MAT 445/1196 - INTRODUCTION TO REPRESENTATION THEORY

CHAPTER 1

Representation Theory of Groups - Algebraic Foundations

1.1. Basic definitions, Schur's Lemma

We assume that the reader is familiar with the fundamental concepts of abstract group theory and linear algebra. A *representation* of a group G is a homomorphism from G to the group $GL(V)$ of invertible linear operators on V, where V is a nonzero complex vector space. We refer to V as the *representation space* of π. If V is finite-dimensional, we say that π is finite-dimensional, and the *degree* of π is the dimension of V. Otherwise, we say that π is infinite-dimensional. If π is one-dimensional, then $V \simeq \mathbb{C}$ and we view π as a homomorphism from G to the multiplicative group of nonzero complex numbers. In the above definition, G is not necessarily finite. The notation (π, V) will often be used when referring to a representation.

Examples:

(1) If G is a group, we can define a one-dimensional representation of G by $\pi(g) = 1$, $g \in G$. This representation is called the trivial representation of G.

(2) Let $G = \mathbb{R}$ and $z \in \mathbb{C}$. The function $t \mapsto e^{zt}$ defines a one-dimensional representation of G.

If n is a positive integer and $\mathbb{C}$ is the field of complex numbers, let $GL_n(\mathbb{C})$ denote the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. If (π, V) is a finite-dimensional representation of G, then, via a choice of ordered basis β for V, the operator $\pi(g) \in GL(V)$ is identified with the element $[\pi(g)]_\beta$ of $GL_n(\mathbb{C})$, where n is the degree of π. Hence we may view a finite-dimensional representation of G as a homomorphism from G to the group $GL_n(\mathbb{C})$.

Examples:

(1) The *self-representation* of $GL_n(\mathbb{C})$ is the n-dimensional representation defined by $\pi(g) = g$.

(2) The function $g \mapsto \det g$ is a one-dimensional representation of $GL_n(\mathbb{C})$.

(3) Let V be a space of functions from G to some complex vector space. Suppose that V has the property that whenever $f \in V$, the function $g_0 \mapsto f(g_0 g)$ also belongs to V for all $g \in G$. Then we may define a representation (π, V) by $(\pi(g)f)(g_0) = f(gg_0)$, $f \in V$, g, $g_0 \in G$. For example, if G is a finite group, we may take V to be the space of all complex-valued functions on G. In this case, the resulting representation is called the *right regular representation* of G.

Let (π, V) be a representation of G. A subspace W of V is *stable* under the action of G, or *G-invariant*, if $\pi(g)w \in W$ for all $g \in G$ and $w \in W$. In this case, denoting the restriction of $\pi(g)$ to W by $\pi\,|_W(g)$, $(\pi\,|_W, W)$ is a representation of G, and we call it a *subrepresentation* of π (or a *subrepresentation* of V).

If $W' \subset W$ are subrepresentations of π, then each $\pi\,|_W(g)$, $g \in G$, induces an invertible linear operator $\pi_{W/W'}(g)$ on the quotient space W/W', and $(\pi_{W/W'}, W/W')$ is a representation of G, called a *subquotient* of π. In the special case $W = V$, it is called a *quotient* of π.

A representation (π, V) of G is *finitely-generated* if there exist finitely many vectors $v_1, \ldots, v_m \in V$ such that $V = \mathrm{Span}\{\,\pi(g)v_j \mid 1 \le j \le m,\ g \in G\,\}$. A representation (π, V) of G is *irreducible* if $\{0\}$ and V are the only G-invariant subspaces of V. If π is not irreducible, we say that π is *reducible*.

Suppose that (π_j, V_j), $1 \le j \le \ell$, are representations of a group G. Recall that an element of the direct sum $V = V_1 \oplus \cdots \oplus V_\ell$ can be represented uniquely in the form $v_1 + v_2 + \cdots + v_\ell$, where $v_j \in V_j$. Set

$$\pi(g)(v_1 + \cdots + v_\ell) = \pi_1(g)v_1 + \cdots + \pi_\ell(g)v_\ell, \qquad g \in G,\ v_j \in V_j,\ 1 \le j \le \ell.$$

This defines a representation of G, called the *direct sum* of the representations $\pi_1, \ldots, \pi_\ell$, sometimes denoted by $\pi_1 \oplus \cdots \oplus \pi_\ell$. We may define infinite direct sums similarly. We say that a representation π is *completely reducible* (or *semisimple*) if π is (equivalent to) a direct sum of irreducible representations.

Lemma. *Suppose that (π, V) is a representation of G.*
(1) If π is finitely-generated, then π has an irreducible quotient.
(2) π has an irreducible subquotient.

Proof. For (1), consider all proper G-invariant subspaces W of V. This set is nonempty and closed under unions of chains (uses finitely-generated). By Zorn's Lemma, there is a maximal such W. By maximality of W, $\pi_{V/W}$ is irreducible.

Part (2) follows from part (1) since of v is a nonzero vector in V, part (1) says that if $W = \mathrm{Span}\{\,\pi(g)v \mid g \in G\,\}$, then $\pi\,|_W$ has an irreducible quotient. qed

Lemma. *Let (π, V) be a finite-dimensional representation of G. Then there exists an irreducible subrepresentation of π.*

Proof. If V is reducible, there exists a nonzero G-invariant proper subspace W_1 of V. If $\pi\,|_{W_1}$ is irreducible, the proof is complete. Otherwise, there exists a nonzero G-invariant subspace W_2 of W_1. Note that $\dim(W_2) < \dim(W_1) < \dim(V)$. Since $\dim(V) < \infty$, this process must eventually stop, that is there exist nonzero subspaces $W_k \subsetneq W_{k-1} \subsetneq \cdots \subsetneq W_1 \subsetneq V$, where $\pi\,|_{W_k}$ is irreducible. qed

Lemma. *Let (π, V) be a representation of G. Assume that there exists an irreducible subrepresentation of π. The following are equivalent:*

(1) (π, V) *is completely reducible.*

(2) For every G*-invariant subspace* $W \subset V$*, there exists a* G*-invariant subspace* W' *such that* $W \oplus W' = V$*.*

Proof. Assume that π is completely reducible. Without loss of generality, π is reducible. Let W be a proper nonzero G-invariant subspace of V. Consider the set of G-invariant subspaces U of V such that $U \cap W = \{0\}$. This set is nonempty and closed under unions of chains, so Zorn's Lemma implies existence of a maximal such U. Suppose that $W \oplus U \neq V$. Since π is completely reducible, there exists some irreducible subrepresentation U' such that $U' \not\subset W \oplus U$. By irreduciblity of U', $U' \cap (W \oplus U) = \{0\}$. This contradicts maximality of U.

Suppose that (2) holds. Consider the partially ordered set of direct sums of families of irreducible subrepresentations: $\sum_\alpha W_\alpha = \oplus_\alpha W_\alpha$. Zorn's Lemma applies. Let $W = \oplus_\alpha W_\alpha$ be the direct sum for a maximal family. By (2), there exists a subrepresentation U such that $V = W \oplus U$. If $U \neq \{0\}$, according to a lemma above, there exists an irreducible subquotient: $U \supset U_1 \supset U_2$ such that π_{U_1/U_2} is is irreducible. By (2), $W \oplus U_2$ has a G-invariant complement U_3: $V = W \oplus U_2 \oplus U_3$. Now

$$U_3 \simeq V/(W \oplus U_2) = (W \oplus U)/(W \oplus U_2) \simeq U/U_2 \supset U_1/U_2.$$

Identifying π_{U_1/U_2} with an irreducible subrepresentation $\pi\,|_{U_4}$ of $\pi\,|_{U_3}$, we have $W \oplus U_4$ contradicting maximality of the family W_α. qed

Lemma. *Subrepresentations and quotient representations of completely reducible representations are completely reducible.*

Proof. Let (π, V) be a completely reducible representation of G. Suppose that W is a proper nonzero G-invariant subspace of W. Then, according to the above lemma, there exists a G-invariant subspace U of V such that $V = W \oplus U$. It follows that the subrepresentation $\pi\,|_W$ is equivalent to the quotient representation $\pi_{V/U}$. Therefore it suffices to prove that any quotient representation of π is completely reducible.

Let $\pi_{V/U}$ be an arbitrary quotient representation of π. We know that $\pi = \oplus_{\alpha \in I} \pi_\alpha$, where I is some indexing set, and each π_α is irreducible. Let $pr : V \to V/U$ be the canonical map. Then $V/U = pr(V) = \oplus_{\alpha \in I} pr(V_\alpha)$. Because $pr(V_\alpha)$ is isomorphic to a quotient of V_α ($pr(V_\alpha) \simeq V_\alpha / \ker(pr \,|\, V_\alpha)$) and π_α is irreducible, we have that $pr(V_\alpha)$ is either 0 or irreducible. Hence $\pi_{V/U}$ is completely reducible. qed

Exercises:

(1) Show that the self-representation of $GL_n(\mathbb{C})$ is irreducible.

