CHAPTER 2

Representations of Finite Groups

In this chapter we consider only finite-dimensional representations.

2.1. Unitarity, complete reducibility, orthogonality relations

Theorem 1. *A representation of a finite group is unitary. c Proof. Let (π, V) be a (finite-dimensional) representation of a finite group $G = \{ g_1, g_2, \ldots g_n \}$. Let $\langle \cdot, \cdot \rangle_1$ be any inner product on V. Set*

$$\langle v, w \rangle = \sum_{j=1}^{n} \langle \pi(g_j)v, \pi(g_j)w \rangle_1, \qquad v,\, w \in V.$$

Then it is clear from the definition that

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle, \quad v,\, w \in V,\, g \in G.$$

Note that if $v \in V$, then $\langle v, v \rangle = \sum_{j=1}^{n} \langle \pi(g_j)v, \pi(g_j)v \rangle_1$ and if $v \neq 0$, then $\pi(g_j)v \neq 0$ for all j implies $\langle \pi(g_j)v, \pi(g_j)v \rangle_1 > 0$ for all j. Hence $v \neq 0$ implies $\langle v, v \rangle > 0$. The other properties of inner product are easy to verify for $\langle \cdot, \cdot \rangle$, using the fact that $\langle \cdot, \cdot \rangle_1$ is an inner product, and each $\pi(g_j)$ is linear. The details are left as an exercise. qed

The following is an immediate consequence of Theorem 1 and a result from Chapter I stating that a finite-dimensional unitary representation is completely reducible.

Theorem 2. *A representation of a finite group is completely reducible.*

Example. Let G be a finite group acting on a finite set X. Let V be a complex vector space having a basis $\{ v_{x_1}, \ldots, v_{x_m} \}$ indexed by the elements $x_1, \ldots, x_m$ of X. If $g \in G$, let $\pi(g)$ be the operator sending v_{x_j} to $v_{g \cdot x_j}$, $1 \leq j \leq m$. Then (π, V) is a representation of G, called the *permutation representation* associated with X.

Let $\mathcal{A}(G)$ be the set of complex-valued functions on G. Often $\mathcal{A}(G)$ is called the *group algebra* of G - see below. Let $R_{\mathcal{A}}$ be the (right) regular representation of G on the space $\mathcal{A}(G)$: Given $f \in \mathcal{A}(G)$ and $g \in G$, $R_{\mathcal{A}}(g)f$ is the function defined by $(R_{\mathcal{A}}(g)f)(g_0) = f(g_0 g)$, $g_0 \in G$. Note that $R_{\mathcal{A}}$ is equivalent to the the permutation representation associated to the set $X = G$. Let $L_{\mathcal{A}}$ be the left regular representation of G on the space $\mathcal{A}(G)$: Given $f \in \mathcal{A}(G)$ and $g \in G$, $L_{\mathcal{A}}(g)f$ is defined by $(L_{\mathcal{A}}(g)f)(g_0) = f(g^{-1}g_0)$, $g_0 \in G$. It is easy to check that the operator $f \mapsto \dot{f}$, where $\dot{f}(g) = f(g^{-1})$, is a unitary equivalence in $\mathrm{Hom}_G(R_{\mathcal{A}}, L_{\mathcal{A}})$.

If $f_1,\ f_2 \in \mathcal{A}(G)$, the *convolution* $f_1 * f_2$ of f_1 with f_2 is defined by

$$(f_1 * f_2)(g) = \sum_{g_0 \in G} f_1(g g_0^{-1}) f_2(g_0), \qquad g \in G.$$

With convolution as multiplication, $\mathcal{A}(G)$ is an algebra. It is possible to study the representations of G in terms of $\mathcal{A}(G)$-modules.

We define an inner product on $\mathcal{A}(G)$ as follows:

$$\langle f_1, f_2 \rangle = |G|^{-1} \sum_{g \in G} f_1(g) \overline{f_2(g)}, \qquad f_1,\ f_2 \in \mathcal{A}(G).$$

Theorem 3 (Orthogonality relations for matrix coefficients). *Let (π_1, V_1) and (π_2, V_2) be irreducible (unitary) representations of G. Let $a^i_{jk}(g)$ be the matrix entries of the matrix of $\pi_i(g)$ relative to a fixed orthonormal basis of V_i, $i = 1, 2$ (relative to an inner product which makes π_i unitary). Then*

(1) If $\pi_1 \not\simeq \pi_2$, then $\langle a^1_{jk}, a^2_{\ell m} \rangle = 0$ for all j, k, ℓ and m.

(2) $\langle a^1_{jk}, a^1_{\ell m} \rangle = \delta_{j\ell}\delta_{km}/n_1$, where $n_1 = \dim V_1$.

Proof. Let B be a linear transformation from V_2 to V_1. Then $A := |G|^{-1} \sum_{g \in G} \pi_1(g) B \pi_2(g)^{-1}$ is also a linear transformation from V_2 to V_1. Let $g' \in G$. Then

$$\pi_1(g')A = |G|^{-1} \sum_{g \in G} \pi_1(g'g) B \pi_2(g^{-1}) = |G|^{-1} \sum_{g \in G} \pi_1(g) B \pi_2(g^{-1}g') = A\pi_2(g').$$

Hence $A \in \mathrm{Hom}_G(\pi_2, \pi_1)$,

Let $n_i = \dim V_i$, $i = 1, 2$. Letting $b_{j\ell}$ be the $j\ell$th matrix entry of B (relative to the orthonormal bases of V_2 and V_1 in the statement of the theorem). Then the $j\ell$th entry of A (relative to the same bases) is equal to

$$|G|^{-1} \sum_{g \in G} \sum_{\mu=1}^{n_1} \sum_{\nu=1}^{n_2} a^1_{j\mu}(g) b_{\mu\nu} a^2_{\nu\ell}(g^{-1}).$$

Suppose that $\pi_1 \not\simeq \pi_2$. By the corollary to Schur's Lemma, $A = 0$. Since this holds for all choices of B, we may choose B such that $b_{\mu\nu} = \delta_{\mu k}\delta_{\nu m}$, $1 \le \mu \le n_1$, $1 \le \nu \le n_2$. Then $|G|^{-1} \sum_{g \in G} a^1_{jk}(g) a^2_{m\ell}(g^{-1}) = 0$. Since the matrix coefficients $a^2_{m\ell}(g)$ are chosen relative to an orthonormal basis of V_2 which makes π_2 unitary, it follows that $a_{m\ell}(g^{-1}) = \overline{a^2_{\ell m}(g)}$. Hence $\langle a^1_{jk}, a^2_{\ell m} \rangle = |G|^{-1} \sum_{g \in G} a^1_{jk}(g) \overline{a^2_{\ell m}(g)} = 0$. This proves (1).

Now suppose that $\pi_1 = \pi_2$. In this case, Schur's Lemma implies that $A = \lambda I$ for some scalar λ. Hence $\mathrm{tr}\, A = |G|^{-1} \sum_{g \in G} \mathrm{tr}\,(\pi_1(g) B \pi_1(g)^{-1}) = \mathrm{tr}\, B = n_1 \lambda$.

That is, the $j\ell$th entry of the matrix A is equal to

$$|G|^{-1} \sum_{g \in G} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} a^1_{j\mu}(g) b_{\mu\nu} a^1_{\nu\ell}(g^{-1}) = \mathrm{tr}\, B \delta_{j\ell}/n_1.$$

Taking B so that $b_{\mu\nu} = \delta_{\mu k}\delta_{\nu m}$, we have $|G|^{-1} \sum_{g \in G} a^1_{jk}(g) a^1_{m\ell}(g^{-1}) = \delta_{j\ell}\delta_{km}/n_1$. qed

Corollary. *Let π_1 and π_2 be irreducible representations of G such that $\pi_1 \not\simeq \pi_2$. The susbpace of $\mathcal{A}(G)$ spanned by all matrix coefficients of π_1 is orthorgonal to the subspace spanned by all matrix coefficients of π_2.*

Proof. Let $a^1_{jk}(g)$ be as in Theorem 3. Let γ be a basis of the space V_1 of π_1, and let $b_{jk}(g)$ be the jkth entry of the matrix $[\pi_1(g)]_\gamma$. Then there exists a matrix $C \in GL_{n_1}(\mathbb{C})$ such that $[b_{jk}(g)] = C[a_{jk}(g)]_{1\le j,k\le n_1}C^{-1}$ for all $g \in G$ (C is the change of basis matrix from the β to γ). It follows that

$$b_{jk} \in \mathrm{Span}\{\, a^1_{\ell m} \mid 1 \le m, \ell \le n_1 \,\}.$$

Hence the subspace spanned by all matrix coefficients of π_1 coincides with the subspace spanned by the matrix coefficients $a_{\ell m}$, $1 \le \ell, m \le n_1$. Hence the corollary follows from Theorem 3(1). qed

Corollary. *There are finitely many equivalence classes of representations of a finite group G.*

Proof. This is an immediate consequence of the preceding corollary, together with $\dim \mathcal{A}(G) = |G|$. qed

For the remainder of this chapter, let G be a finite group, and let $\{\, \pi_1, \dots, \pi_r \,\}$ be a complete set of irreducible representations of G, that is, a set of irreducible representations of G having the property that each irreducible representation of G is equivalent to exactly one π_j. Let n_j be the degree of π_j, $1 \le j \le r$. Let $a^j_{\ell m}(g)$ be the ℓmth entry of the matrix of $\pi_j(g)$ relative to an orthonormal basis of the space of π_j with respect to which each matrix of π_j is unitary.

