f_{\tilde{v},v}(g)$, $g \in G$.

Let $\tilde{v}$ and v be as above. Let $v' \in V$. By irreducibility of π, there exist finitely many $g_i \in G$ and $c_i \in \mathbb{C}$ such that $v' = \sum_i c_i \pi(g_i)v$. Let $E = E_z$ be as above. We have

$$f_{\tilde{v},v'}(g) = \sum_i c_i \langle \tilde{v}, \pi(gg_i)v\rangle = \sum_i c_i E(gg_i),$$

so $f_{\tilde{v},v'} \in V_E$. Define $A : V \mapsto V_E$ by $Av' = f_{\tilde{v},v'}$. Observe that $f_{\tilde{v},\pi(g_0)v'}(g) = (\pi_E(g_0)f_{\tilde{v},v'})(g)$, $g \in G$. Hence A intertwines π and π_E. Since $Av = E$, A is nonzero, so by irreducibility of π and π_E, must be an isomorphism. Therefore $\pi \simeq \pi_E$. □

Given $\mu \in X_*(A)^+$, let f_μ be the characteristic function of $Ka_\mu K$. Then

$$f_\mu^\vee(z) = \int_{Ka_\mu K} E_z(g)\,dg = m(Ka_\mu K)\,E_z(a_\mu).$$

Next, we discuss Macdonald's formula for $f_\mu^\vee(z)$. By properties of $f_\mu^\vee$, for fixed μ, $E_z(a_\mu)$ is a regular function of z and $E_{w(z)}(a_\mu) = E_z(a_\mu)$ for $w \in W$.

Lemma. *Let $V = i_B^G(\chi_z)$. Then if $\langle\cdot,\cdot\rangle$ denotes the usual pairing between $\widetilde{V}$ and V, and $\langle\cdot,\cdot\rangle_N$ denotes the pairing between the (unnormalized) Jacquet modules $\widetilde{V}_{N^-}$ and V_N given in Casselman's theorem on asymptotic behaviour of matrix coefficients. Given $\widetilde{v} \in \widetilde{V}$, and $v \in V$, let $\bar{\widetilde{v}}$ and $\bar{v}$ be the images of $\widetilde{v}$ and v in $\widetilde{V}_{N^-}$ and V_N, respectively. Then*

$$\langle \widetilde{v}, av\rangle = \langle \bar{\widetilde{v}}, a\bar{v}\rangle_N, \qquad a \in A^+,\ v \in V,\ \widetilde{v} \in \widetilde{V}.$$

The lemma is a stronger version of Casselman's theorem. The proof will be omitted.

Suppose that $z \in (\mathbb{C}^\times)^d$ is such that $\mathrm{Stab}_W(z) = \{1\}$. As shown in an earlier section, the composition factors of V_N are $\chi_{wz}\delta_B^{1/2}$, $w \in W$. When $\mathrm{Stab}_W(z) = \{1\}$, these factors are all distinct and the A-module V_N is a direct sum:

$$V_N = \bigoplus_{w \in W} \chi_{wz}\delta_B^{1/2}.$$

Setting $v = f_{z,B}$ and $\widetilde{v} = f_{z^{-1},B}$ and writing $\bar{v}$ as a sum of eigenvectors, we see that $a \in A^+ \mapsto \langle \bar{\widetilde{v}}, a\bar{v}\rangle_N = E_z(a)$ is a linear combination of functions of the form $a \in A^+ \mapsto \chi_{wz}(a)\delta_B^{1/2}(a)$. That is,

$$(*) \qquad E_z(a) = \sum_{w \in W} d(w,z)\,\delta_B^{1/2}(a)\chi_{wz}(a), \qquad a \in A^+.$$

Example: $G = SL_2(F)$. In this case $X_*(A) = A/A_0 \simeq \mathbb{Z}$ and $X_*(A)^+ = \mathbb{Z}_{\geq 0}$. Given $z \in \mathbb{C}^\times$, $\chi_z(\mathrm{diag}(\varpi_F^j, \varpi_F^{-j})) = z^j$. Evaluating $(*)$ for $a_0 = 1$ and $a_1 = \mathrm{diag}(\varpi_F, \varpi_F^{-1})$, we have, denoting the non trivial element of W by w,

$$\begin{aligned} E_z(a_0) &= d(1,z) + d(w,z) \\ E_z(a_1) &= d(1,z)\, zq^{-1} + d(w,z)\, z^{-1}q^{-1} \end{aligned}$$

which we can rewrite as

$$\begin{pmatrix} E_z(a_0) \\ q\, E_z(a_1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ z & z^{-1} \end{pmatrix} \begin{pmatrix} d(1,z) \\ d(w,z) \end{pmatrix}.$$

From these equations, and the fact that $E_z(a_0)$ and $E_z(a_1)$ are regular functions of z, it follows that $d(1,z)$ and $d(w,z)$ can be expressed in terms of regular functions of z and rational functions of z and are therefore rational functions of z.