(2) Verify that $\pi : t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ defines a representation of $\mathbb{R}$, with space $\mathbb{C}^2$, that is a two-dimensional representation of $\mathbb{R}$. Show that there is exactly one one-dimensional

subrepresentation, hence π is not completely reducible. Prove that the restriction of π to the unique one-dimensional invariant subspace W is the trivial representation, and the quotient representation $\pi_{V/W}$ is the trivial representation.

If (π_1, V_1) and (π_2, V_2) are representations of a group G, a linear transformation $A : V_1 \to V_2$ *intertwines* π_1 and π_2 if $A\pi_1(g)v = \pi_2(g)Av$ for all $v \in V_1$ and $g \in G$. The notation $\mathrm{Hom}_G(\pi_1, \pi_2)$ or $\mathrm{Hom}_G(V_1, V_2)$ will be used to denote the set of linear transformations from V_1 to V_2 that intertwine π_1 and π_2. Two representations (π_1, V_1) and (π_2, V_2) of a group G are said to be *equivalent* (or *isomorphic*) whenever $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains an isomorphism, that is, whenever there exists an invertible linear tranformation $A : V_1 \to V_2$ that intertwines π_1 and π_2. In this case, we write $\pi_1 \simeq \pi_2$. It is easy to check that the notion of equivalence of representations defines an equivalence relation on the set of representations of G. It follows from the definitions that if π_1 and π_2 are equivalent representations, then π_1 is irreducible if and only if π_2 is irreducible. More generally, π_1 is completely reducible if and only if π_2 is completely reducible.

Lemma. *Suppose that (π_1, V_1) and (π_2, V_2) are finite-dimensional representations of G. Then the following are equivalent:*
(1) π_1 and π_2 are equivalent.
(2) $\dim V_1 = \dim V_2$ and there exist ordered bases β_1 and β_2 of V_1 and V_2, respectively, such that $[\pi_1(g)]_{\beta_1} = [\pi_2(g)]_{\beta_2}$ for all $g \in G$.

Proof. Assume (1). Fix ordered bases γ_1 for V_1 and γ_2 for V_2. Via these bases, identifying any invertible operator in $\mathrm{Hom}_G(\pi_1, \pi_2)$ as a matrix A in $GL_n(\mathbb{C})$, we have

$$[\pi_1(g)]_{\gamma_1} = A^{-1}[\pi_2(g)]_{\gamma_2}A, \qquad \forall\ g \in G.$$

Let $\beta_1 = \gamma_1$. Because $A \in GL_n(\mathbb{C})$, there exists an ordered basis β_2 of V_2 such that A is the change of basis matrix from β_2 to γ_2. With these choices of β_1 and β_2, (2) holds.

Now assume that (2) holds. Let A be the unique linear transformation from V_1 to V_2 which maps the jth vector in β_1 to the jth vector in β_2. qed

A representation (π, V) of G has a (finite) *composition series* if there exist G-invariant subspaces V_j of V such that

$$\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$$

each subquotient π_{V_{j+1}/V_j}, $1 \le j \le r-1$, is irreducible. The subquotients π_{V_{j+1}/V_j} are called the *composition factors* of π.

Lemma. *Let (π, V) be a finite-dimensional represntation of G. Then π has a composition series. Up to reordering and equivalence, the composition factors of π are unique.*

Proof left as an exercise.

Schur's Lemma. *Let* (π_1, V_1) *and* (π_2, V_2) *be irreducible representations of* G*. Then any nonzero operator in* $\mathrm{Hom}_G(\pi_1, \pi_2)$ *is an isomorphism.*

Proof. If $\mathrm{Hom}_G(\pi_1, \pi_2) = \{0\}$ there is nothing to prove, so assume that it is nonzero. Suppose that $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$ is nonzero. Let $g \in G$ and $v_2 \in A(V_1)$. Writing $v_2 = A(v_1)$, for some $v_1 \in V_1$, we have $\pi_2(g)v_2 = A\pi(g)v_1 \in A(V_1)$. Hence $A(V_1)$ is a nonzero G-invariant subspace of V_2. By irreducibility of π_2, we have $A(V_1) = V_2$.

Next, let W be the kernel of A. Let $v_1 \in W$. Then $A(\pi_1(g)v_1) = \pi_2(g)(A(v_1)) = \pi_1(g)0 = 0$ for all $g \in G$. Hence W is a G-invariant proper subspace of V_1. By irreducibility of π_1, $W = \{0\}$. qed

Corollary. *Let* (π, V) *be a finite-dimensional irreducible representation of* G*. Then* $\mathrm{Hom}_G(\pi, \pi)$ *consists of scalar multiples of the identity operator, that is,* $\mathrm{Hom}_G(\pi, \pi) \simeq \mathbb{C}$.

Proof. Let $A \in \mathrm{Hom}_G(\pi, \pi)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A (such an eigenvalue exists, since V is finite-dimensional and $\mathbb{C}$ is algebraically closed). It is easy to see that $A - \lambda I \in \mathrm{Hom}_G(\pi, \pi)$. But $A - \lambda I$ is not invertible. By the previous lemma, $A = \lambda I$. qed

Corollary. *If* G *is an abelian group, then every irreducible finite-dimensional representation of* G *is one-dimensional.*

Proof left as an exercise.

Exercise: Prove that an irreducible representation of the cyclic group of order $n > 1$, with generator g_0, has the form $g_0^k \mapsto e^{2\pi imk/n}$ for some $m \in \{\,0, 1, \ldots, n-1\,\}$. (Here i is a complex number such that $i^2 = -1$ and π denotes the area of a circle of radius one).

Let (π, V) be a representation of G. A *matrix coefficient* of π is a function from G to $\mathbb{C}$ of the form $g \mapsto \lambda(\pi(g)v)$, for some fixed $v \in V$ and λ in the dual space $V^\vee$ of linear functionals on V. Suppose that π is finite-dimensional. Choose an ordered basis $\beta = \{\,v_1, \ldots, v_n\,\}$ of V. Let $\beta^\vee = \{\,\lambda_1, \ldots, \lambda_n\,\}$ be the basis of $V^\vee$ which is dual to β: $\lambda_j(v_i) = \delta_{ij}$, $1 \le i, j \le n$. Define a function $a_{ij} : G \to \mathbb{C}$ by $[\pi(g)]_\beta = (a_{ij}(g))_{1 \le i,j \le n}$. Then it follows from $\pi(g)v_i = \sum_{\ell=1}^n a_{i\ell}(g)v_\ell$ that $a_{ij}(g) = \lambda_j(\pi(g)v_i)$, so a_{ij} is a matrix coefficient of π.

If $g \in G$ and $\lambda \in V^\vee$, define $\pi^\vee(g)\lambda \in V^\vee$ by $(\pi^\vee(g)\lambda)(v) = \lambda(\pi(g^{-1})v)$, $v \in V$. Then $(\pi^\vee, V^\vee)$ is a representation of G, called the *dual* (or *contragredient*) of π.

Exercises:

(1) Let (π, V) be a finite-dimensional representation of G. Choose β and $\beta^\vee$ as above. Show that $[\pi^\vee(g)]_{\beta^\vee} = [\pi(g^{-1})]_\beta^t$, for all $g \in G$. Here the superscript t denotes transpose.

(2) Prove that if (π, V) is finite-dimensional then π is irreducible if and only if $\pi^\vee$ is irreducible.

(3) Determine whether the self-representation of $GL_n(\mathbb{R})$ (restrict the self-representation of $GL_n(\mathbb{C})$ to the subgroup $GL_n(\mathbb{R})$) is equivalent to its dual.
(4) Prove that a finite-dimensional representation of a finite abelian group is the direct sum of one-dimensional representations.

1.2. Tensor products

Let (π_j, V_j) be a representation of a group G_j, $j = 1, 2$. Recall that $V_1 \otimes V_2$ is spanned by elementary tensors, elements of the form $v_1 \otimes v_2$, $v_1 \in V_1$, $v_2 \in V_2$. We can define a representation $\pi_1 \otimes \pi_2$ of the direct product $G_1 \times G_2$ by setting

$$(\pi_1 \otimes \pi_2)(g_1, g_2)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2, \qquad g_j \in G_j,\ v_j \in V_j,\ j = 1, 2,$$

and extending by linearity to all of $V_1 \otimes V_2$. The representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$ is called the *(external or outer) tensor product* of π_1 and π_2. Of course, when π_1 and π_2 are finite-dimensional, the degree of $\pi_1 \otimes \pi_2$ is equal to the product of the degrees of π_1 and π_2.

Lemma. *Let (π_j, V_j) and G_j, $j = 1, 2$ be as above. Assume that each π_j is finite-dimensional. Then $\pi_1 \otimes \pi_2$ is an irreducible representation of $G_1 \times G_2$ if and only if π_1 and π_2 are both irreducible.*

Proof. If π_1 or π_2 is reducible, it is easy to see that $\pi_1 \otimes \pi_2$ is also reducible.