Theorem 4. *The set $\{\, \sqrt{n_j}a^j_{\ell m} \mid 1 \le \ell, m \le n_j,\ 1 \le j \le r \,\}$ is an orthonormal basis of $\mathcal{A}(G)$.*

Proof. According to Theorem 3, the set is orthormal. Hence it suffices to prove that the set spans $\mathcal{A}(G)$. The regular representation $R_{\mathcal{A}}$ is completely reducible. So $\mathcal{A}(G) = \oplus_{k=1}^t V_k$, where each V_k is an irreducible G-invariant subspace. Fix k. There exists j such that $R_{\mathcal{A}}\,|_{V_k} \simeq \pi_j$. Choose an orthonormal basis $\beta = \{\, f_1, \dots, f_{n_j} \,\}$ of V_k such that $[R_{\mathcal{A}}(g)\,|_{V_k}]_\beta = [a^j_{\ell m}(g)]$, $g \in G$. Then

$$f_\ell(g_0) = (R_{\mathcal{A}}(g_0)f_\ell)(1) = \sum_{i=1}^{n_j} a^j_{i\ell}(g_0)f_i(1), \qquad 1 \le \ell \le n_j.$$

Hence $f_\ell = \sum_{i=1}^{n_j} c_i a^j_{i\ell}$, with $c_i = f_i(1)$. It follows that

$$V_k \subset \mathrm{Span}\{\, a^j_{\ell m} \mid 1 \le \ell, m \le n_j \,\}.$$

qed

Theorem 5. *Let $1 \le j \le r$. The representation π_j occurs as a subrepresentation of $R_{\mathcal{A}}$ with multiplicity n_j.*

Proof. Fix $m \in \{1, \ldots, n_j\}$. Let $W_m^j = \mathrm{Span}\{a_{m\ell}^j \mid 1 \le \ell \le n_j\}$. Then $\{a_{m\ell}^j \mid 1 \le \ell \le n_j\}$ is an orthogonal basis of W_m^j. And W_m^j is orthogonal to $W_{m'}^{j'}$ whenever $j \ne j'$ or $m \ne m'$. Hence $\mathcal{A}(G) = \oplus_{j=1}^r \oplus_{m=1}^{n_j} W_m^j$.

Let g, $g_0 \in G$. Then

$$R_{\mathcal{A}}(g_0)a_{m\ell}^j(g) = a_{m\ell}^j(gg_0) = \sum_{\mu=1}^{n_j} a_{m\mu}^j(g)a_{\mu\ell}^j(g_0), \qquad 1 \le \ell \le n_j.$$

It follows that the matrix of $R_{\mathcal{A}}(g_0)$ relative to the basis $\{a_{m\ell}^j \mid 1 \le \ell \le n_j\}$ of W_m^j coincides with the matrix of π_j. Therefore the restriction of $R_{\mathcal{A}}$ to the subspace $\oplus_{m=1}^{n_j} W_m^j$ is equivalent to the n_j-fold direct sum of π_j. qed

Corollary. $n_1^2 + \cdots + n_r^2 = |G|$.

Corollary. *$\mathcal{A}(G)$ equals the span of all matrix coefficients of all irreducible representations of G.*

Theorem 6 (Row orthogonality relations for irreducible characters). *Let $\chi_j = \chi_{\pi_j}$, $1 \le j \le r$. Then $\langle \chi_k, \chi_j \rangle = \delta_{jk}$.*

Proof.

$$\langle \chi_k, \chi_j \rangle = \sum_{\mu=1}^{n_k} \sum_{\nu=1}^{n_j} \langle a_{\mu\mu}^j, a_{\nu\nu}^j \rangle = \begin{cases} 0, & \text{if } k \ne j \\ \sum_{\mu=1}^{n_j} 1/n_j = 1, & \text{if } k = j \end{cases}.$$

Lemma. *A finite-dimensional representation of a finite group is determined up to equivalence by its character.*

Proof. If m is positive integer, let $m\pi_j = \pi_j \oplus \cdots \oplus \pi_j$, where π_j occurs m times in the direct sum. Let $\pi = m_1\pi_1 \oplus m_2 \oplus \cdots \oplus m_r\pi_r$. Then $\chi_\pi = \sum_{j=1}^r m_j\chi_j$. Let $\pi' = \ell_1\pi_1 \oplus \cdots \oplus \ell_r\pi_r$. We know that $\pi \simeq \pi'$ if and only if $m_j = \ell_j$ for $1 \le j \le r$. By linear independence of the functions χ_j, this is equivalent to $\chi_\pi = \chi_{\pi'}$. qed

Lemma. *Let $\pi = m_1\pi_1 \oplus \cdots \oplus m_r\pi_r$. Then $\langle \chi_\pi, \chi_\pi \rangle = \sum_{j=1}^r m_j^2$.*

Corollary. *π is irreducible if and only if $\langle \chi_\pi, \chi_\pi \rangle = 1$.*

A complex-valued function on G is a *class function* if it is constant on conjugacy classes in G. Note that the space of class functions on G is a subspace of $\mathcal{A}(G)$ and the inner product on $\mathcal{A}(G)$ restricts to an inner product on the space of class functions.

Theorem 7. *The set $\{ \chi_j \mid 1 \leq j \leq r \}$ is an orthonormal basis of the space of class functions on G. Consequently the number r of equivalence classes of irreducible representations of G is equal to the number of conjugacy classes in G.*

Proof. By Theorem 6, the set $\{ \chi_j \mid 1 \leq j \leq r \}$ is orthonormal. It suffices to prove that the functions χ_j span the class functions.

Let f be a class function on G. Since $f \in \mathcal{A}(G)$, we can apply Theorem 4 to conclude that

$$f = \sum_{j=1}^{r} \sum_{m,\ell=1}^{n_j} \langle f, \sqrt{n_j} a^j_{m\ell} \rangle \sqrt{n_j} a^j_{m\ell} = \sum_{j=1}^{r} n_j \sum_{m,\ell=1}^{n_j} \langle f, a^j_{m\ell} \rangle a^j_{m\ell}.$$

Next,

$$(*) \qquad f(g) = |G|^{-1} \sum_{g' \in G} f(g'gg'^{-1}) = |G|^{-1} \sum_{j=1}^{r} n_j \sum_{m,\ell=1}^{n_j} \langle f, a^j_{m\ell} \rangle \sum_{g' \in G} a^j_{m\ell}(g'gg'^{-1}).$$

Note that

$$\begin{aligned} |G|^{-1} \sum_{g' \in G} a^j_{m\ell}(g'gg'^{-1}) &= |G|^{-1} \sum_{g' \in G} \sum_{\mu,\nu=1}^{n_j} a^j_{m\nu}(g') a^j_{\mu\nu}(g) a^j_{\nu\ell}(g'^{-1}) \\ &= \sum_{\mu,\nu=1}^{n_j} \left(|G|^{-1} \sum_{g' \in G} a^j_{m\mu}(g') \overline{a^j_{\ell\nu}(g)} \right) a^j_{\mu\nu}(g) = \sum_{\mu,\nu=1}^{n_j} \langle a^j_{m\mu}, a^j_{\nu\ell} \rangle a^j_{\mu\nu}(g) \\ &= n_j^{-1} \delta_{m\ell} \sum_{\mu=1}^{n_j} a^j_{\mu\mu}(g) = \delta_{m\ell} n_j^{-1} \chi_j(g). \end{aligned}$$

Substituting into (*) results in

$$f(g) = \sum_{j=1}^{r} \sum_{m,\ell=1}^{n_j} \langle f, a^j_{m\ell} \rangle \delta_{m\ell} \chi_j(g) = \sum_{j=1}^{r} \left(\sum_{\ell=1}^{n_j} \langle f, a^j_{\ell\ell} \rangle \right) \chi_j(g) = \sum_{j=1}^{r} \langle f, \chi_j \rangle \chi_j(g).$$

qed

If $g \in G$, let $|cl(g)|$ be the number of elements in the conjugacy class of g in G.

Theorem 8 (Column orthogonality relations for characters).

$$\sum_{j=1}^{r} \chi_j(g) \overline{\chi_j(g')} = \begin{cases} |G|/|cl(g)|, & \textit{if } g' \textit{ is conjugate to } g \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $g_1, \ldots, g_r$ be representatives for the the distinct conjugacy classes in G. Let $A = [\chi_j(g_k)]_{1 \leq j,k \leq r}$. Let $c_j = |cl(g_j)|$, $1 \leq j \leq r$. Let D be the diagonal matrix with diagonal entries c_j, $1 \le j \le r$. Then

$$
\begin{aligned}
(ADA^*)_{m\ell} &= \sum_{j=1}^{r} (AD)_{mj} A^*_{j\ell} = \sum_{j=1}^{r}\sum_{t=1}^{r} \chi_m(g_t) D_{tj} \overline{\chi_\ell(g_j)} \\
&= \sum_{j=1}^{r} \chi_m(g_j) c_j \overline{\chi_\ell(g_j)} = \sum_{g\in G} \chi_m(g)\overline{\chi_\ell}(g) = |G|\delta_{m\ell}
\end{aligned}
$$