In general, to see that the $d(w,z)$ are rational functions of z, we work with $\mu \in X_*(A)^+$ (μ dominant) and not on any wall. So $w \cdot \mu = w' \cdot \mu$ implies $w = w'$. For $PGL_3(F) = GL_3(F)/Z$, these look like:

Evaluating $(*)$ at a_ν for $\nu \in \{0, \mu, 2\,\mu, \ldots, (|W|-1)\,\mu\}$. Solving for the functions $d(w,z)$ results in expressions involving the regular functions $E_z(a_\nu)$ and the inverse of a van der Monde determinant $\pm \prod (\chi_{wz}(a_\mu) - \chi_{w'z}(a_\mu))$.

As $d(w,z)$ is a rational function of z, the values of $d(w,z)$ are determined by its values on the open set

$$\mathcal{S} = \{\, z \in (\mathbb{C}^\times)^d \mid |\chi_z(\alpha^\vee(\varpi_F)| < 1 \text{ for every positive root } \alpha \,\}$$

We say that a root α is positive, resp. negative, if α is positive, resp. negative, for B. Note that if $z \in \mathcal{S}$, then z is *regular*: $w(z) = z$ implies $w = 1$. (For $G = GL_n(F)$, $z \in \mathcal{S}$ is equivalent to $|z_j||z_{j+1}|^{-1} < 1$ for $1 \leq j \leq n-1$).

Let w_0 be the longest element (relative to B) of the Weyl group W. It is clear from the definition of roots that W acts on the set of roots. The length of $w \in W$ is equal to the number of roots α such that α is positive (for B) and α^w is negative (for B). The element w_0 has the property that, for any α, α^{w_0} is negative if and only if α is positive. This implies that $w_0 N w_0^{-1} = N^-$.

We rewrite $(*)$ as follows:

$$d(w_0,z)=\delta_B^{-1/2}(a)\chi_{w_0z}(a)^{-1}E_z(a)+\sum_{w\neq w_0}\chi_{w_0z}(a)\,\chi_{wz}(a)\,d(w,z).$$

For $G=SL_2(F)$, $z\in\mathcal{S}$ if and only if $|\chi_z(\alpha^\vee(\varpi_F)|=|\chi_z(\mathrm{diag}(\varpi_F,\varpi_F^{-1}))|=|z|<1$. The above expression for $d(w_0,z)$ has the form, for $a_j=\mathrm{diag}(\varpi_F^j,\varpi_F^{-j})$,

$$d(w_0,z)=q^{-j}z^jE_z(a)+d(1,z)z^{2j}.$$

For $z\in\mathcal{S}$, $\lim_{j\to\infty}z^{2j}=0$. In the general case, $z\in\mathcal{S}$ implies that for $w\neq w_0$, $\chi_{w_0z}(a)^{-1}\chi_{wz}(a)$ is a decreasing exponential as $a\in A^+$ approaches infinity. Thus

$$(**)\qquad d(w_0,z)=\lim_{a\to\infty\; a\in A^+}\delta_B^{-1/2}(a)\chi_{w_0z}(a)^{-1}E_z(a),$$

The idea is to use analyze the behaviour of $E_z(a)$ as $a\to\infty$, for $z\in\mathcal{S}$, and then use $(**)$ to get a formula for $d(w_0,z)$.

Let I be an Iwahori subgroup contained in K. We have a projection of $K=G(\mathfrak{o}_F)$ onto $G(\mathfrak{o}_F/\mathfrak{p}_F)$. The Iwahori subgroup I is the inverse image in K of some Borel subgroup B_q of $G(\mathfrak{o}_F/\mathfrak{p}_F)$. For example, if $G=GL_n(F)$, I is conjugate to the subgroup of matrices of the form

$$\begin{pmatrix}\mathfrak{o}_F^\times & \mathfrak{o}_F & \cdots & \mathfrak{o}_F\\ \mathfrak{p}_F & \mathfrak{o}_F^\times & \cdots & \mathfrak{o}_F\\ \vdots & \vdots & \ddots & \vdots\\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{o}_F^\times\end{pmatrix}.$$

We can (and do) choose I such that $N\cap I=N\cap K$ and I has an Iwahori factorization with respect to $B=AN$: $I=(N\cap I)(A\cap I)(N^-\cap I)=(N^-\cap I)(A\cap I)(N\cap I)$.