Assume that π_1 is irreducible. Let $n = \dim V_2$. Let

$$\mathrm{Hom}_{G_1}(\pi_1, \pi_1)^n = \mathrm{Hom}_{G_1}(\pi_1, \pi_1) \oplus \cdots \oplus \mathrm{Hom}_{G_1}(\pi_1, \pi_1),$$

and $\pi_1^n = \pi_1 \oplus \cdots \oplus \pi_1$, where each direct sum has n summands. Then $\mathrm{Hom}_{G_1}(\pi_1, \pi_1)^n \simeq \mathrm{Hom}_{G_1}(\pi_1, \pi_1^n)$, where the isomorphism is given by $A_1 \oplus \cdots \oplus A_n \mapsto B$, with $B(v) = A_1(v) \oplus \cdots \oplus A_n(v)$. By (the corollary to) Schur's Lemma, $\mathrm{Hom}_{G_1}(\pi_1, \pi_1) \simeq \mathbb{C}$. Irreducibility of π_1 guarantees that given any nonzero $v \in V_1$, $V_1 = \mathrm{Span}\{\, \pi_1(g_1)v \mid g_1 \in G_1 \,\}$, and this implies surjectivity.

Because $V_2 \simeq \mathbb{C}^n$ and $\mathbb{C} \simeq \mathrm{Hom}_{G_1}(\pi_1, \pi_1)$, we have

$$(i) \qquad V_2 \simeq \mathrm{Hom}_{G_1}(\pi_1, \pi_1 \otimes 1^n),$$

where $\pi_1 \otimes 1^n$ is the representation of G_1 on $V_1 \otimes V_2$ defined by $(\pi_1 \otimes 1^n)(g_1)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes v_2$, $v_1 \in V_1$, $v_2 \in V_2$. (Note that this representation can be identified with the restriction of $\pi_1 \otimes \pi_2$ to the subgroup $G_1 \times \{1\}$ of $G_1 \times G_2$).

If m is a positive integer, then

$$(ii) \qquad \begin{aligned} V_1 \otimes \mathrm{Hom}_{G_1}(\pi_1, \pi_1^m) &\to V_1^m \\ v \otimes A &\mapsto A(v) \end{aligned}$$

is an isomorphism.

Next, we can use (i) and (ii) to show that

$$\begin{aligned} \{\, G_1 - \text{invariant subspaces of } V_1 \otimes V_2 \,\} &\leftrightarrow \{\, \mathbb{C} - \text{subspaces of } V_2 \,\} \\ V_1 \otimes W &\leftarrow W \\ X &\rightarrow \mathrm{Hom}_{G_1}(\pi_1, X) \subset \mathrm{Hom}_{G_1}(\pi_1, \pi_1 \otimes 1^n) = V_2 \end{aligned}$$

As any $(G_1 \times G_2)$-invariant subspace X of $V_1 \otimes V_2$ is also a G_1-invariant subspace, we have $X = V_1 \otimes W$ for some complex subspace W of V_2. If $X \neq \{0\}$ and π_2 is irreducible, then

$$\mathrm{Span}\{\, (\pi_1 \otimes \pi_2)(1, g_2)X \mid g_2 \in G_2 \,\} = V_1 \otimes \mathrm{Span}\{\, \pi_2(g_2)W \mid g_2 \in G_2 \,\} = V_1 \otimes V_2.$$

But $G_1 \times G_2$-invariance of X then forces $X = V_1 \otimes V_2$. It follows that if π_1 and π_2 are irreducible, then $\pi_1 \otimes \pi_2$ is irreducible (as a representation of $G_1 \times G_2$). qed

Proposition. *Let (π, V) be an irreducible finite-dimensional representation of $G_1 \times G_2$. Then there exist irreducible representations π_1 and π_2 of G_1 and G_2, respectively, such that $\pi \simeq \pi_1 \otimes \pi_2$.*

Proof. Note that $\pi_1'(g_1)v = \pi((g_1, 1))v$, $g_1 \in G_1$, $v \in V$, and $\pi_2'(g_2)v = \pi((1, g_2))v$, $g_2 \in G_2$, $v \in V$, define representations of G_1 and G_2, respectively. Choose a nonzero G_1-invariant subspace V_1 such that $\pi_1' \,|_{V_1}$ is an irreducible representation of G_1. Let v_0 be a nonzero vector in V_1. Let

$$V_2 = \mathrm{Span}\{\, \pi_2'(g_2)v_0 \mid g_2 \in G_2 \,\}.$$

Then V_2 is G_2-invariant and $\pi_2 := \pi_2' \,|_{V_2}$ is a representation of G_2, which might be reducible.

Define $A : V_1 \otimes V_2 \to V$ as follows. Let $v_1 \in V_1$ and $v_2 \in V_2$. Then there exist complex numbers c_j and elements $g_1^{(j)} \in G_1$ such that $v_1 = \sum_{j=1}^m c_j \pi_1(g_1^{(j)})v_0$, as well as complex numbers b_ℓ and elements $g_2^{(\ell)} \in G_2$ such that $v_2 = \sum_{\ell=1}^n b_\ell \pi_2(g_2^{(\ell)})$. Set

$$A(v_1 \otimes v_2) = \sum_{j=1}^{m} \sum_{\ell=1}^{n} c_j b_\ell \pi(g_1^{(j)}, g_2^{(\ell)})v_0.$$

Now $\pi(g_1^{(j)}, g_2^{(\ell)})v_0 = \pi_1(g_1^{(j)})\pi_2(g_2^{(\ell)})v_0 = \pi_2(g_2^{(\ell)})\pi_1(g_1^{(j)})v_0$. Check that the map A is well-defined, extending to a linear transformation from $V_1 \otimes V_2$ to V. Also check that $A \in \mathrm{Hom}_{G_1 \times G_2}(V_1 \otimes V_2, V)$.

Because $A(v_0 \otimes v_0) = v_0$, we know that A is nonzero. Combining $G_1 \times G_2$-invariance of $A(V_1 \otimes V_2)$ with irreducibility of π, we have $A(V_1 \otimes V_2) = V$. If A also happens to be one-to-one, then we have $\pi_1 \otimes \pi_2 \simeq \pi$.

Suppose that A is not one-to-one. Then $\operatorname{Ker} A$ is a $G_1 \times G_2$-invariant subspace of $V_1 \otimes V_2$. In particular, $\operatorname{Ker} A$ is a G_1-invariant subspace of $V_1 \otimes V_2$. Using irreducibility of π_1 and arguing as in the previous proof, we can conclude that $\operatorname{Ker} A = V_1 \otimes W$ for some complex subspace W of V_2. We have an equivalence of the representations $(\pi_1 \otimes \pi_2)_{(V_1 \otimes V_2)/\operatorname{Ker} A}$ and π of $G_1 \times G_2$. To finish the proof, we must show that the quotient representation $(\pi_1 \otimes \pi_2)_{V/(V_1 \otimes W)}$ is a tensor product. If $v_1 \in V_1$ and $v_2 \in V_2$, define

$$B(v_1 \otimes (v_1 + W)) = v_1 \otimes v_2 + V_1 \otimes W.$$

This extends by linearity to a map from $V_1 \otimes (V_2/W)$ to the quotient space $(V_1 \otimes V_2)/(V_1 \otimes W)$ and it is a simple matter to check that B is an isomorphism and $B \in \operatorname{Hom}_{G_1 \times G_1}(\pi_1 \otimes (\pi_2)_{V_2/W}, (\pi_1 \otimes \pi_2)_{(V_1 \otimes V_2)/(V_1 \otimes W)}$. The details are left as an exercise. qed

If (π_1, V_1) and (π_2, V_2) are representations of a group G, then we may form the tensor product representation $\pi_1 \otimes \pi_2$ of $G \times G$ and restrict to the subgroup $\delta G = \{\, (g, g) \mid g \in G \,\}$ of $G \times G$. This restriction is then a representation of G, also written $\pi_1 \otimes \pi_2$. It is called the *(inner) tensor product* of π_1 and π_2. Using inner tensor products gives ways to generate new representations of a group G. However, it is important to note that even if π_1 and π_2 are both irreducible, the inner tensor product representation $\pi_1 \otimes \pi_2$ of G can be reducible.

Exercise: Let π_1 and π_2 be finite-dimensional irreducible representations of a group G. Prove that the trivial representation of G occurs as a subrepresentation of the (inner) tensor product representation $\pi_1 \otimes \pi_2$ of G if and only if π_2 is equivalent to the dual $\pi_1^\vee$ of π.

1.3. Unitary representations

Suppose that (π, V) is a representation of G. If V is a finite-dimensional inner product space and there exists an inner product $\langle \cdot, \cdot \rangle$ on V such that

$$\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle, \qquad \forall\ v_1, v_2 \in V,\ g \in G.$$

then we say that π is a *unitary* representation. If V is infinite-dimensional, we say that π is *pre-unitary* if such an inner product exists, and if V is complete with respect to the norm induced by the inner product (that is, V is a Hilbert space), then we say that π is *unitary.*

Now assume that π is finite-dimensional. Recall that if T is a linear operator on V, the adjoint T^* of T is defined by $< T(v), w > = < v, T^*(w) >$ for all v, $w \in V$. Note that π is unitary if and only if each operator $\pi(g)$ satisfies $\pi(g)^* = \pi(g)^{-1}$, $g \in G$.