Thus $ADA^* = |G|I$. Since $A(DA^*)$ is a scalar matrix, $A(DA^*) = (DA^*)A$. So $DA^*A = |G|I$. That is,

$$
|G|\delta_{m\ell} = \sum_{j=1}^{r} (DA^*)_{mj} A_{j\ell} = \sum_{j=1}^{r} c_j \overline{\chi_j(g_m)} \chi_j(g_\ell).
$$

qed

Example: Let G be a nonabelian group of order 8. Because G is nonabelian, we have $Z(G) \neq G$, where $Z(G)$ is the centre of G. Because G is a 2-group, $Z(G) \neq \{1\}$. If $|Z(G)| = 4$, then $|G/Z(G)| = 2$, so $G/Z(G)$ is cyclic. That is, $G/Z(G) = \langle gZ(G)\rangle$, $g \in G$. Hence $G = \langle Z(G) \cup \{x\}\rangle$. But this implies that G is abelian, which is impossible. Therefore $|Z(G)| = 2$. Now $|G/Z(G)| = 4$ implies $G/Z(G)$ is abelian. Combining G nonabelian and $G/Z(G)$ abelian, we get $G_{der} \subset Z(G)$. We cannot have G_{der} trivial, since G is nonabelian. So we have $G_{der} = Z(G)$. Now, we saw above that $G/Z(G)$ cannot be cyclic. Thus

$$
G/Z(G) = G/G_{der} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
$$

Suppose that χ is a linear character of G (that is, a one-dimensional representation). Then $\chi \,|\, G_{der} \equiv 1$, because $\chi(g_1 g_2 g_1^{-1} g_2^{-1}) = \chi(g_1)\chi(g_2)\chi(g_1)^{-1}\chi(g_2)^{-1} = 1$. Now G_{der} is a normal subgroup of G. So we can consider χ as a linear character of G/G_{der}. Now, in view of results on tensor products of representations, we know that G/G_{der} has 4 irreducible (one-dimensional) representations, each one being the tensor product of two characters of $\mathbb{Z}/2\mathbb{Z}$. Hence

$$
1^2 + 1^2 + 1^2 + 1^2 + n_5^2 + \cdots + n_r^2 = |G| = 8,
$$

with $n_j \ge 2$, $j \ge 5$. It follows that $r = 5$ and $n_5 = 2$.

Since $G_{der} = Z(G)$ has order 2, there are two conjugacy classes consisting of single elements. There are 5 conjugacy classes altogether. Let a, b, and c be the orders of the conjugacy classes containing more than 1 element. Then $2 + a + b + c = 8$ implies $a = b = c = 2$. Let x_1, x_2 and x_3 be representatives of the conjugacy classes containing 2 elements. Let z be the nontrivial element of $Z(G)$. Then 1, y, x_1, x_2, x_3 are representatives of the 5 conjugacy classes. The character table of G takes the form:

	1	1	2	2	2
	1	y	x_1	x_2	x_3
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	$\chi_5(y)$	$\chi_5(x_1)$	$\chi_5(x_2)$	$\chi_5(x_3)$

Using column orthogonality relations, we see that

$$0 = \sum_{j=1} \chi_j(y)\overline{\chi_j(1)} = 4 + 2\chi_5(y),$$

implying $\chi_5(y) = -2$. And

$$0 = \sum_{j=1}^{5} \chi_j(x_k)\overline{\chi_j(1)} = \sum_{j=1}^{4} \chi_j(x_k) + 2\chi_5(x_k) = 2\chi_5(x_k),$$

implying $\chi_5(x_k) = 0$, $1 \le k \le 3$.

Note that (up to isomorphism) there are two nonabelian groups of order 8, the dihedral group D_8, and the quaternion group Q_8. We see from this example that both groups have the same character table.

Exercises:

1. Using orthogonality relations, prove that if (π_j, V_j) is an irreducible representation of a finite group G_j, $j = 1, 2$, then $\pi_1 \otimes \pi_2$ is an irreducible representation of $G_1 \times G_2$. Then prove that every irreducible representation of $G_1 \times G_2$ arises in this way.
2. Let D_{10} be the dihedral group of order 10.
   a) Describe the conjugacy classes in D_{10}.
   b) Compute the character table of D_{10}.
3. Let B be the upper triangular Borel subgroup in $GL_3(\mathbb{F}_p)$, where $\mathbb{F}_p$ is a finite field containing p elements, p prime. Let N be the subgroup of B consisting of the upper triangular matrices having ones on the diagonal.
   a) Identify the set of one-dimensional representations of B.
   b) Suppose that π is an irreducible representation of B, having the property that $\pi(x)v \neq v$ for some $x \in N$ and $v \in V$. Show that $\pi\,|_N$ is a reducible representation of N. (*Hint*: One approach is to start by considering the action of the centre of N on V.)
4. Suppose that G is a finite group. Let $n \in \mathbb{N}$. Define $\theta_n : G \to \mathbb{N}$ by

$$\theta_n(g) = |\,\{\, h \in G \mid h^n = g \,\}\,|, \qquad g \in G.$$

Let χ_i, $1 \leq i \leq r$ be the distinct irreducible (complex) characters of G. Set

$$\nu_n(\chi_i) = |G|^{-1} \sum_{g \in G} \chi_i(g^n).$$

Prove that $\theta_n = \sum_{1 \leq i \leq r} \nu_n(\chi_i)\chi_i$.

5. Let (π, V) be an irreducible representation of a finite group G. Prove *Burnside's Theorem*:

$$\mathrm{Span}\{\, \pi(g) \mid g \in G \,\} = \mathrm{End}_{\mathbb{C}}(V).$$

(*Hint*: Of course the theorem is equivalent to $\mathrm{Span}\{\, [\pi(g)]_\beta \mid g \in G \,\} = M_{n\times n}(\mathbb{C})$, where β is a basis of V. This can be proved using properties of matrix coefficients of π (Theorem 3)).

6. Let (π, V) be a finite-dimensional representation of a finite group G. Let

$$\begin{aligned} W_{ext} &= \mathrm{Span}\{\, v \otimes v \mid v \in V \,\} \subset V \otimes V \\ W_{sym} &= \mathrm{Span}\{\, v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1,\, v_2 \in V \,\} \subset V \otimes V \end{aligned}.$$

   a) Prove that W_{ext} and W_{sym} are G-invariant subspaces of $V \otimes V$ (considered as the space of the inner tensor product representation $\pi \otimes \pi$ of G).
   b) Let $(\wedge^2\pi, \wedge^2 V)$ be the quotient representation, where $\wedge^2 V = (V \otimes V)/W_{ext}$. Then $\wedge^2\pi$ is called the exterior square of π. Compute the character $\chi_{\wedge^2\pi}$.
   c) Let $(\mathrm{Sym}^2\pi, \mathrm{Sym}^2 V)$ be the quotient representation, where $\mathrm{Sym}^2 V = (V \otimes V)/W_{sym}$. Then $\mathrm{Sym}^2\pi$ is called the symmetric square of π. Prove that (the inner tensor product) $\pi \otimes \pi$ is equivalent to $\wedge^2\pi \oplus \mathrm{Sym}^2\pi$.

7. Let (π, V) be the permutation representation associated to an action of a finite group G on a set X. Show that $\chi_\pi(g)$ is equal to the number of elements of X that are fixed by g.

8. Let f be a function from a finite group G to the complex numbers. For each finite dimensional representation (π, V) of G, define a linear operator $\pi(f) : V \to V$ by $\pi(f)v = \sum_{g \in G} f(g)\pi(g)v$, $v \in V$. Prove that $\pi(f) \in \mathrm{Hom}_G(\pi, \pi)$ for all finite-dimensional representations (π, V) of G if and only if f is a class function.

9. A (finite-dimensional) representation (π, V) of a finite group is called *faithful* if the homomorphism $\pi : G \to GL(V)$ is injective (one-to-one). Prove that every irreducible representation of G occurs as a subrepresentation of the set of representations

$$\{\, \pi, \pi \otimes \pi, \pi \otimes \pi \otimes \pi, \pi \otimes \pi \otimes \pi \otimes \pi, ... \,\}.$$

10. Show that the character of any irreducible representation of dimension greater than 1 takes the value 0 on some conjugacy class.

11. A finite-dimensional representation (π, V) of a finite group is *multiplicity-free* if each irreducible representation occurring in the decomposition of π into a direct sum of irreducibles occurs exactly once. Prove that π is multiplicity-free if and only if the ring $\mathrm{Hom}_G(\pi, \pi)$ is commutative.

2.2. Character values as algebraic integers, degree of an irreducible represnetation divides the order of the group

A complex number z is an *algebraic integer* if $f(z) = 0$ for some monic polynomial f having integer coefficients. The proof of the following lemma is found in many standard references in algebra.

Lemma.

(1) Let $z \in \mathbb{C}$. The following are equivalent:

(a) z is an algebraic integer

(b) z is algebraic over $\mathbb{Q}$ and the minimal polynomial of z over $\mathbb{Q}$ has integer coefficients.

(c) The subring $\mathbb{Z}[z]$ of $\mathbb{C}$ generated by $\mathbb{Z}$ and z is a finitely generated $\mathbb{Z}$-module.

(2) The algebraic integers form a ring. The only rational numbers that are algebraic integers are the elements of $\mathbb{Z}$.