Lemma. *Let $z\in\mathcal{S}$. Set $Q=\sum_{w\in W}q(w)$, where $q(w)$ is the number of cosets of I in the double coset IwI. Then $d(w_0,z)=Q^{-1}\int_{w_0Nw_0^{-1}}f_{z,B}(n)\,dn$.*

Assuming the lemma for now, we state Macdonald's formula.

Theorem. *Let $\mu\in X_*(A)^+$. Then*

$$f_\mu^\vee(z)=Q^{-1}\delta_B(a_\mu)^{-1/2}\sum_{w\in W}\chi_{wz}(a_\mu)\prod_{\alpha>0}\frac{1-q^{-1+\langle\alpha^\vee,w(z)\rangle}}{1-q^{\langle\alpha^\vee,w(z)\rangle}}.$$

Proof. Let $B'=AN^-$. Recall that $w_0Nw_0^{-1}=N^-$ and the roots which are positive for B are all negative for B'. By a proposition from the section on intertwining maps,

$$\int_{w_0Nw_0^{-1}}f_{z,B}(n)\,dn=(I_{B',B}f_{z,B})(1)=\prod_{\alpha>0}\frac{1-q^{-(1+\langle\alpha^\vee,z\rangle)}}{1-q^{-\langle\alpha^\vee,z\rangle}}f_{z,B'}(1)=\prod_{\alpha>0}\frac{1-q^{-(1+\langle\alpha^\vee,z\rangle)}}{1-q^{-\langle\alpha^\vee,z\rangle}}.$$

By the above lemma, this gives a formula for $d(w_0, z)$.

Because $E_{w(z)} = E_z$ for all $w \in W$, it follows from $(*)$ that $d(w,z) = d(1, w(z))$ for $w \in W$. This then gives a formula for each $d(w,z)$ and hence (by $(*)$) for $E_z(a_\mu)$ and $f_\mu(z)$ (recall that $m(Ka_\mu K) = \delta_B(a_\mu)^{-1}$ by an earlier lemma). $\square$

Sketch of proof of lemma. We assume that representatives for elements of W have been chosen in K. Then $K = \coprod_{w\in W} IwI$. As $w_0 \in K$,

$$K = Kw_0 = \coprod_{w\in W} IwIw_0 = \coprod_{w\in W} I(ww_0)w_0^{-1}Iw_0 = \coprod_{w\in W} Iw(w_0^{-1}Iw_0) = \coprod_{w\in W} Iw(N^-\cap K).$$

The final equality is obtained as follows. From the Iwahori decomposition and $N \cap I = N \cap K$ we have

$$\begin{aligned} w_0^{-1}Iw_0 &= w_0^{-1}(N^-\cap I)w_0^{-1}(A\cap I)w_0^{-1}(N\cap I)w_0^{-1} \\ &= w_0^{-1}(N^-\cap I)w_0^{-1}(A\cap I)w_0^{-1}(N\cap K)w_0 = w_0^{-1}(N\cap I)w_0(A\cap I)(N^-\cap K). \end{aligned}$$

Next, note that $N \cap I$ is a subgroup of K_1 (where K_1 is the set of x in G such that every entry of $x-1$ belongs to $\mathfrak{p}_F$). Since K_1 is normal in K and $K_1 \subset I$, we have $Iww_0^{-1}(N\cap I)w_0 \subset IK_1w = Iw$. Note also that w normalizes $A \cap I = A \cap K$.