Let n be a positive integer. Recall that if A is an $n \times n$ matrix with entries in $\mathbb{C}$, the adjoint A^* of A is just $A^* = {}^t\bar{A}$.

Lemma. *If (π, V) is a finite-dimensional unitary representation of G and β is an orthonormal basis of V, then $[\pi(g)]_\beta^* = [\pi(g)]_\beta^{-1}$.*

Proof. Results from linear algebra show that if T is a linear operator on V and β is an orthonormal basis of V, then $[T^*]_\beta = [T]_\beta^*$. Combining this with $\pi(g)^* = \pi(g)^{-1}$, $g \in G$, proves the lemma. qed

Exercises:

(1) If (π, V) is a representation, form a new vector space $\bar{V}$ as follows. As a set, $V = \bar{V}$, and $\bar{V}$ has the same vector addition as V. If $c \in \mathbb{C}$ and $v \in \bar{V}$, set $c \cdot v = \bar{c}v$, where $\bar{c}$ is the complex conjugate of c and $\bar{c}v$ is the scalar multiplication in V. If $g \in G$, and $v \in \bar{V}$, $\bar{\pi}(g)v = \pi(g)v$. Show that $(\bar{\pi}, \bar{V})$ is a representation of V.

(2) Assume that (π, V) is a finite-dimensional unitary representation. Prove that $\pi^\vee \simeq \bar{\pi}$.

Lemma. *Let W be a subspace of V, where (π, V) is a unitary representation of G. Then W is G-invariant if and only if $W^\perp$ is G-invariant.*

Proof. W is G-invariant if and only if $\pi(g)w \in W$ for all $g \in G$ and $w \in W$ if and only if $\langle \pi(g)w, w^\perp \rangle = 0$ for all $w \in W$, $w^\perp \in W^\perp$ and $g \in G$ if and only if $\langle w, \pi(g^{-1})w^\perp \rangle = 0$ for all $w \in W$, $w^\perp \in W^\perp$ and $g \in G$, if and only if $W^\perp$ is G-invariant. qed

Corollary. *A finite-dimensional unitary representation is completely reducible.*

Lemma. *Suppose that (π, V) is a finite-dimensional unitary represetantion of G. Let W be a proper nonzero G-invariant subspace of V, and let P_W be the orthogonal projection of V onto W. Then P_W commutes with $\pi(g)$ for all $g \in G$.*

Proof. Let $w \in W$ and $w^\perp \in W^\perp$. Then

$$P_W\pi(g)(w + w^\perp) = P_W\pi(g)w + P_W\pi(g)w^\perp = \pi(g)w + 0 = \pi(g)P_W(w + w^\perp).$$

qed

Lemma. *Let (π, V) be a finite-dimensional unitary representation of G. Then π is irreducible if and only if $\mathrm{Hom}_G(\pi, \pi) \simeq \mathbb{C}$ (every operator which commutes with all $\pi(g)$'s is a scalar multiple of the identity operator).*

Proof. One direction is simply the corollary to Schur's Lemma (using irreducibility of π). For the other, if π is reducible, and W is a proper nonzero G-invariant subspace of V, Then $P_W \in \mathrm{Hom}_G(\pi, \pi)$ and P_W is not a scalar multiple of the identity operator. qed

Suppose that (π_1, V_1) and (π_2, V_2) are representations of G and V_1 and V_2 are complex inner product spaces, with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Then π_1 and π_2 are *unitarily equivalent* if there exists an invertible linear operator $A : V_1 \to V_2$ such that $\langle Av, Aw \rangle_2 = \langle v, w \rangle_1$ for all v and $w \in V_1$ and $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$.

Lemma. *Let (π_1, V_1) and (π_2, V_2) be finite-dimensional unitary representations of G. Then $\pi_1 \simeq \pi_2$ if and only if π_1 and π_2 are unitarily equivalent.*

Proof. Assume that $\pi_1 \simeq \pi_2$. Let $A : V_1 \to V_2$ be an isomorphism such that $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$. Recall that the adjoint $A^* : V_2 \to V_1$ is defined by the condition $\langle A^* v_2, v_1\rangle_1 = \langle v_2, Av_1\rangle_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. By assumption, we have

$$(i) \qquad \pi_1(g) = A^{-1}\pi_2(g)A, \quad \forall\, g \in G.$$

Taking adjoints, we have $\pi_1(g)^* = A^*\pi_2(g)^*(A^*)^{-1}$ for all $g \in G$. Since π_j is unitary, we have $\pi_j(g)^* = \pi_j(g^{-1})$. Replacing g^{-1} by g, we have

$$(ii) \qquad \pi_1(g) = A^*\pi_2(g)(A^*)^{-1}, \quad \forall\, g \in G.$$

Expressing $\pi_2(g)$ in terms of $\pi_1(g)$ using (i), we can rewrite (ii) as

$$\pi_1(g) = A^*A\pi_1(g)A^{-1}(A^*)^{-1}, \quad \forall\, g \in G,$$

or

$$\pi_1(g)^{-1}A^*A\pi_1(g) = A^*A, \quad \forall\, g \in G.$$

Now A^*A is positive definite (that is, self-adjoint and having positive (real) eigenvalues), and so has a unique positive definite square root, say B. Note that $\pi_1(g)^{-1}B\pi_1(g)$ is also a square root of A^*A and it is positive definite, using $\pi_1(g)^* = \pi_1(g)^{-1}$. Hence $\pi_1(g)^{-1}B\pi_1(g) = B$ for all $g \in G$. Writing A in terms of the polar decomposition, we have $A = UB$, with B as above, and with U an isomorphism from $V_1 \to V_2$ such that $\langle Uv, Uw\rangle_2 = \langle v, w\rangle_1$ for all v and $w \in V_1$. Next, note that

$$\pi_2(g) = UB\pi_1(g)B^{-1}U^{-1} = U\pi_1(g)U^{-1}, \quad \forall\, g \in G.$$

Hence $U \in \mathrm{Hom}_G(\pi_1, \pi_2)$, and π_1 and π_2 are unitarily equivalent. qed

1.4. Characters of finite-dimensional representations

Let (π, V) be a finite-dimensional representation of a group G. The function $g \mapsto \mathrm{tr}\,\pi(g)$ from G to $\mathbb{C}$ is called the *character* of π. We use the notation $\chi_\pi(g) = \mathrm{tr}\,\pi(g)$. Note that we can use any ordered basis of V to compute $\chi_\pi(g)$, since the trace of an operator depends only on the operator itself. Note that if π were infinite-dimensional, the operator $\pi(g)$ would not have a trace.

Lemma. *Let (π, V) be a finite-dimensional representation of G.*

(1) If $\pi' \simeq \pi$, then $\chi_\pi = \chi_{\pi'}$.

(2) The function χ_π is constant on conjugacy classes in G.

(3) Let $\pi^\vee$ be the representation dual to π. Then $\chi_{\pi^\vee}(g) = \chi_\pi(g^{-1})$, $g \in G$.

(4) If π is unitary, then $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$, $g \in G$.

(5) Suppose that (π, V) has a composition series $\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$, with composition factors $\pi_{V_1}, \pi_{V_2/V_1}, \ldots, \pi_{V_r/V_{r-1}}$ (see page 5). Then $\chi_\pi = \chi_{\pi_{V_1}} + \chi_{\pi_{V_2/V_1}} + \cdots + \chi_{\pi_{V_r/V_{r-1}}}$.

(6) The character $\chi_{\pi_1 \otimes \cdots \otimes \pi_r}$ of a tensor product of finite-dimensional representations $\pi_1, \ldots, \pi_r$ of $G_1, \ldots, G_r$, respectively, is given by

$$\chi_{\pi_1 \otimes \cdots \otimes \pi_r}(g_1, \ldots, g_r) = \chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2) \cdots \chi_{\pi_r}(g_r), \qquad g_1 \in G_1, \ldots, g_r \in G_r.$$

Proof. By an earlier result, if $\pi' \simeq \pi$, then π' and π have the same matrix realization (for some choice of bases). Part (1) follows immediately.

Note that

$$\chi_\pi(g_1 g g_1^{-1}) = \mathrm{tr}\,(\pi(g_1)\pi(g)\pi(g_1)^{-1}) = \mathrm{tr}\,\pi(g) = \chi_\pi(g), \quad g,\, g_1 \in G.$$

Recall that if β is an ordered basis of V and $\beta^\vee$ is the basis of $V^\vee$ dual to β, then $[\pi(g)]_\beta = {}^t[\pi^\vee(g^{-1})]_{\beta^\vee}$. This implies (3).

Suppose that π is unitary. Let β be an orthonormal basis of V. Then $[\pi(g^{-1})]_\beta = [\pi(g)]_\beta^* = {}^t\overline{[\pi(g)]}$ implies part (4).