Lemma. *Let π be a finite-dimensional representation of G and let $g \in G$. Then $\chi_\pi(g)$ is an algebraic integer.*

Proof. Because G is finite, we must have $g^k = 1$ for some positive integer k. Hence $\pi(g)^k = 1$. It follows that every eigenvalue of $\pi(g)$ is a kth root of unity. Clearly a kth root of unity is an algebraic integer. Since $\chi_\pi(g)$ is the sum of the eigenvalues of $\pi(g)$, it follows from part (2) of the above lemma that $\chi_\pi(g)$, being a sum of algebraic integers, is an algebraic integer. qed

Lemma. *Let $g_1, \ldots g_r$ be representatives of the conjugacy classes in G. Let c_j be the number of elements in the conjugacy class of g_j, $1 \le j \le r$. Define $f_i \in \mathcal{A}(G)$ by $f_i(g) = c_j\chi_i(g)/\chi_i(1)$, if g is conjugate to g_j. Then $f_i(g)$ is an algebraic integer for $1 \le i \le r$ and all $g \in G$.*

Proof. Let $g_0 \in G$. As g ranges over the elements in the conjugacy class of g_j, so does $g_0 g g_0^{-1}$. Therefore

$$\sum_{g \in \mathrm{cl}(g_j)} \pi_i(g) = \sum_{g \in \mathrm{cl}(g_j)} \pi_i(g_0)\pi_i(g)\pi_i(g_0)^{-1} = \pi_i(g_0)\left(\sum_{g \in \mathrm{cl}(g_j)} \pi_i(g)\right)\pi_i(g_0)^{-1}.$$

So $T := \sum_{g\in \mathrm{cl}(g_j)} \pi_i(g)$ belongs to $\mathrm{Hom}_G(\pi_i,\pi_i)$. By irreducibility of π_i, $T = zI$ for some $z \in \mathbb{C}$. Note that

$$\mathrm{tr}\, T = \sum_{g\in \mathrm{cl}(g_j)} \chi_i(g) = c_j\chi_i(g_j) = z\chi_i(1),$$

so $z = c_j\chi_i(g_j)/\chi_i(1)$.

Let g be an element of $\mathrm{cl}(g_s)$. Let a_{ijs} be the number of ordered pairs $(g', \hat{g})$ such that $g'\hat{g} = g$. Note that a_{ijs} is independent of the choice of $g \in \mathrm{cl}(g_s)$.

$$\begin{aligned}
(c_i\chi_t(g_i)/\chi_t(1))\,(c_j\chi_t(g_j)/\chi_t(1))\,I &= \left(\sum_{g'\in \mathrm{cl}(g_i)} \pi_t(g')\right)\left(\sum_{\hat{g}\in \mathrm{cl}(g_j)} \pi_t(\hat{g})\right) \\
&= \sum_{g'\in \mathrm{cl}(g_i)} \sum_{\hat{g}\in \mathrm{cl}(g_j)} \pi_t(g'\hat{g}) = \sum_{s=1}^{r} \sum_{g\in \mathrm{cl}(g_s)} a_{ijs}\pi_t(g) \\
&= \left(\sum_{s=1}^{r} a_{ijs}c_s\chi_t(g_s)/\chi_t(1)\right) I
\end{aligned}$$

Hence

$$(c_i\chi_t(g_i)/\chi_t(1))\,(c_j\chi_t(g_j)/\chi_t(1)) = \sum_{s=1}^{r} a_{ijs}c_s\chi_t(g_s)/\chi_t(1).$$

This implies that the subring of $\mathbb{C}$ generated by the scalars $c_s\chi_t(g_s)/\chi_t(1)$, $1 \le s \le r$, and $\mathbb{Z}$ is a finitely-generated $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a principal ideal domain, any submodule of a finitely-generated $\mathbb{Z}$-module is also a finitely-generated $\mathbb{Z}$-module, the submodule $\mathbb{Z}[c_i\chi_t(g_i)/\chi_t(1)]$ is finitely-generated. Applying part (2) of one of the above lemmas, the result of this lemma follows. qed

Theorem 9. *n_j divides $|G|$, $1 \le j \le r$.*

Proof. Note that

$$\begin{aligned}
|G|/\chi_i(1) &= |G|\langle \chi_i, \chi_i\rangle/\chi_i(1) \\
&= \sum_{j=1}^{r} c_j\chi_i(g_j)\overline{\chi_i(g_j)}/\chi_i(1) = \sum_{j=1}^{r} (c_j\chi_i(g_j)/\chi_i(1))\,\overline{\chi_i(g_j)}.
\end{aligned}$$

Because the right side above is an algebraic integer, the left side is a rational number which is also an algebraic integer, hence it is an integer. qed

2.3. Decomposition of finite-dimensional representations

In this section we describe how to decompose a representation π into a direct sum of irreducible representations, assuming that the functions $a^j_{m\ell}$ are known.

Lemma. *Let (π, V) be a finite-dimensional representation of G. For $1 \le k \le r$, $1 \le j, \ell \le n_k$, define $P^k_{j\ell} : V \to V$ by*

$$P^k_{j\ell} = n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{j\ell}(g)} \pi(g).$$

Then
(1) $\pi(g) P^k_{j\ell} = \sum_{\nu=1}^{n_k} a^k_{\nu j}(g) P^k_{\nu\ell}$ *and* $P^k_{j\ell}\pi(g) = \sum_{\nu=1}^{n_k} a_{\ell\nu}(g) P^k_{j\nu}, \quad g \in G.$
(2) $P^k_{j\ell} P^{k'}_{\mu\nu} = P^k_{j\nu}$ *if* $k = k'$ *and* $\ell = \mu$, *and equals* 0 *otherwise.*
(3) $(P^k_{j\ell})^* = P^k_{\ell j}$.

Proof. For the first part of (1),

$$\begin{aligned} \pi(g) P^k_{j\ell} &= |G|^{-1} n_k \sum_{g' \in G} \overline{a^k_{j\ell}(g')} \pi(gg') = |G|^{-1} n_k \sum_{g' \in G} \overline{a^k_{j\ell}(g^{-1}g')} \pi(g') \\ &= |G|^{-1} n_k \sum_{g' \in G} \sum_{\nu=1}^{n_k} \overline{a^k_{j\nu}(g^{-1}) a^k_{\nu\ell}(g')} \pi(g') = \sum_{\nu=1}^{n_k} a^k_{\nu j}(g) P^k_{\nu\ell} \end{aligned}$$

The second part of (1) is proved similarly.

For (2),

$$\begin{aligned} P^k_{j\ell} P^{k'}_{\mu\nu} &= n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{j\ell}(g)} \pi(g) P^{k'}_{\mu\nu} = n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{j\ell}(g)} \sum_{t=1}^{n_k} a^{k'}_{t\mu}(g) P^{k'}_{t\nu} \\ &= n_k \sum_{t=1}^{n_k} \langle a^{k'}_{t\mu}, a^k_{j\ell} \rangle P^{k'}_{t\nu} = \delta_{kk'} \delta_{\ell\mu} P^k_{j\nu}. \end{aligned}$$

For (3),

$$(P^k_{j\ell})^* = n_k |G|^{-1} \sum_{g \in G} a^k_{j\ell}(g) \pi(g^{-1}) = n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{\ell j}(g^{-1})} \pi(g^{-1}) = P^k_{\ell j}.$$

Set $V^k_j = P^k_{jj}(V)$. Note that , since $(P^k_{jj})^* = P^k_{jj} = (P^k_{jj})^2$, P^k_{jj} is the orthogonal projection of V on V^k_j. From property (2) of the above lemma, it follows that $V^k_j \perp V^{k'}_{j'}$ if $k \ne k'$ or $j \ne j'$.

Let $W = \oplus_{k=1}^r \oplus_{j=1}^{n_k} V^k_j$. Note that $P^k_{j\ell}(V) = P^k_{jj} P^k_{j\ell}(V) \subset V^k_j \subset W$. Fix $v_0 \in W^\perp$. Then

$$0 = \langle v_0, P^k_{j\ell}(v) \rangle = n_k |G|^{-1} \sum_{g \in G} a^k_{j\ell}(g) \langle v_0, \pi(g) v \rangle_V = n_k \langle a^k_{j\ell}, f \rangle_{\mathcal{A}(G)},$$

where $f(g) = \langle v_0, \pi(g) v \rangle$, $g \in G$. It follows that $f \in \mathcal{A}(G)^\perp$. Hence $f(g) = 0$ for all $g \in G$. Setting $v = v_0$ and $g = 1$, we have $\langle v_0, v_0 \rangle_V = 0$. Thus $v_0 = 0$. That is, $W = V$.

Next, note that $P^k_{j\ell}V^{k'}_t = 0$ if $k \neq k'$ or $t \neq \ell$, by part (2) of the above lemma.