The correspondence $W \longleftrightarrow I\backslash K/(N^-\cap K)$ is used to decompose the integral $E_z(a)$. We write

$$E_z(a) = \int_K f_{z,B}(ka)\,dk = [K:I]^{-1}\sum_{k\in I\backslash K}\int_I f_{z,B}(ika)\,dk.$$

Take $a \in A^+$. Conjugation by a^{-1} shrinks N^-, so $a^{-1}(N^-\cap K)a \subset N^-\cap K$. As $f_{z,B}$ is right K invariant, this implies that the function $k \mapsto \int_I f_{z,B}(ika)\,di$ is left I invariant and right $N^-\cap K$ invariant. Thus, denoting the number of left I cosets in K contained in a double coset $Iw(N^-\cap K)$ ($w \in W$) by $c(w)$,

$$\begin{aligned} E_z(a) &= [K:I]^{-1}\sum_{w\in W} c(w)\int_I f_{z,B}(iwa)\,di \\ &= [K:I]^{-1}\sum_{w\in W} c(w)(\delta_B^{1/2}\chi_z)(w(a))\int_I f_{z,B}(w(a)^{-1}iw(a))\,di, \qquad a\in A^+, \end{aligned}$$

where $w(a) = waw^{-1}$, the second inequality following from $w \in K$, right K invariance of $f_{z,B}$, and the way $f_{z,B}$ transforms under left A translation. Since $a \in A^+$, conjugation by a^{-1} shrinks N^-. Thus conjugation by $w(a)^{-1}$ shrinks wN^-w^{-1}. We have

$$\begin{aligned} E_z(a) &= [K:I]^{-1}\sum_{w\in W} c(w)(\delta_B^{1/2}\chi_z)(w(a))\int_I f_{z,B}(w(a)^{-1}iw(a))\,di \\ &= [K:I]^{-1}\sum_{w\in W} c(w)(\delta_B^{1/2}\chi_z)(w(a))\int_{I\cap N} f_{z,B}(w(a)^{-1}xw(a))\,dx, \qquad a\in A^+, \end{aligned}$$

the second equality following from the Iwahori decomposition and right K invariance of $f_{z,B}$. Set $J_{w(a)} = (N \cap wa(I \cap N)a^{-1}w^{-1}) \backslash wa(I \cap N)a^{-1}w^{-1}$. As $f_{z,B}$ is left N invariant, the above expression for $E_z(a)$ can be rewritten as

$$E_z(a) = [K : I]^{-1} \sum_{w \in W} c(w)(\delta_B^{1/2}\chi_z)(w(a)) \int_{J_{w(a)}} f_{z,B}(x)dx, \qquad a \in A^+.$$

Consequently (∗∗) becomes

$$d(w_0, z) = [K : I]^{-1} \lim_{a \to \infty \ a \in A^+} \sum_{w \in W} c(w)\chi_{w_0 z}(a)^{-1}\chi_z(w(a)) \int_{J_{w(a)}} f_{z,B}(x)\,dx.$$

Recall that if $w \neq w_0$ and $z \in \mathcal{S}$, then $\chi_{w_0 z}(a)^{-1}\chi_{wz}(a)$ is a decreasing exponential as $a \in A^+$ tends to infinity. Note that $Iw_0(N^- \cap K) = Iw_0(w_0 I w_0) = Iw_0$ ($w_0^2 = 1$), so $c(w_0) = 1$. Therefore we have

$$d(w_0, z) = [K : I]^{-1} \lim_{a \to \infty \ a \in A^+} \int_{J_{w_0(a)}} f_{z,B}(x)\,dx.$$

For $a \in A$, conjugation by $w_0 a$ takes N to $w_0 N w_0^{-1} = N^-$, so $N \cap w_0 a(I \cap N)a^{-1}w_0 = \{1\}$. For $a \in A^+$, as $a \to \infty$ the set $w_0 a(I \cap N)a^{-1}w_0^{-1}$ expands to fill out $w_0 N w_0^{-1} = N^-$. Therefore in the limit, $J_{w_0(a)}$ becomes $w_0 N w_0^{-1}$. The lemma follows. □

References

[A] J.D. Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math., **185** (1998), 1–32.
(**Construction of certain supercuspidal representations using Moy-Prasad filtrations of parahoric subgroups - see also [Y]**)

[Ba] G. Bachman, *Introduction to p-adic numbers and valuation theory*, Academic Press, London, 1964.

[Bo] A. Borel, *Properties and linear representations of Chevalley groups*, in *Seminar on Algebraic Groups and Related Finite Groups*, Springer Lecture Notes in Math. **131** (1970), 1–55.

[B] A. Borel, *Admissible Representations of a Semi-Simple Group over a Local Field with Vectors Fixed under an Iwahori Subgroup*, Invent. Math. **35** (1976), 233–259. (**Iwahori spherical Hecke algebra**)

[BJ] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, in Proc. Symp. Pure Math. **33**, part 1, AMS, 1979, pp.189–202.