For (5), it is enough to do the case $r = 2$. Let β be an ordered basis for V_1. Extend β to an ordered basis γ for $V_2 = V$. Let $\dot\gamma$ be the ordered basis for V_2/V_1 which is the image of γ under the canonical map $V \to V_2/V_1$. Then it is easy to check that $[\pi(g)]_\gamma$ is equal to

$$\begin{pmatrix} [\pi\,|_{V_1}(g)]_\beta & * \\ 0 & [\pi_{V_2/V_1}(g)]_{\dot\gamma} \end{pmatrix}.$$

For (6), it is enough to do the case $r = 2$. Let $\beta = \{\, v_1, \ldots, v_n \}$ and $\gamma = \{\, w_1, \ldots, w_m \,\}$ be ordered bases of V_1 and V_2, respectively. Then

$$\{\, v_j \otimes w_\ell \mid 1 \le j \le n,\ 1 \le \ell \le m \,\}$$

is an ordered basis of $V_1 \otimes V_2$. Let $a_{ij}(g_1)$ be the ijth entry of $[\pi_1(g_1)]_\beta$, $g_1 \in G_1$, and let $b_{ij}(g_2)$ be the ijth entry of $[\pi_2(g_2)]_\gamma$, $g_2 \in G_2$. We have

$$\begin{aligned} \pi_1(g_1)v_j &= a_{1j}(g_1)v_1 + a_{2j}(g_1)v_2 + \cdots + a_{nj}(g_1)v_n, \quad g_1 \in G_r \\ \pi_2(g_2)w_\ell &= b_{1\ell}(g_2)w_1 + b_{2\ell}(g_2)w_2 + \cdots + b_{m\ell}(g_2)w_m, \quad g_2 \in G_2. \end{aligned}$$

Hence

$$\pi_1(g_1)v_j \otimes \pi_2(g_2)w_\ell = \sum_{t=1}^{n}\sum_{s=1}^{m} a_{tj}(g_1)b_{s\ell}(g_2)(v_t \otimes w_s),$$

and, as the coefficient of $v_j \otimes w_\ell$ on the right side equals $a_{jj}(g_1)b_{\ell\ell}(g_2)$, we have

$$\chi_{\pi_1\otimes\pi_2}(g_1, g_2) = \sum_{j=1}^{n}\sum_{\ell=1}^{m} a_{jj}(g_1)b_{\ell\ell}(g_2) = \chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2), \quad g_1 \in G_1,\ g_2 \in G_2.$$

Example: The converse to part (1) is false. Consider the Example (2) on page 4. We have $\chi_\pi(t) = 2$ for all $t \in \mathbb{R}$. Now take $\pi_0 \oplus \pi_0$, where π_0 is the trivial representation of $\mathbb{R}$. This clearly has the same character as π, though $\pi_0 \oplus \pi_0$ is not equivalent to π.

In many cases, for example, if G is finite, or compact, two irreducible finite-dimensional representations having the same character must be equivalent.

CHAPTER 2

Representations of Finite Groups

In this chapter we consider only finite-dimensional representations.

2.1. Unitarity, complete reducibility, orthogonality relations

Theorem 1. *A representation of a finite group is unitary. c Proof. Let (π, V) be a (finite-dimensional) representation of a finite group $G = \{ g_1, g_2, \dots g_n \}$. Let $\langle \cdot, \cdot \rangle_1$ be any inner product on V. Set*

$$\langle v, w \rangle = \sum_{j=1}^{n} \langle \pi(g_j)v, \pi(g_j)w \rangle_1, \qquad v,\, w \in V.$$

Then it is clear from the definition that

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle, \quad v,\, w \in V,\, g \in G.$$

Note that if $v \in V$, then $\langle v, v \rangle = \sum_{j=1}^{n} \langle \pi(g_j)v, \pi(g_j)v \rangle_1$ and if $v \neq 0$, then $\pi(g_j)v \neq 0$ for all j implies $\langle \pi(g_j)v, \pi(g_j)v \rangle_1 > 0$ for all j. Hence $v \neq 0$ implies $\langle v, v \rangle > 0$. The other properties of inner product are easy to verify for $\langle \cdot, \cdot \rangle$, using the fact that $\langle \cdot, \cdot \rangle_1$ is an inner product, and each $\pi(g_j)$ is linear. The details are left as an exercise. qed

The following is an immediate consequence of Theorem 1 and a result from Chapter I stating that a finite-dimensional unitary representation is completely reducible.

Theorem 2. *A representation of a finite group is completely reducible.*

Example. Let G be a finite group acting on a finite set X. Let V be a complex vector space having a basis $\{ v_{x_1}, \dots, v_{x_m} \}$ indexed by the elements $x_1, \dots, x_m$ of X. If $g \in G$, let $\pi(g)$ be the operator sending v_{x_j} to $v_{g \cdot x_j}$, $1 \leq j \leq m$. Then (π, V) is a representation of G, called the *permutation representation* associated with X.

Let $\mathcal{A}(G)$ be the set of complex-valued functions on G. Often $\mathcal{A}(G)$ is called the *group algebra* of G - see below. Let $R_{\mathcal{A}}$ be the (right) regular representation of G on the space $\mathcal{A}(G)$: Given $f \in \mathcal{A}(G)$ and $g \in G$, $R_{\mathcal{A}}(g)f$ is the function defined by $(R_{\mathcal{A}}(g)f)(g_0) = f(g_0 g)$, $g_0 \in G$. Note that $R_{\mathcal{A}}$ is equivalent to the the permutation representation associated to the set $X = G$. Let $L_{\mathcal{A}}$ be the left regular representation of G on the space $\mathcal{A}(G)$: Given $f \in \mathcal{A}(G)$ and $g \in G$, $L_{\mathcal{A}}(g)f$ is defined by $(L_{\mathcal{A}}(g)f)(g_0) = f(g^{-1}g_0)$, $g_0 \in G$. It is easy to check that the operator $f \mapsto \dot{f}$, where $\dot{f}(g) = f(g^{-1})$, is a unitary equivalence in $\mathrm{Hom}_G(R_{\mathcal{A}}, L_{\mathcal{A}})$.

If $f_1,\ f_2 \in \mathcal{A}(G)$, the *convolution* $f_1 * f_2$ of f_1 with f_2 is defined by

$$(f_1 * f_2)(g) = \sum_{g_0 \in G} f_1(gg_0^{-1}) f_2(g_0), \qquad g \in G.$$

With convolution as multiplication, $\mathcal{A}(G)$ is an algebra. It is possible to study the representations of G in terms of $\mathcal{A}(G)$-modules.

We define an inner product on $\mathcal{A}(G)$ as follows:

$$\langle f_1, f_2 \rangle = |G|^{-1} \sum_{g \in G} f_1(g)\overline{f_2(g)}, \qquad f_1,\ f_2 \in \mathcal{A}(G).$$

Theorem 3 (Orthogonality relations for matrix coefficients). *Let (π_1, V_1) and (π_2, V_2) be irreducible (unitary) representations of G. Let $a^i_{jk}(g)$ be the matrix entries of the matrix of $\pi_i(g)$ relative to a fixed orthonormal basis of V_i, $i = 1, 2$ (relative to an inner product which makes π_i unitary). Then*

(1) If $\pi_1 \not\simeq \pi_2$, then $\langle a^1_{jk}, a^2_{\ell m} \rangle = 0$ for all j, k, ℓ and m.

(2) $\langle a^1_{jk}, a^1_{\ell m} \rangle = \delta_{j\ell}\delta_{km}/n_1$, where $n_1 = \dim V_1$.

Proof. Let B be a linear transformation from V_2 to V_1. Then $A := |G|^{-1} \sum_{g \in G} \pi_1(g) B \pi_2(g)^{-1}$ is also a linear transformation from V_2 to V_1. Let $g' \in G$. Then

$$\pi_1(g')A = |G|^{-1} \sum_{g \in G} \pi_1(g'g) B \pi_2(g^{-1}) = |G|^{-1} \sum_{g \in G} \pi_1(g) B \pi_2(g^{-1}g') = A\pi_2(g').$$

Hence $A \in \mathrm{Hom}_G(\pi_2, \pi_1)$,

Let $n_i = \dim V_i$, $i = 1, 2$. Letting $b_{j\ell}$ be the $j\ell$th matrix entry of B (relative to the orthonormal bases of V_2 and V_1 in the statement of the theorem). Then the $j\ell$th entry of A (relative to the same bases) is equal to

$$|G|^{-1} \sum_{g \in G} \sum_{\mu=1}^{n_1} \sum_{\nu=1}^{n_2} a^1_{j\mu}(g) b_{\mu\nu} a^2_{\nu\ell}(g^{-1}).$$

Suppose that $\pi_1 \not\simeq \pi_2$. By the corollary to Schur's Lemma, $A = 0$. Since this holds for all choices of B, we may choose B such that $b_{\mu\nu} = \delta_{\mu k}\delta_{\nu m}$, $1 \leq \mu \leq n_1$, $1 \leq \nu \leq n_2$. Then $|G|^{-1} \sum_{g \in G} a^1_{jk}(g) a^2_{m\ell}(g^{-1}) = 0$. Since the matrix coefficients $a^2_{m\ell}(g)$ are chosen relative to an orthonormal basis of V_2 which makes π_2 unitary, it follows that $a_{m\ell}(g^{-1}) = \overline{a^2_{\ell m}(g)}$. Hence $\langle a^1_{jk}, a^2_{\ell m} \rangle = |G|^{-1} \sum_{g \in G} a^1_{jk}(g) \overline{a^2_{\ell m}(g)} = 0$. This proves (1).