Let $v \in V^k_\ell$. Then $v = P^k_{\ell\ell}(v')$ for some $v' \in V$. Now $v = P^k_{\ell\ell}(v') = P^k_{\ell\ell}(P^k_{\ell\ell}(v')) = P^k_{\ell\ell}(v)$, so we have

$$P^k_{j\ell}(v) = P^k_{j\ell}P^k_{\ell\ell}(v) = P^k_{jj}P^k_{j\ell}(v) \subset V^k_j.$$

Thus $P^k_{j\ell}(V^k_\ell) \subset V^k_j$. Now

$$V^k_\ell = P^k_{\ell\ell}V^k_\ell = P^k_{\ell j}P^k_{j\ell}V^k_\ell \subset P^k_{\ell j}V^k_j \subset V^k_\ell.$$

Hence we have $P^k_{\ell j}V^k_j = V^k_\ell$. Let v, $v' \in V^k_\ell$. Then

$$\langle P^k_{j\ell}(v), P^k_{j\ell}(v')\rangle = \langle (P^k_{j\ell})^* P^k_{j\ell}(v), v'\rangle = \langle P^k_{\ell j}P^k_{j\ell}(v), v'\rangle = \langle P^k_{\ell,\ell}(v), v'\rangle = \langle v, v'\rangle.$$

We have shown

Lemma. *$P^k_{j\ell}$ is an isometry of V^k_ℓ onto V^k_j.*

Choose an orthonormal basis $e^k_{11}, e^k_{21}, \ldots, e^k_{r_k,1}$ of $V^k_1 = P^k_{11}(V)$. Then $e^k_{j\ell} := P^k_{\ell 1}(e^k_{j1})$, $1 \leq j \leq r_k$ is an orthonormal basis of V^k_ℓ. It follows that the set

$$\{\, e^k_{j\ell} \mid 1 \leq j \leq r_k,\, 1 \leq \ell \leq n_k,\, 1 \leq k \leq r \,\}$$

is an orthonormal basis of V. Set $Y^k_j = \mathrm{Span}\{\, e^k_{j\ell} \mid 1 \leq \ell \leq n_k \,\}$. If $g \in G$, then

$$\pi(g)e^k_{j\ell} = \pi(g)P^k_{\ell 1}(e^k_{j1}) = \sum_{\nu=1}^{n_k} a^k_{\nu\ell}(g)P^k_{\nu 1}e^k_{j1} = \sum_{\nu=1}^{n_k} a^k_{\nu\ell}(g)e^k_{j\nu}.$$

This shows that Y^k_j is G-invariant and has the matrix $[a^k_{\nu\ell}(g)]_{\{1\leq\nu,\ell\leq n_k\}}$ relative to the given orthonormal basis of Y^k_j. This implies that $\pi\,|_{Y^k_j} \simeq \pi_k$, $1 \leq j \leq r_k$. Now $V = \oplus_{1\leq k\leq r} \oplus_{1\leq j\leq r_k} Y^k_j$, so we have decomposed π into a direct sum of irreducible representations. This decomposition is not unique.

Set $Y^k = \oplus_{j=1}^{r_k} Y^k_j$. Now $\{\, e^k_{j\ell} \mid 1 \leq j \leq r_k,\, 1 \leq \ell \leq n_k \,\}$ is an orthonormal basis of Y^k, and $\pi\,|_{Y^k} \simeq r_k\pi_k$. Because $P^k_{\ell\ell}$ is the orthogonal projection of V on V^k_ℓ and $Y^k = \oplus_{\ell=1}^{n_k} V^k_\ell$, it follows that $P^k := \sum_{\ell=1}^{n_k} P^k_{\ell\ell}$ is the orthogonal projection of V on Y^k. Looking at the definitions, we see that this orthogonal projection P^k is defined by

$$P^k = \sum_{\ell=1}^{n_k} n_k|G|^{-1} \sum_{g\in G} \overline{a^k_{\ell\ell}(g)}\pi(g) = n_k|G|^{-1} \sum_{g\in G} \overline{\chi_k(g)}\pi(g).$$

Suppose that W is a G-invariant subspace of V such that $\pi\,|_W$ is equivalent to a direct sum of π_k with itself some number of times. Let $\{\, v_1, \ldots, v_{n_k} \,\}$ be an orthonormal basis of an irreducible G-invariant subspace of W, chosen so that the matrix of the restriction of $\pi(g)$ to this subspace is $[a^k_{\ell m}(g)]$. Then

$$\begin{aligned} P^k(v_j) &= n_k|G|^{-1}\sum_{g\in G}\overline{\chi_k(g)}\pi(g)v_j = n_k|G|^{-1}\sum_{g\in G}\sum_{\ell=1}^{n_k}a^k_{\ell\ell}(g)\sum_{\mu=1}^{n_k}a^k_{\mu j}(g)v_\mu \\ &= n_k|G|^{-1}\sum_{g\in G}\sum_{\ell,\mu=1}^{n_k}\overline{a^k_{\ell\ell}(g)}a^k_{\mu j}(g)v_\mu = n_k\sum_{\ell,\mu=1}^{n_k}\langle a^k_{\mu j}, a^k_{\ell\ell}\rangle v_\mu \\ &= n_k n_k^{-1}v_j = v_j \end{aligned}$$

Therefore $P^k\,|\,W$ is the identity. Because P^k is the orthogonal projection of V on Y^k, we know that $P^k(v) = v$ if and only if $v \in Y^k$. It follows that $W \subset Y^k$. Now we may conclude that if we have a G-invariant subspace of V such that the restriction of π to that subspace is equivalent to $r_k\pi_k$, then that subspace must equal Y^k.

Lemma. *The subspaces Y^k are unique.*

The subspace Y^k is called the π_k-*isotypic* subspace of V. It is the (unique) largest subspace of V on which the restriction of π is a direct sum of representations equivalent to π_k. Of course, we will have $Y^k = \{0\}$ if no irreducible constituent of π is equivalent to π_k.

2.4. Induced representations

One method of producing representations of a finite group G is the process of induction: given a representation of a subgroup of G, we can define a related representation of G. Let (π, V) be a (finite-dimensional) representation of a subgroup H of G. Define

$$\mathcal{V} = \{\, f : G \to V \mid f(hg) = \pi(h)f(g),\ h \in H,\ g \in G\,\}.$$

We define the induced representation $i^G_H\pi = \mathrm{Ind}^G_H(\pi)$ by $(i^G_H\pi(g)f)(g_0) = f(g_0g)$, g, $g_0 \in G$. Observe that if $h \in H$, then

$$(i^G_H\pi(g)f)(hg_0) = f(hg_0g) = \pi(h)f(g_0g) = \pi(h)(i^G_H\pi(g)f)(g_0).$$

It follows from the definitions that the degree of $i^G_H\pi$ equals $|G||H|^{-1}$ times the degree of π. Let $\langle\cdot,\cdot\rangle_V$ be any inner product on V. Set $\langle f_1, f_2\rangle_{\mathcal{V}} = |G|^{-1}\sum_{g\in G}\langle f_1(g), f_2(g)\rangle_V$, f_1, $f_2 \in \mathcal{V}$. It is easy to check that this defines an inner product on $\mathcal{V}$ with respect to which $i^G_H\pi$ is unitary.

Example If $H = \{1\}$ and π is the trivial representation of H, then $i^G_H\pi$ is the right regular representation of G.

The Frobenius character formula expresses the character of $i^G_H\pi$ in terms of the character of π.

Theorem 10 (Frobenius character formula). *Let (π, V) be a representation of a subgroup H of G. Fix $g \in G$. Let $h_1, \dots, h_m$ be representatives for the conjugacy classes in H which lie inside the conjugacy class of g in G. Then*

$$\chi_{i_H^G \pi}(g) = |G||H|^{-1} \sum_{i=1}^{m} |cl_H(h_i)||cl_G(g)|^{-1} \chi_\pi(h_i).$$

Proof. Let $g \in G$. Define $T : \mathcal{V} \to \mathcal{V}$ by $T = \sum_{g' \in \mathrm{cl}(g)} i_H^G \pi(g')$. Note that $\mathrm{tr}\, T = |cl(g)|\chi_{i_H^G \pi}(g)$.

Let $\beta = \{ v_1, \dots, v_n \}$ be an orthonormal basis of V such that each $[\pi(h)]_\beta$, $h \in H$, is a unitary matrix. For each $j \in \{1, \dots, n\}$, define

$$f_j(g) = \begin{cases} |G|^{1/2}|H|^{-1/2}\pi(h)v_j, & \text{if } g = h \in H \\ 0, & \text{if } g \notin H. \end{cases}$$

Then $f_j(h_0 g) = |G|^{1/2}|H|^{-1/2}\pi(h_0 h)v = \pi(h_0)f_j(h)v$, if $g = h \in H$, and $f_j(h_0 g) = f_j(g) = 0$ if $g \notin H$. Thus $f_j \in \mathcal{V}$. Note that

$$\begin{aligned} \langle f_j, f_k \rangle_{\mathcal{V}} &= |G|^{-1} \sum_{g \in G} \langle f_j(g), f_k(g) \rangle_V = |H|^{-1} \sum_{h \in H} \langle \pi(h)v_j, \pi(h)v_k \rangle_V \\ &= |H|^{-1} \sum_{h \in H} \langle v_j, v_k \rangle_V = \langle v_j, v_k \rangle_V = \delta_{jk}. \end{aligned}$$

Therefore $\{ f_1, \dots, f_n \}$ is an orthonormal set in $\mathcal{V}$. Pick representatives $g_1, \dots, g_\ell$ of the cosets in $H \backslash G$ (that is, of the right H cosets in G). Then

$$\begin{aligned} \langle i_H^G \pi(g_i) f_j, i_H^G \pi(g_k)) f_s \rangle_{\mathcal{V}} &= |G|^{-1} \sum_{g \in G} \langle f_j(g g_i), f_s(g g_k) \rangle_V = |G|^{-1} \sum_{h \in H} \langle f_j(h), f_s(h g_i^{-1} g_k) \rangle_V \\ &= \delta_{ik}|G|^{-1} \sum_{h \in H} \langle \pi(h) f_j(1), \pi(h) f_s(1) \rangle_V \delta_{ik} \langle f_j, f_s \rangle_V = \delta_{ik}\delta_{js} \end{aligned}$$

Hence $\{ i_H^G \pi(g_i) f_j \}_{1 \le i \le \ell,\, 1 \le j \le n}$ is an orthonormal set. Since $\dim \mathcal{V} = |G||H|^{-1} \dim V = \ell n$, it is an orthonormal basis of $\mathcal{V}$ (with respect to which $i_H^G \pi$ is a unitary representation).