[BK1] C.J. Bushnell and P.C. Kutzko, *The admissible dual of $GL(N)$ via compact open subgroups*, Annals Math. Studies, Princeton Univ. Press (1993)

[BK2] C.J. Bushnell and P.C. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. **77** (1998), 582–634.

[BK3] C.J. Bushnell and P.C. Kutzko, *Semisimple types in GL_n*, Comp. Math. **119** (1999), 53–97.

[BT1] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local I. Données radicielles valuées*, Publ. Math. I.H.E.S. **42** (1972), 1–251.

[BT2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local II. Schémas en groupes*, Publ. Math. I.H.E.S. **60** (1984), 1–251.

[C] H. Carayol, *Représentations cuspidales du groupe linéaire*, Ann. Scient. Ec. Norm. Sup. t. **17** (1984), 191–225. (**Construction of certain supercuspidal representations of general linear groups**)

[Cart] R. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Pure and Applied Mathematics, Wiley, 1985.

[Car] P. Cartier, *Representations of p-adic groups: a survey*, in Proc. Symp. Pure Math. **33**, part 1, AMS, 1979, pp.111–155.

[Ca] W. Casselman, *The Steinberg character as a true character*, in Harmonic Analysis on Homogeneous Spaces, Proc. Math. Symp. XXVI, AMS, (1973), 413–417.

[Cass] W. Casselman, *The unramified principal series of p-adic groups I. The spherical function*, Comp. Math. **40** (1980), 387–406.

[Ca'] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, notes (unpublished).

[Cas1] J.W.S. Cassels, *Local Fields*, London Mathematical Society Student Texts. **3**, Cambridge University Press, 1986.

[Cas2] J.W.S. Cassels, *Global Fields*, in Algebraic Number Theory, Academic Press, London, 1967.

[D] S. DeBacker, *Parameterizing conjugacy classes of maximal unramified tori via Bruhat-Tits Theory*, Michigan Math. J., **54** (2006), 157–178.

[DR] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Annals of Math., to appear.

[DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. Math. **103** (1976), 103–161.

[vD] G. van Dijk,*Computation of certain induced characters of p-adic groups*, Math. Ann. **199** (1972), 229-240.

[Fr] A. Frohlich, *Local Fields*, in Algebraic Number Theory, Academic Press, London, 1967.

[FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics, **129**, Springer-Verlag, New York, 1991.

[GR] B.H. Gross and M. Reeder, *From Laplace to Langlands via representations of orthogonal groups*, Bull. Amer. Math. Soc., **43** (2006), 163–205.

[HC] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Math. vol. 162, Springer 1970.

[HC'] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Preface and notes by Stephen DeBacker and Paul J. Sally, Jr., University Lecture Series **16** AMS 1999.

[H] R. Howe, *Tamely ramified supercuspidal representations of GL_n*, Pacific J. Math. **73** (1977), 479–538.

[HM] R. Howe and A. Moy, *Hecke Algebra Isomorphisms for $GL(n)$ over a p-adic field*, J. Algebra **131** (1990), 388–424.

[IM] N. Iwahori, H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke ring of p-adic Chevalley groups*, Publ. Math. IHES **25** (1965), 5–48. (**Hecke algebras**)

[J] H. Jacquet, *Représentations des groupes linéaires p-adiques*, Theory of Group Representations and Harmonic Analysis (C.I.M.E. II Ciclo, Montecatini terme, 1970), Edizioni Cremonese, Roma, 1971, pp.119-220.

[JL] H. Jacquet, R. Langlands, *Automorphic forms on GL(2)*, Lecture Notes in Math. vol. **260** Springer Verlag 1962.

[JS] H. Jacquet and J. Shalika, *The Whittaker models of induced representations*, Pacific J. Math. **109** (1983), 107–120. (See p. 109 for info on discrete series for GL_n).

[Ja1] C. Jantzen, *On square-integrable representations of classical p-adic groups*, Canad. J. Math. **52** (2000), 539–581.

[Ja2] C. Jantzen, *On square-integrable representations of classical p-adic groups, II*, Represent. Theory **4** (2000), 127–180.

[K] D. Keys, *Reducibility of unramified unitary principal series representations of p-adic groups and class-1 representations*, Math. Ann. **260** (1982), 397–402.

[Ki1] J.-L. Kim, *Hecke algebras of classical groups over p-adic fields and supercuspidal representations*, Amer. J. Math. **121** (1999), 967–1029.