Now suppose that $\pi_1 = \pi_2$. In this case, Schur's Lemma implies that $A = \lambda I$ for some scalar λ. Hence $\mathrm{tr}\, A = |G|^{-1} \sum_{g \in G} \mathrm{tr}\,(\pi_1(g) B \pi_1(g)^{-1}) = \mathrm{tr}\, B = n_1\lambda$.

That is, the $j\ell$th entry of the matrix A is equal to

$$|G|^{-1} \sum_{g \in G} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} a^1_{j\mu}(g) b_{\mu\nu} a^1_{\nu\ell}(g^{-1}) = \mathrm{tr}\, B \delta_{j\ell}/n_1.$$

Taking B so that $b_{\mu\nu} = \delta_{\mu k}\delta_{\nu m}$, we have $|G|^{-1} \sum_{g \in G} a^1_{jk}(g) a^1_{m\ell}(g^{-1}) = \delta_{j\ell}\delta_{km}/n_1$. qed

Corollary. *Let π_1 and π_2 be irreducible representations of G such that $\pi_1 \not\simeq \pi_2$. The susbpace of $\mathcal{A}(G)$ spanned by all matrix coefficients of π_1 is orthorgonal to the subspace spanned by all matrix coefficients of π_2.*

Proof. Let $a^1_{jk}(g)$ be as in Theorem 3. Let γ be a basis of the space V_1 of π_1, and let $b_{jk}(g)$ be the jkth entry of the matrix $[\pi_1(g)]_\gamma$. Then there exists a matrix $C \in GL_{n_1}(\mathbb{C})$ such that $[b_{jk}(g)] = C[a_{jk}(g)]_{1\le j,k\le n_1}C^{-1}$ for all $g \in G$ (C is the change of basis matrix from the β to γ). It follows that

$$b_{jk} \in \mathrm{Span}\{\, a^1_{\ell m} \mid 1 \le m, \ell \le n_1 \,\}.$$

Hence the subspace spanned by all matrix coefficients of π_1 coincides with the subspace spanned by the matrix coefficients $a_{\ell m}$, $1 \le \ell, m \le n_1$. Hence the corollary follows from Theorem 3(1). qed

Corollary. *There are finitely many equivalence classes of representations of a finite group G.*

Proof. This is an immediate consequence of the preceding corollary, together with $\dim \mathcal{A}(G) = |G|$. qed

For the remainder of this chapter, let G be a finite group, and let $\{\, \pi_1, \ldots, \pi_r \,\}$ be a complete set of irreducible representations of G, that is, a set of irreducible representations of G having the property that each irreducible representation of G is equivalent to exactly one π_j. Let n_j be the degree of π_j, $1 \le j \le r$. Let $a^j_{\ell m}(g)$ be the ℓmth entry of the matrix of $\pi_j(g)$ relative to an orthonormal basis of the space of π_j with respect to which each matrix of π_j is unitary.

Theorem 4. *The set $\{\, \sqrt{n_j} a^j_{\ell m} \mid 1 \le \ell, m \le n_j,\ 1 \le j \le r \,\}$ is an orthonormal basis of $\mathcal{A}(G)$.*

Proof. According to Theorem 3, the set is orthormal. Hence it suffices to prove that the set spans $\mathcal{A}(G)$. The regular representation $R_\mathcal{A}$ is completely reducible. So $\mathcal{A}(G) = \oplus_{k=1}^t V_k$, where each V_k is an irreducible G-invariant subspace. Fix k. There exists j such that $R_\mathcal{A}\,|_{V_k} \simeq \pi_j$. Choose an orthonormal basis $\beta = \{\, f_1, \ldots, f_{n_j} \,\}$ of V_k such that $[R_\mathcal{A}(g)\,|_{V_k}]_\beta = [a^j_{\ell m}(g)]$, $g \in G$. Then

$$f_\ell(g_0) = (R_\mathcal{A}(g_0)f_\ell)(1) = \sum_{i=1}^{n_j} a^j_{i\ell}(g_0) f_i(1), \qquad 1 \le \ell \le n_j.$$

Hence $f_\ell = \sum_{i=1}^{n_j} c_i a^j_{i\ell}$, with $c_i = f_i(1)$. It follows that

$$V_k \subset \mathrm{Span}\{\, a^j_{\ell m} \mid 1 \le \ell, m \le n_j \,\}.$$

qed

Theorem 5. *Let $1 \leq j \leq r$. The representation π_j occurs as a subrepresentation of $R_{\mathcal{A}}$ with multiplicity n_j.*

Proof. Fix $m \in \{1, \dots, n_j\}$. Let $W_m^j = \mathrm{Span}\{a_{m\ell}^j \mid 1 \leq \ell \leq n_j\}$. Then $\{a_{m\ell}^j \mid 1 \leq \ell \leq n_j\}$ is an orthogonal basis of W_m^j. And W_m^j is orthogonal to $W_{m'}^{j'}$ whenever $j \neq j'$ or $m \neq m'$. Hence $\mathcal{A}(G) = \oplus_{j=1}^r \oplus_{m=1}^{n_j} W_m^j$.

Let g, $g_0 \in G$. Then

$$R_{\mathcal{A}}(g_0)a_{m\ell}^j(g) = a_{m\ell}^j(gg_0) = \sum_{\mu=1}^{n_j} a_{m\mu}^j(g)a_{\mu\ell}^j(g_0), \qquad 1 \leq \ell \leq n_j.$$

It follows that the matrix of $R_{\mathcal{A}}(g_0)$ relative to the basis $\{a_{m\ell}^j \mid 1 \leq \ell \leq n_j\}$ of W_m^j coincides with the matrix of π_j. Therefore the restriction of $R_{\mathcal{A}}$ to the subspace $\oplus_{m=1}^{n_j} W_m^j$ is equivalent to the n_j-fold direct sum of π_j. qed

Corollary. $n_1^2 + \cdots + n_r^2 = |G|$.

Corollary. *$\mathcal{A}(G)$ equals the span of all matrix coefficients of all irreducible representations of G.*

Theorem 6 (Row orthogonality relations for irreducible characters). *Let $\chi_j = \chi_{\pi_j}$, $1 \leq j \leq r$. Then $\langle \chi_k, \chi_j \rangle = \delta_{jk}$.*

Proof.

$$\langle \chi_k, \chi_j \rangle = \sum_{\mu=1}^{n_k} \sum_{\nu=1}^{n_j} \langle a_{\mu\mu}^j, a_{\nu\nu}^j \rangle = \begin{cases} 0, & \text{if } k \neq j \\ \sum_{\mu=1}^{n_j} 1/n_j = 1, & \text{if } k = j \end{cases}.$$

Lemma. *A finite-dimensional representation of a finite group is determined up to equivalence by its character.*

Proof. If m is positive integer, let $m\pi_j = \pi_j \oplus \cdots \oplus \pi_j$, where π_j occurs m times in the direct sum. Let $\pi = m_1\pi_1 \oplus m_2 \oplus \cdots \oplus m_r\pi_r$. Then $\chi_\pi = \sum_{j=1}^r m_j\chi_j$. Let $\pi' = \ell_1\pi_1 \oplus \cdots \oplus \ell_r\pi_r$. We know that $\pi \simeq \pi'$ if and only if $m_j = \ell_j$ for $1 \leq j \leq r$. By linear independence of the functions χ_j, this is equivalent to $\chi_\pi = \chi_{\pi'}$. qed

Lemma. *Let $\pi = m_1\pi_1 \oplus \cdots \oplus m_r\pi_r$. Then $\langle \chi_\pi, \chi_\pi \rangle = \sum_{j=1}^r m_j^2$.*

Corollary. *π is irreducible if and only if $\langle \chi_\pi, \chi_\pi \rangle = 1$.*

A complex-valued function on G is a *class function* if it is constant on conjugacy classes in G. Note that the space of class functions on G is a subspace of $\mathcal{A}(G)$ and the inner product on $\mathcal{A}(G)$ restricts to an inner product on the space of class functions.

Theorem 7. *The set $\{\, \chi_j \mid 1 \leq j \leq r \,\}$ is an orthonormal basis of the space of class functions on G. Consequently the number r of equivalence classes of irreducible representations of G is equal to the number of conjugacy classes in G.*

Proof. By Theorem 6, the set $\{\, \chi_j \mid 1 \leq j \leq r \,\}$ is orthonormal. It suffices to prove that the functions χ_j span the class functions.