The kkth entry of T with respect to this basis is

$$\langle T(k\text{th basis element}), k\text{th basis element} \rangle_{\mathcal{V}}.$$

Therefore

$$\mathrm{tr}\, T = \sum_{i=1}^{\ell} \sum_{j=1}^{n} \langle T(i_H^G \pi(g_i) f_j), i_H^G \pi(g_i) f_j \rangle_{\mathcal{V}}.$$

As g' ranges over the conjugacy class $\mathrm{cl}(g)$, $g_i g' g_i^{-1}$ also ranges over $\mathrm{cl}(g)$. Hence

$$(i_H^G \pi)(g_i) T \, i_H^G \pi(g_i^{-1}) = T,$$

and the above expression for $\operatorname{tr} T$ becomes

$$\sum_{i=1}^{\ell}\sum_{j=1}^{n}\langle (i_H^G\pi)(g_i)Tf_j, (i_H^G\pi)(g_0)f_j\rangle_{\mathcal{V}} = \ell\sum_{j=1}^{n}\langle Tf_j, f_j\rangle_{\mathcal{V}}.$$

Now we rewrite each $\langle Tf_j, f_j\rangle_{\mathcal{V}}$ using the definitions of f_j and T.

$$\begin{aligned}
\langle Tf_j, f_j\rangle_{\mathcal{V}} &= \sum_{g'\in \mathrm{cl}(g)}|G|^{-1}\sum_{g_0\in G}\langle f_j(g_0g'), f_j(g_0)\rangle_V = \sum_{g'\in\mathrm{cl}(g)\cap H}|G|^{-1}\sum_{h\in H}\langle f_j(hg'), f_j(h)\rangle_V\\
&= \sum_{s=1}^{m}|\mathrm{cl}_H(h_s)||G|^{-1}\sum_{h\in H}\langle f_j(hh_s), f_j(h)\rangle_V\\
&= |H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\sum_{h\in H}\langle \pi(h)\pi(h_s)v_j, \pi(h)v_j\rangle_V = \sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\langle\pi(h_s)v_j, v_j\rangle_V
\end{aligned}$$

It follows that

$$\operatorname{tr} T = |G||H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\sum_{j=1}^{n}\langle\pi(h_s)v_j, v_j\rangle_V = |G||H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\chi_\pi(h_s).$$

Thus $\chi_{i_H^G\pi}(g) = |G||H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_G(g)|^{-1}|\mathrm{cl}_H(h_s)|\chi_\pi(h_s)$. qed

Example: Applying the Frobenius character formula with π the trivial representation of the trivial subgroup of G, we see that the character of the regular representation of G vanishes on all elements except for the identity element.

The inner product on $\mathcal{A}(G)$ restricts to an inner product on the space $\mathcal{C}(G)$ of class functions on G. When we wish to identify the fact that we are taking the inner product on $\mathcal{A}(G)$, we will sometimes write $\langle\cdot,\cdot\rangle_G$. Let H be a subgroup of G. We may view i_H^G as a map from $\mathcal{C}(H)$ to $\mathcal{C}(G)$, mapping χ_π to $i_H^G(\chi_\pi) := \chi_{i_H^G\pi}$, for π any irreducible representation of H. As the characters of the irreducible representations of H form a basis of $\mathcal{C}(H)$, the map extends by linearity to all of $\mathcal{C}(H)$. We can define a linear map r_G^H from $\mathcal{C}(G)$ to $\mathcal{C}(H)$ by restricting a class function on G to H. The next result, Frobenius Reciprocity, tells us that r_G^H is the adjoint of the map i_H^G.

If (π, V) is a representation of G and τ is an irreducible representation of G, the *multiplicity* of τ in π is defined to be the number of times that τ occurs in the decomposition of π as a direct sum of irreducible representations of G. This multiplicity is equal to $\langle\chi_\tau, \chi_\pi\rangle_G = \langle\chi_\pi, \chi_\tau\rangle_G$.

Theorem 11 (Frobenius Reciprocity). *Let (π, V) be an irreducible representation of H and let (τ, W) be an irreducible representation of G. Then $\langle\chi_\tau, \chi_{i_H^G\pi}\rangle_G = \langle r_G^H\chi_\tau, \chi_\pi\rangle_H$.*

Proof. Let π and τ be as in the statement of the theorem. Let $g \in G$ be such that $\mathrm{cl}_G(g) \cap H \neq \emptyset$. Choose $h_1, \ldots, h_m$ as in the previous theorem. Then, using $\chi_\tau(g) = \chi_\tau(h_i)$, $1 \leq i \leq m$,

$$\begin{aligned}\chi_{i_H^G \pi}(g)\overline{\chi_\tau(g)} &= \left(|G||H|^{-1} \sum_{i=1}^{m} |\mathrm{cl}_H(h_i)||\mathrm{cl}_G(h_i)|^{-1} \chi_\pi(h_i)\right) \overline{\chi_\tau(g)} \\ &= |G||H|^{-1} \sum_{i=1}^{m} |\mathrm{cl}_H(h_i)||\mathrm{cl}_G(h_i)|^{-1} \chi_\pi(h_i)\overline{\chi_\tau(h_i)}\end{aligned}$$

Now when evaluating $\langle \chi_{i_H^G \pi}, \chi_\tau \rangle_G = |G|^{-1} \sum_{g \in G} \chi_{i_H^G \pi}(g)\overline{\chi_\tau(g)}$, we need only sum over those $g \in G$ such that $\mathrm{cl}(g) \cap H \neq \emptyset$. Then

$$\langle \chi_{i_H^G \pi}, \chi_\tau \rangle_G = |H|^{-1} \sum_{h \in H} \chi_\pi(h)\overline{\chi_\tau(h)} = \langle \chi_\pi, r_{G^H}(\chi_\tau) \rangle_H.$$

qed

Corollary (Transitivity of induction). *Suppose that $K \subset H$ are subgroups of G. Let (π, V) be a representation of K. Then $i_K^G \pi = i_H^G(i_K^H \pi)$.*

Proof. Note that it follows from the definitions that $r_G^K = r_H^K \circ r_G^H$. Taking adjoints, we have

$$i_K^G = (r_G^K)^* = (r_G^H)^* \circ (r_H^K)^* = i_H^G \circ i_K^H.$$

qed

Lemma. *Let (π, V) be a representation of a subgroup H of G. Fix $g \in G$. Let π' be the representation of gHg^{-1} defined by $\pi'(ghg^{-1}) = \pi(h)$, $h \in H$. Then $i_H^G \pi \simeq i_{gHg^{-1}}^G \pi'$.*

Proof. Let f be in the space of $i_H^G \pi$. Set $(Af)(g_0) = f(g^{-1}g_0)$, $g_0 \in G$. Let $h \in H$. Then

$$(Af)(ghg^{-1}g_0) = f(hg^{-1}g_0) = \pi(h)(Af)(g_0) = \pi'(ghg^{-1})(Af)(g_0).$$

Therefore Af belongs to the space of $i_{gHg^{-1}}^G \pi'$. It is clear that A is invertible. Note that

$$i_{gHg^{-1}}(g_1)(Af)(g_0) = (Af)(g_0 g_1) = f(g^{-1}g_0 g_1) = (A\, i_H^G \pi(g_1) f)(g_0).$$

qed

Let (π, V) be a finite-dimensional representation of a subgroup H of G. Let K be a subgroup of G, and let $g \in G$. Then $K \cap gHg^{-1}$ is a subgroup of G. Define a representation π^g of this subgroup by $\pi^g(k) = \pi(g^{-1}kg)$, $k \in K \cap gHg^{-1}$. Let $h \in H$. Then $K \cap ghH(gh)^{-1} = K \cap gHg^{-1}$ and $\pi^{gh}(k) = \pi(h^{-1}g^{-1}kgh) = \pi(h)^{-1}\pi^g(k)\pi(h)$. Hence $\pi^{gh} \simeq \pi^g$. This certainly implies that $i_{K \cap ghHhg^{-1}}^K \pi^{gh} \simeq i_{K \cap gHg^{-1}}^K \pi^g$.

Changing notation slightly, we see that the above lemma tells us that $i_{K \cap gHg^{-1}}^K \pi^g \simeq i_{K \cap kgHg^{-1}k^{-1}} \pi^{kg}$. We now know that the equivalence class of $i_{K \cap gHg^{-1}}^K \pi^g$ is independent of the choice of element g inside its K-H-double coset (that is, we may replace g by kgh, $k \in K$, $h \in H$, without changing the equivalence class).

Theorem 12. *(Mackey) K and H be subgroups of G, and let (π, V) be a representation of H. Then*

$$(i_H^G\pi)_K = r_G^K(i_H^G\pi) \simeq \bigoplus_{g\in K\backslash G/H} i_{K\cap gHg^{-1}}^K(\pi^g).$$

Proof. Let $\rho = i_H^G\pi$. Let $\mathcal{V}$ be the space of ρ. Define a map $A : V \to \mathcal{V}$ by $(Av)(g) = \begin{cases} \pi(h)v, & \text{if } g = h \in H \\ 0, & \text{if } g \notin H. \end{cases}$, $v \in V$. Let $v_1 \in V$, and g_1, $g_2 \in G$. Then $\rho(g_1)Av_1 \in \rho(g_2)AV$ if and only if $\rho(g_2^{-1}g_1)Av_1 = Av_2$ for some $v_2 \in V$. Now $\rho(g_2^{-1}g_1)Av_1$ is supported in $Hg_1^{-1}g_2$ and Av_2 is supported in H. Hence the two functions are equal if and only if $g_1H = g_2H$. It follows that $\sum_{g\in G/H}\rho(g)AV = \oplus_{g\in G/H}\rho(g)AV$. Now $\rho(g)$ is invertible and A is one-to-one, so $\dim\rho(g)AV = \dim V$. Therefore the dimension of the latter direct sum equals $|G||H|^{-1}\dim V = \dim\mathcal{V}$. Thus $\mathcal{V} = \oplus_{g\in G/H}\rho(g)AV$.