[Ki2] J.-L. Kim, *Supercuspidal reprsentations, an exhaustion theorem*, J. Amer. Math. Soc. **20** (2007), 273–320.

[L] G. Lusztig, *Classification of unipotent representations of simple p-adic groups*, IMRN **11** (1995), 517–589.

[Mac'] I.G. Macdonald, *Spherical functions on a p-adic Chevalley group*, Bull. Amer. Math. Soc. **74** (1968), 520–525.

[Mac] I.G. Macdonald, *Spherical functions on a group of p-adic type*, Ramanujan Institute Lecture Notes, University of Madras, 1971. (**Hecke algebras**)

[MT] C. Moeglin and M. Tadic, *Construction of discrete series for classical p-adic groups*, J.A.M.S. **15** (2002), 715–786.

[Mo1] L. Morris, *Some tamely ramified supercuspidal representations of symplectic groups*, Proc. London Math. Soc. **63** (1991), 519–551.

[Mo2-3] L. Morris, *Tamely ramified supercuspidal representations of classical groups I, II*, Ann. scient. Ec. Norm. Sup. **25** (1991), 705–738, and **26** (1992), 233–274.

[Mo4] L. Morris, *P-cuspidal representations of level one*, Proc. London Math. Soc., **58** (1989), 550–558.

[Mo5] L. Morris, *Tamely ramified interwining algebras*, Invent. Math., **114** (1993), 1–54.

[My1] A. Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108** (1986), 863–930. (**Supercuspidal representations of general linear groups**)

[My2] A. Moy, *Representations of $U(2,1)$ over a p-adic field*, J. Reine. Angew. Math. **372** (1986), 178–208. (**Classification of irreducible admissible representations of 3×3 unramified unitary group via Hecke algebra isomorphisms**)

[My3-4] A. Moy, *Representations of GSp_4 over a p-adic field: parts 1 and 2*, Comp. Math. **66** (1988), 237–284, 285–328. (**Classification of irreducible admissible representations of GSp_4 via Hecke algebra isomorphisms**)

[MP1] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), 393–408.

[MP2] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helvetici **71** (1996), 98–121.

[MS] A. Moy, P. Sally, *Supercuspidal representations of SL_n over a p-adic field: the tame case*, Duke J. Math. **51** (1984), 149–161.

[PS] I. Piatetski-Shapiro, *Classical and adelic automorphic forms. An Introduction.* in Proc. Symp. Pure Math. **33**, part 1, AMS, 1979, pp.185–188.

[S] S. Shatz, *Profinite groups, arithmetic, and geometry*, Annals of Mathematics Studies, **67**, Princeton University Press, 1972.

[Si] B. Simon, *Representations of Finite and Compact Groups*, Graduate Studies in Mathematics, **10**, AMS, 1996.

[Ta1] M. Tadic, *Square integrable representations of calssical p-adic groups corresponding to segments*, Represent. Theory **3** (1999), 58–89.

[Ta2] M. Tadic, *A family of square integrable representations of classical p-adic groups in the case of half-integrable reducibilities*, Glas. Mat. Ser III **37(57)** (2002), 21–57.

[T] J.T. Tate, *Fourier anslysis in number fields and Hecke's zeta functions*, Algebraic Number Theory (J.W.S. Cassels and A. Frohlich, eds.), Academic Press, 1967, pp. 305-347.

[Ti] J. Tits, *Reductive groups over local fields*, in Automorphic Forms, Representations, and L-Functions, Proc. Symp. Pure Math. **33**, Part 1, Amer. Math. Soc., Providence, RI, 1979, pp. 29–69.

[W] J.-L. Waldspurger, *Algèbres de Hecke et induites de représentations cuspidales, pour GL(N)*, J. fur die reine und angew. Math. **370** (1986), 127–191. (**Section on computing formal degress of discrete series representations of $GL_n(F)$; representations of Hecke algebras**)

[Wi] N. Winarsky, *Reducibility of principal series representations of p-adic Chevalley groups*, Amer. J. Math. **100** (1978), 941–956.

[Y] J.-K. Yu, *Construction of tame supercuspidal representations*, J.A.M.S. **14** (2001), 579–622.
(**Construction of tame supercuspidal representations using Moy-Prasad filtrations of parahoric subgroups.**)

[Z] A. Zelevinsky, *Induced representations of p-adic reductive groups II: Irreducible representations of $GL(n)$*, Ann. Sci. Ec. Norm. Sup. **3** (1980), 165–210.