Let f be a class function on G. Since $f \in \mathcal{A}(G)$, we can apply Theorem 4 to conclude that

$$f = \sum_{j=1}^{r} \sum_{m,\ell=1}^{n_j} \langle f, \sqrt{n_j} a^j_{m\ell} \rangle \sqrt{n_j} a^j_{m\ell} = \sum_{j=1}^{r} n_j \sum_{m,\ell=1}^{n_j} \langle f, a^j_{m\ell} \rangle a^j_{m\ell}.$$

Next,

$$(*) \qquad f(g) = |G|^{-1} \sum_{g' \in G} f(g' g g'^{-1}) = |G|^{-1} \sum_{j=1}^{r} n_j \sum_{m,\ell=1}^{n_j} \langle f, a^j_{m\ell} \rangle \sum_{g' \in G} a^j_{m\ell}(g' g g'^{-1}).$$

Note that

$$\begin{aligned}
|G|^{-1} \sum_{g' \in G} a^j_{m\ell}(g' g g'^{-1}) &= |G|^{-1} \sum_{g' \in G} \sum_{\mu,\nu=1}^{n_j} a^j_{m\nu}(g') a^j_{\mu\nu}(g) a^j_{\nu\ell}(g'^{-1}) \\
&= \sum_{\mu,\nu=1}^{n_j} \left(|G|^{-1} \sum_{g' \in G} a^j_{m\mu}(g') \overline{a^j_{\ell\nu}(g)} \right) a^j_{\mu\nu}(g) = \sum_{\mu,\nu=1}^{n_j} \langle a^j_{m\mu}, a^j_{\nu\ell} \rangle a^j_{\mu\nu}(g) \\
&= n_j^{-1} \delta_{m\ell} \sum_{\mu=1}^{n_j} a^j_{\mu\mu}(g) = \delta_{m\ell} n_j^{-1} \chi_j(g).
\end{aligned}$$

Substituting into (*) results in

$$f(g) = \sum_{j=1}^{r} \sum_{m,\ell=1}^{n_j} \langle f, a^j_{m\ell} \rangle \delta_{m\ell} \chi_j(g) = \sum_{j=1}^{r} \left(\sum_{\ell=1}^{n_j} \langle f, a^j_{\ell\ell} \rangle \right) \chi_j(g) = \sum_{j=1}^{r} \langle f, \chi_j \rangle \chi_j(g).$$

qed

If $g \in G$, let $|cl(g)|$ be the number of elements in the conjugacy class of g in G.

Theorem 8 (Column orthogonality relations for characters).

$$\sum_{j=1}^{r} \chi_j(g) \overline{\chi_j(g')} = \begin{cases} |G|/|cl(g)|, & \textit{if } g' \textit{ is conjugate to } g \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $g_1, \ldots, g_r$ be representatives for the the distinct conjugacy classes in G. Let $A = [\chi_j(g_k)]_{1 \leq j,k \leq r}$. Let $c_j = |cl(g_j)|$, $1 \leq j \leq r$. Let D be the diagonal matrix with diagonal entries c_j, $1 \le j \le r$. Then

$$\begin{aligned}(ADA^*)_{m\ell} &= \sum_{j=1}^{r}(AD)_{mj}A^*_{j\ell} = \sum_{j=1}^{r}\sum_{t=1}^{r}\chi_m(g_t)D_{tj}\overline{\chi_\ell(g_j)} \\ &= \sum_{j=1}^{r}\chi_m(g_j)c_j\overline{\chi_\ell(g_j)} = \sum_{g\in G}\chi_m(g)\overline{\chi_\ell}(g) = |G|\delta_{m\ell}\end{aligned}$$

Thus $ADA^* = |G|I$. Since $A(DA^*)$ is a scalar matrix, $A(DA^*) = (DA^*)A$. So $DA^*A = |G|I$. That is,

$$|G|\delta_{m\ell} = \sum_{j=1}^{r}(DA^*)_{mj}A_{j\ell} = \sum_{j=1}^{r}c_j\overline{\chi_j(g_m)}\chi_j(g_\ell).$$

qed

Example: Let G be a nonabelian group of order 8. Because G is nonabelian, we have $Z(G) \ne G$, where $Z(G)$ is the centre of G. Because G is a 2-group, $Z(G) \ne \{1\}$. If $|Z(G)| = 4$, then $|G/Z(G)| = 2$, so $G/Z(G)$ is cyclic. That is, $G/Z(G) = \langle gZ(G)\rangle$, $g \in G$. Hence $G = \langle Z(G) \cup \{x\}\rangle$. But this implies that G is abelian, which is impossible. Therefore $|Z(G)| = 2$. Now $|G/Z(G)| = 4$ implies $G/Z(G)$ is abelian. Combining G nonabelian and $G/Z(G)$ abelian, we get $G_{der} \subset Z(G)$. We cannot have G_{der} trivial, since G is nonabelian. So we have $G_{der} = Z(G)$. Now, we saw above that $G/Z(G)$ cannot be cyclic. Thus

$$G/Z(G) = G/G_{der} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Suppose that χ is a linear character of G (that is, a one-dimensional representation). Then $\chi \,|\, G_{der} \equiv 1$, because $\chi(g_1g_2g_1^{-1}g_2^{-1}) = \chi(g_1)\chi(g_2)\chi(g_1)^{-1}\chi(g_2)^{-1} = 1$. Now G_{der} is a normal subgroup of G. So we can consider χ as a linear character of G/G_{der}. Now, in view of results on tensor products of representations, we know that G/G_{der} has 4 irreducible (one-dimensional) representations, each one being the tensor product of two characters of $\mathbb{Z}/2\mathbb{Z}$. Hence

$$1^2 + 1^2 + 1^2 + 1^2 + n_5^2 + \cdots + n_r^2 = |G| = 8,$$

with $n_j \ge 2$, $j \ge 5$. It follows that $r = 5$ and $n_5 = 2$.

Since $G_{der} = Z(G)$ has order 2, there are two conjugacy classes consisting of single elements. There are 5 conjugacy classes altogether. Let a, b, and c be the orders of the conjugacy classes containing more than 1 element. Then $2 + a + b + c = 8$ implies $a = b = c = 2$. Let x_1, x_2 and x_3 be representatives of the conjugacy classes containing 2 elements. Let z be the nontrivial element of $Z(G)$. Then 1, y, x_1, x_2, x_3 are representatives of the 5 conjugacy classes. The character table of G takes the form:

	1	1	2	2	2
	1	y	x_1	x_2	x_3
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	$\chi_5(y)$	$\chi_5(x_1)$	$\chi_5(x_2)$	$\chi_5(x_3)$

Using column orthogonality relations, we see that

$$0 = \sum_{j=1} \chi_j(y)\overline{\chi_j(1)} = 4 + 2\chi_5(y),$$

implying $\chi_5(y) = -2$. And

$$0 = \sum_{j=1}^{5} \chi_j(x_k)\overline{\chi_j(1)} = \sum_{j=1}^{4} \chi_j(x_k) + 2\chi_5(x_k) = 2\chi_5(x_k),$$

implying $\chi_5(x_k) = 0$, $1 \leq k \leq 3$.

Note that (up to isomorphism) there are two nonabelian groups of order 8, the dihedral group D_8, and the quaternion group Q_8. We see from this example that both groups have the same character table.

Exercises:

1. Using orthogonality relations, prove that if (π_j, V_j) is an irreducible representation of a finite group G_j, $j = 1, 2$, then $\pi_1 \otimes \pi_2$ is an irreducible representation of $G_1 \times G_2$. Then prove that every irreducible representation of $G_1 \times G_2$ arises in this way.
2. Let D_{10} be the dihedral group of order 10.
   a) Describe the conjugacy classes in D_{10}.
   b) Compute the character table of D_{10}.
3. Let B be the upper triangular Borel subgroup in $GL_3(\mathbb{F}_p)$, where $\mathbb{F}_p$ is a finite field containing p elements, p prime. Let N be the subgroup of B consisting of the upper triangular matrices having ones on the diagonal.
   a) Identify the set of one-dimensional representations of B.
   b) Suppose that π is an irreducible representation of B, having the property that $\pi(x)v \neq v$ for some $x \in N$ and $v \in V$. Show that $\pi\,|_N$ is a reducible representation of N. (*Hint*: One approach is to start by considering the action of the centre of N on V.)
4. Suppose that G is a finite group. Let $n \in \mathbb{N}$. Define $\theta_n : G \to \mathbb{N}$ by

$$\theta_n(g) = |\{\, h \in G \mid h^n = g \,\}|, \qquad g \in G.$$

Let χ_i, $1 \leq i \leq r$ be the distinct irreducible (complex) characters of G. Set

$$\nu_n(\chi_i) = |G|^{-1} \sum_{g \in G} \chi_i(g^n).$$

Prove that $\theta_n = \sum_{1 \leq i \leq r} \nu_n(\chi_i)\chi_i$.