Now we want to study $\mathcal{V}$ as a K-space and $\rho(g)AV$ is not K-stable. Given $g \in G$, the double coset KgH is a disjoint union of certain cosets $g'H$. So we group together those $\rho(g')AV$ such that $g'H \subset KgH$. Let $X(g) = \sum_{g'H\subset KgH}\rho(g')AV$. It should be understood that the above sum is taken over a set of representatives g' of the left H-cosets which lie in KgH. Now we have regrouped things and we have $\mathcal{V} = \oplus_{g\in K\backslash G/H}X(g)$. Now $\rho(k)X(g) = X(g)$ for all $k \in K$. We will prove that

$$(**)\qquad \rho_K\,|_{X(g)} = r_G^K\rho\,|_{X(g)} \simeq i_{K\cap gHg^{-1}}^K\pi^g.$$

The theorem is a consequence of (**) and the above direct sum decomposition of $\mathcal{V}$.

Now suppose that $g' = kgh$, $k \in K$, $h \in H$. Then $\rho(g')AV = \rho(kg)AV$. Let $k_0 \in K \cap gHg^{-1}$.

$$\rho(k_0g')AV = \rho(k_0g)AV = \rho(g(g^{-1}k_0g))AV = \rho(g)AV$$

. This implies that $X(g) = \sum_{K/(K\cap gHg^{-1})}\rho(k)\rho(g)AV$. Now we can easily check that if $k \in K$, then $\rho(k)\rho(g)AV = \rho(g)AV$ if and only if $\rho(g^{-1}kg)AV = AV$ if and only if $g^{-1}kg \in H$, that is $k \in K \cap gHg^{-1}$. So $X(g) = \oplus_{K/(K\cap gHg^{-1})}\rho(k)\rho(g)AV$.

Let $\mathcal{W}$ be the space of $i_{K\cap gHg^{-1}}^K\pi^g$. Now define $B : X(G) \to \mathcal{W}$ as follows. Let $v \in V$ and $k \in K$. Set $\varphi_v(k) = \pi^g(k)v$ if $k \in K\cap gHg^{-1}$ and $\varphi_v(k) = 0$ otherwise. Then $\varphi_v \in \mathcal{W}$. Given $v \in V$ and $k \in K$, set $B\rho(k)\rho(g)Av = i_{K\cap gHg^{-1}}^K\pi^g(k)\varphi_v$. It is a simple matter to check that B is invertible. Since $r_G^K\rho$ acts by right translation on $X(g)$ and $i_{K\cap gHg^{-1}}^K\pi^g$ acts by right translation on $\mathcal{W}$, we see that B intertwines these representations. Hence (**) holds. qed

Theorem 13. *Let H and K be subgroups of a finite group G. Let (π, V) and (ρ, W) be (finite-dimensional) representations of H and K, respectively. Then $\mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$ is isomorphic to*

$$\{\,\varphi : G \to \mathrm{End}_{\mathbb{C}}(V, W)\ |\varphi(kgh) = \rho(k)\circ\varphi(g)\circ\pi(h),\ k \in K,\, g \in G,\, h \in H\,\}.$$

Sketch of proof. Let $A \in \mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$. Define $\varphi_A : G \to \mathrm{End}_{\mathbb{C}}(V,W)$ by $\varphi_A(g)v = (Af_v)(g)$, $v \in V$, where $f_v(h) = \pi(h)v$, $h \in H$, and $f_v(g) = 0$ if $g \notin H$. Note that f_v is in the space of $i_H^G\pi$. Let $k \in K$, $g \in G$, $h \in H$, and $v \in V$. Then

$$\begin{aligned}\varphi_A(kgh)v &= (Af_v)(kgh) = \rho(k)(Af_v)(gh) = \rho(k)(\rho(h)Af_v)(g)\\ &= \rho(k)(Af_{\pi(h)v})(g) = \rho(k)\varphi_A(g)\pi(h)v\end{aligned}$$

where the second equality holds because Af_v is in the space of $i_K^G\rho$, the fourth equality holds because $A \in \mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$, and the fifth because $\pi(h)f_v = f_{\pi(h)v}$ for all $h \in H$. Hence φ_A has the desired properties relative to left translation by elements of K and right translation by elements of H.

Given $\varphi : G \to \mathrm{End}_{\mathbb{C}}(V,W) \mid \varphi(kgh) = \rho(k) \circ \varphi(g) \circ \pi(h)$, $g \in G$, $h \in H$, $k \in K$. Let f be in the space of $i_H^G\pi$. Define $A_\varphi f$ in the space of $i_K^G\rho$ by $(A_\varphi f)(g) = \sum_{g_0} \varphi(gg_0^{-1})f(g_0)$, where in the sum g_0 runs over a set of coset representatives for $K\backslash G$. Suppose that $k \in K$ and $g \in G$. Then

$$(A_\varphi f)(kg) = \sum_{g_0} \varphi(kgg_0^{-1})f(g_0) = \rho(k)\sum_{g_0} \varphi(gg_0^{-1})f(g_0) = \rho(k)(A\varphi f)(g).$$

Hence $A_\varphi f$ belongs to the space of $i_K^G\rho$.

Next, let g, $g_1 \in G$. Then

$$\begin{aligned}(A_\varphi i_H^G\pi(g_1)f)(g) &= \sum_{g_0} \varphi(gg_0^{-1})(i_H^G\pi(g_1)f)(g_0) = \sum_{g_0} \varphi(gg_0^{-1})f(g_0g_1)\\ &= \sum_{g_0} \varphi(gg_1g_0^{-1})f(g_0) = (A_\varphi f)(gg_1) = (i_K^G\rho(g_1)A_\varphi f)(g)\end{aligned}$$

Therefore $A_\varphi \in \mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$.

To finish the proof, check that $A \mapsto \varphi_A$ and $\varphi \mapsto A_\varphi$ are inverses of each other. The details are left as an exercise. qed

Corollary. *Let π be an irreducible representation of a subroup H of G. Then $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ is isomorphic to*

$$\mathcal{H}(G,\pi) := \{\, \varphi : G \to \mathrm{End}_{\mathbb{C}}(V) \mid \varphi(hgh') = \pi(h) \circ \varphi(g) \circ \pi(h'),\ g \in G,\ h, h' \in H \,\}.$$

Lemma. *The subspace of $\mathcal{H}(G,\pi)$ consisting of functions supported on the double coset HgH is isomorphic to $\mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi)$. where $H^g = H \cap gHg^{-1}$ and $\pi^g(h) = \pi(g^{-1}hg)$, $h \in H^g$.*

Proof. Fix $g \in G$. Given $\varphi \in \mathcal{H}(G,\pi)$ such that φ is supported on HgH, define a linear operator $B_\varphi : V \to V$ by $B_\varphi(v) = \varphi(g)v$, $v \in V$. Then, if $h \in H^g$, we have, using the fact that $g^{-1}hg \in H$ and $h \in H$, and properties of φ,

$$B_\varphi(\pi^g(h)v) = \varphi(g)(\pi(g^{-1}hg)v) = \varphi(hg)v = \pi(h)\varphi(g)v = \pi(h)B_\varphi(v).$$

Hence $B_\varphi \in \mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi)$.

Given $B \in \mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi)$, set $\varphi_B(h_1gh_2)v = \pi(h_1)B\pi(h_2)v$, for h_1, $h_2 \in H$ and $v \in V$, and $\varphi_B(g_1)v = 0$ if $g_1 \notin HgH$. Check that $\varphi_B \in \mathcal{H}(G,\pi)$, and also that the map $B \mapsto \varphi_B$ is the inverse of the map $\varphi \mapsto B_\varphi$. The details are left as an exercise. qed

Corollary. *Let (π, V) be a representation of a subgroup H of G. If $g \in G$, let $H^g = H \cap gHg^{-1}$ and set $\pi^g(h) = \pi(g^{-1}hg)$, $g \in H^g$. Then*

$$\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi) \simeq \bigoplus_{g \in H\backslash G/H} \mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi).$$

Corollary(Mackey irreducibility criterion). *Let (π, V) be an irreducible representation of a subgroup H of G. Then $i_H^G\pi$ is irreducible if and only if $\mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi) = 0$ for all $g \notin H$.*

Proof. Note that if $g = h \in H$, then $H^h = H$ and $r_H^{H^h}\pi = \pi$. Hence, by irreducibility of π, $\mathrm{Hom}_H(\pi, \pi^h) \simeq \mathbb{C}$. Therefore, since $i_H^G\pi$ is irreducible if and only if $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi) \simeq \mathbb{C}$, by the above proposition, $i_H^G\pi$ is irreducible if and only if $\mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi) = 0$ whenever $g \notin H$. qed

Corollary. *If π is the trivial representation of a subgroup H of G, then $\dim \mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ equals the number of H-H-double cosets in G.*

Note that π^g and $r_H^{H^g}\pi$ are both the trivial representation of H^g (for any $g \in G$). Acccording to the above proposition, there is a one-dimensional contribution to $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ for each double coset HgH. qed

Given φ_1, $\varphi_2 \in \mathcal{H}(G,\pi)$, set

$$(\varphi_1 * \varphi_2)(g) = |G|^{-1} \sum_{g_0 \in G} \varphi_1(g_0) \circ \varphi_2(g_0^{-1}g).$$

This product makes $\mathcal{H}(G,\pi)$ into an algebra, known as a *Hecke algebra*.

Proposition. *The algebra $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ is isomorphic to the Hecke algebra $\mathcal{H}(G,\pi)$.*

To prove the proposition, check that the vector space $A \mapsto \varphi_A$ of the the Theorem 13 is an algebra homomorphism: $\varphi_{A_1 \circ A_2} = \varphi_{A_1} * \varphi_{A_2}$. The details are left as an exercise.

Exercises:

1. Adapt the above arguments to prove: Let H and K be subgroups of G, let π be a representation of H, and let τ be a representation of K. Suppose that $i_H^G\pi$ and $i_K^G\tau$ are irreducible. Show that $i_H^G\pi \not\simeq i_K^G\tau$ if and only if for every $g \in G$

$$\mathrm{Hom}_{gHg^{-1}\cap K}(\pi^g, r_K^{gHg^{-1}\cap K}\tau) = 0.$$

2. Let A be an abelian subgroup of a group G. Show that each irreducible representation of G has degree at most $|G||A|^{-1}$.

3. Let π_j be a representation of a subgroup H_j of a group G_j, $j = 1, 2$. Prove that

$$i_{H_1 \times H_2}^{G_1 \times G_2} \pi_1 \otimes \pi_2 \simeq i_{H_1}^{G_1} \pi_1 \otimes i_{H_2}^{G_2} \pi_2.$$

4. Let π and ρ be representations of subgroups H and K of a finite group G. Let $g_1,\ g_2 \in G$. Define representations π^{g_1} and ρ^{g_2} of the group $g_1^{-1} H g_1 \cap g_2^{-1} K g_2$ by $\pi^{g_1}(x) = \pi(g_1 x g_1^{-1})$ and $\rho^{g_2}(x) = \rho(g_2 x g_2^{-1})$, $x \in g_1^{-1} H g_1 \cap g_2 K g_2^{-1}$.
   (i) Prove that the equivalence class of $\tau^{(g_1, g_2)} := i_{g_1^{-1} H g_1 \cap g_2 K g_2^{-1}}^{G} (\pi^{g_1} \otimes \rho^{g_2})$ depends only on the double coset $H g_1 g_2^{-1} K$.
   (ii) Prove that the internal tensor product $i_H^G \pi \otimes i_K^G \rho$ is equivalent to the direct sum of the representations $\tau^{(g_1, g_2)}$ as $g_1 g_2^{-1}$ ranges over a set of representatives for the H-K-double cosets in G. (*Hint*: Consider the restriction of the representation of $G \times G$ which is the outer tensor product $i_H^G \pi \otimes i_K^G \rho$ to the subgroup $\bar{G} = \{ (g, g) \mid g \in G \}$ and apply Theorem 11.)

5. Suppose that the finite group G is a the semidirect product of a subgroup H with an abelian normal subgroup A, that is, $G = H \ltimes A$. Let G act on the set $\hat{A}$ of irreducible (that is, one-dimensional) representations of A by $\sigma \mapsto \sigma^g$, where $\sigma^g(a) = \sigma(g^{-1} a g)$, $a \in A$, $\sigma \in \hat{A}$. Let $\{ \sigma_1, \ldots, \sigma_r \}$ be a set of representatives for the orbits of G on $\hat{A}$. Let $H_i = \{ \sigma_i^h = \sigma_i \}$.
   a) Let $G_i = H_i \ltimes A$. Show that σ_i extends to a representation of G_i via $\sigma_i(ha) = \sigma_i(a)$, $h \in H_i$, $a \in A$.
   b) Let π be an irreducible representation of H_i. Show that we may define an irreducible representation $\rho(\pi, \sigma_i)$ of G_i on the space V of π by: $\rho(\pi, \sigma_i)(ha) = \sigma_i(ha)\pi(h) = \sigma_i(a)\pi(h)$, $h \in H_i$, $a \in A$.
   c) Let π be an irreducible representation of H_i. Set $\theta_{i,\pi} = \mathrm{Ind}_{G_i}^G \rho(\pi, \sigma_i)$. Prove that $\theta_{i,\pi}$ is an irreducible representation of G.
   d) Let π and π' be irreducible representations of H_i and $H_{i'}$. Prove that $\theta_{i,\pi} \simeq \theta_{i',\pi'}$ implies $i = i'$ and $\pi \simeq \pi'$.
   e) Prove that $\{ \theta_{i,\pi} \}$ are all of the (equivalence classes of) irreducible representations of G. (Here, i ranges over $\{1, \ldots, r\}$ and, for i fixed, π ranges over all of the (equivalence classes of) irreducible representations of H_i).

6. Let H be a subgroup of a finite group G. Let $\mathcal{H}(G, 1)$ be the Hecke algebra associated with the trivial representation of the subgroup H.
   a) Show that if (π, V) is an irreducible representation of G, and V^H is the subspace of H-fixed vectors in V, then V^H becomes a representation of $\mathcal{H}(G, 1)$, that is, a

module over the ring $\mathcal{H}(G,1)$, with the action

$$f \cdot v = |G|^{-1} \sum_{g \in G} f(g)\pi(g)v, \qquad v \in V^H.$$

b) Show that if $V^H \neq \{0\}$, then V^H is an irreducible representation of $\mathcal{H}(G,1)$ (an irreducible $\mathcal{H}(G,1)$-module). (*Hint*: If W is a nonzero invariant subspace of V^H, and $v \in V^H$, use irreducibility of π to show that there exists a function f_1 on G such that $v = f_1 \cdot w$, where $w \in W$ and $f_1 \cdot w$ is defined as above even though $f_1 \notin \mathcal{H}(G,1)$. Next, show that if 1_H is the characteristic function of H, then $f := 1_H * f_1 * 1_H$, $f \in \mathcal{H}(G,1)$, and $f \cdot w = v$.)

c) Show that $(\pi, V) \to V^H$ is a bijection between the equivalence classes of irreducible representatations of G such that $V^H \neq \{0\}$ and the equivalence classes of irreducible representations of $\mathcal{H}(G,1)$.

Remarks - Representations of Hecke algebras: Suppose that (π, V) is an irreducible finite-dimensional representation of a subgroup H of G. A representation of the Hecke algebra $\mathcal{H}(G,\pi)$ is defined to be an algebra homomorphism from $\mathcal{H}(G,\pi)$ to $\mathrm{End}_{\mathbb{C}}(V')$ for some finite-dimensional complex vector space V'. (That is, V' is a finite-dimensional $\mathcal{H}(G,\pi)$-module).

Let (ρ, W) be a finite-dimensional representation of G. Then it is easy to check that the internal tensor product $(r_G^H \rho \otimes \pi, W \otimes V)$ contains the trivial representation of H if and only if $r_G^H \rho$ contains the representation $\pi^\vee$ of H that is dual to π.

Assume that the dual representaion $\pi^\vee$ is a subrepresentation of $r_G^H \rho$. Given $f \in \mathcal{H}(G,\pi)$, define a linear operator $\rho'(f)$ on $W \otimes V$ by

$$\rho'(f)(w \otimes v) = |G|^{-1} \sum_{g \in G} \rho(g)w \otimes f(g)v.$$

The fact that $f \in \mathcal{H}(G,\pi)$ can be used to prove that the subspace $(W \otimes V)^H$ of H-invariant vectors in $W \otimes V$ is $\rho'(f)$-invariant, and the map $f \mapsto \rho'(f) \,|\, (W \otimes V)^H$ defines a representation of the Hecke algebra $\mathcal{H}(G,\pi)$.

In this way, we obtain a map $\rho \mapsto \rho'$ from the set of representations of G whose restrictions to H contain $\pi^\vee$ and the set of nonzero representations of the Hecke algebra $\mathcal{H}(G,\pi)$. It can be shown that this map has the following properties:

(i) ρ is irreducible if and only if ρ' is irreducible.

(ii) If ρ_1 and ρ_2 are irreducible, then ρ_1 and ρ_2 are equivalent if and only if ρ_1' and ρ_2' are equivalent.

(iii) For each nonzero irreducible representation (τ, U) of $\mathcal{H}(G,\pi)$, there exists an irreducible representation ρ of G such that ρ' is equivalent to τ.

The study of representations of reductive groups over finite fields (that is, finite groups of Lie type) is sometimes approached via the study of representations of Hecke algebras. In certain cases, $\mathcal{H}(G, \pi)$ may be isomorphic (as an algebra) to another Hecke algebra $\mathcal{H}(G', \pi')$, where G' is a different group (and π' is an irreducible representation of a subgroup H' of G'). In this case, the study of those irreducible representations of G whose restrictions to H contain $\pi^\vee$ reduces to the study of a similar set of representations of the group G'.

Representations of Hecke algebras also play a role in the study of admissible representations of reductive groups over p-adic fields. An example of such a group is $GL_n(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is the field of p-adic numbers. In this setting, the representation ρ of G will be infinite-dimensional (and admissible), the subgroup H will be compact, and open in G, π will be finite-dimensional (since H is compact) and the representation ρ' of the Hecke algebra will be finite-dimensional. In this setting, the definition of the Hecke algebras is slightly different from that for finite groups.