5. Let (π, V) be an irreducible representation of a finite group G. Prove *Burnside's Theorem*:

$$\operatorname{Span}\{\, \pi(g) \mid g \in G \,\} = \operatorname{End}_{\mathbb{C}}(V).$$

(*Hint*: Of course the theorem is equivalent to $\operatorname{Span}\{\, [\pi(g)]_\beta \mid g \in G \,\} = M_{n \times n}(\mathbb{C})$, where β is a basis of V. This can be proved using properties of matrix coefficients of π (Theorem 3)).

6. Let (π, V) be a finite-dimensional representation of a finite group G. Let

$$\begin{aligned} W_{ext} &= \operatorname{Span}\{\, v \otimes v \mid v \in V \,\} \subset V \otimes V \\ W_{sym} &= \operatorname{Span}\{\, v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1,\, v_2 \in V \,\} \subset V \otimes V \end{aligned}.$$

   a) Prove that W_{ext} and W_{sym} are G-invariant subspaces of $V \otimes V$ (considered as the space of the inner tensor product representation $\pi \otimes \pi$ of G).
   b) Let $(\wedge^2\pi, \wedge^2 V)$ be the quotient representation, where $\wedge^2 V = (V \otimes V)/W_{ext}$. Then $\wedge^2\pi$ is called the exterior square of π. Compute the character $\chi_{\wedge^2\pi}$.
   c) Let $(\operatorname{Sym}^2\pi, \operatorname{Sym}^2 V)$ be the quotient representation, where $\operatorname{Sym}^2 V = (V \otimes V)/W_{sym}$. Then $\operatorname{Sym}^2\pi$ is called the symmetric square of π. Prove that (the inner tensor product) $\pi \otimes \pi$ is equivalent to $\wedge^2\pi \oplus \operatorname{Sym}^2\pi$.

7. Let (π, V) be the permutation representation associated to an action of a finite group G on a set X. Show that $\chi_\pi(g)$ is equal to the number of elements of X that are fixed by g.

8. Let f be a function from a finite group G to the complex numbers. For each finite dimensional representation (π, V) of G, define a linear operator $\pi(f) : V \to V$ by $\pi(f)v = \sum_{g \in G} f(g)\pi(g)v$, $v \in V$. Prove that $\pi(f) \in \operatorname{Hom}_G(\pi, \pi)$ for all finite-dimensional representations (π, V) of G if and only if f is a class function.

9. A (finite-dimensional) representation (π, V) of a finite group is called *faithful* if the homomorphism $\pi : G \to GL(V)$ is injective (one-to-one). Prove that every irreducible representation of G occurs as a subrepresentation of the set of representations

$$\{\, \pi, \pi \otimes \pi, \pi \otimes \pi \otimes \pi, \pi \otimes \pi \otimes \pi \otimes \pi, ... \,\}.$$

10. Show that the character of any irreducible representation of dimension greater than 1 takes the value 0 on some conjugacy class.

11. A finite-dimensional representation (π, V) of a finite group is *multiplicity-free* if each irreducible representation occurring in the decomposition of π into a direct sum of irreducibles occurs exactly once. Prove that π is multiplicity-free if and only if the ring $\mathrm{Hom}_G(\pi,\pi)$ is commutative.

2.2. Character values as algebraic integers, degree of an irreducible represnetation divides the order of the group

A complex number z is an *algebraic integer* if $f(z)=0$ for some monic polynomial f having integer coefficients. The proof of the following lemma is found in many standard references in algebra.

Lemma.
(1) Let $z \in \mathbb{C}$. The following are equivalent:
- *(a) z is an algebraic integer*
- *(b) z is algebraic over $\mathbb{Q}$ and the minimal polynomial of z over $\mathbb{Q}$ has integer coefficients.*
- *(c) The subring $\mathbb{Z}[z]$ of $\mathbb{C}$ generated by $\mathbb{Z}$ and z is a finitely generated $\mathbb{Z}$-module.*

(2) The algebraic integers form a ring. The only rational numbers that are algebraic integers are the elements of $\mathbb{Z}$.

Lemma. *Let π be a finite-dimensional representation of G and let $g \in G$. Then $\chi_\pi(g)$ is an algebraic integer.*

Proof. Because G is finite, we must have $g^k = 1$ for some positive integer k. Hence $\pi(g)^k = 1$. It follows that every eigenvalue of $\pi(g)$ is a kth root of unity. Clearly a kth root of unity is an algebraic integer. Since $\chi_\pi(g)$ is the sum of the eigenvalues of $\pi(g)$, it follows from part (2) of the above lemma that $\chi_\pi(g)$, being a sum of algebraic integers, is an algebraic integer. qed

Lemma. *Let $g_1, \dots g_r$ be representatives of the conjugacy classes in G. Let c_j be the number of elements in the conjugacy class of g_j, $1 \le j \le r$. Define $f_i \in \mathcal{A}(G)$ by $f_i(g) = c_j\chi_i(g)/\chi_i(1)$, if g is conjugate to g_j. Then $f_i(g)$ is an algebraic integer for $1 \le i \le r$ and all $g \in G$.*

Proof. Let $g_0 \in G$. As g ranges over the elements in the conjugacy class of g_j, so does $g_0 g g_0^{-1}$. Therefore

$$\sum_{g\in \mathrm{cl}(g_j)} \pi_i(g) = \sum_{g\in \mathrm{cl}(g_j)} \pi_i(g_0)\pi_i(g)\pi_i(g_0)^{-1} = \pi_i(g_0)\left(\sum_{g\in \mathrm{cl}(g_j)} \pi_i(g)\right)\pi_i(g_0)^{-1}.$$

So $T := \sum_{g\in \mathrm{cl}(g_j)} \pi_i(g)$ belongs to $\mathrm{Hom}_G(\pi_i, \pi_i)$. By irreducibility of π_i, $T = zI$ for some $z \in \mathbb{C}$. Note that

$$\mathrm{tr}\, T = \sum_{g\in \mathrm{cl}(g_j)} \chi_i(g) = c_j\chi_i(g_j) = z\chi_i(1),$$

so $z = c_j\chi_i(g_j)/\chi_i(1)$.

Let g be an element of $\mathrm{cl}(g_s)$. Let a_{ijs} be the number of ordered pairs $(g', \hat{g})$ such that $g'\hat{g} = g$. Note that a_{ijs} is independent of the choice of $g \in \mathrm{cl}(g_s)$.

$$\begin{aligned}(c_i\chi_t(g_i)/\chi_t(1))\,(c_j\chi_t(g_j)/\chi_t(1))\,I &= \left(\sum_{g'\in \mathrm{cl}(g_i)} \pi_t(g')\right)\left(\sum_{\hat{g}\in \mathrm{cl}(g_j)} \pi_t(\hat{g})\right)\\ &= \sum_{g'\in \mathrm{cl}(g_i)}\ \sum_{\hat{g}\in \mathrm{cl}(g_j)} \pi_t(g'\hat{g}) = \sum_{s=1}^{r}\ \sum_{g\in \mathrm{cl}(g_s)} a_{ijs}\pi_t(g)\\ &= \left(\sum_{s=1}^{r} a_{ijs}c_s\chi_t(g_s)/\chi_t(1)\right) I\end{aligned}$$

Hence

$$(c_i\chi_t(g_i)/\chi_t(1))\,(c_j\chi_t(g_j)/\chi_t(1)) = \sum_{s=1}^{r} a_{ijs}c_s\chi_t(g_s)/\chi_t(1).$$

This implies that the subring of $\mathbb{C}$ generated by the scalars $c_s\chi_t(g_s)/\chi_t(1)$, $1 \le s \le r$, and $\mathbb{Z}$ is a finitely-generated $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a principal ideal domain, any submodule of a finitely-generated $\mathbb{Z}$-module is also a finitely-generated $\mathbb{Z}$-module, the submodule $\mathbb{Z}[c_i\chi_t(g_i)/\chi_t(1)]$ is finitely-generated. Applying part (2) of one of the above lemmas, the result of this lemma follows. qed

Theorem 9. *n_j divides $|G|$, $1 \le j \le r$.*

Proof. Note that

$$\begin{aligned}|G|/\chi_i(1) &= |G|\langle\chi_i, \chi_i\rangle/\chi_i(1)\\ &= \sum_{j=1}^{r} c_j\chi_i(g_j)\overline{\chi_i(g_j)}/\chi_i(1) = \sum_{j=1}^{r} (c_j\chi_i(g_j)/\chi_i(1))\,\overline{\chi_i(g_j)}.\end{aligned}$$

Because the right side above is an algebraic integer, the left side is a rational number which is also an algebraic integer, hence it is an integer. qed

2.3. Decomposition of finite-dimensional representations

In this section we describe how to decompose a representation π into a direct sum of irreducible representations, assuming that the functions $a^j_{m\ell}$ are known.

Lemma. *Let (π, V) be a finite-dimensional representation of G. For $1 \leq k \leq r$, $1 \leq j, \ell \leq n_k$, define $P^k_{j\ell} : V \to V$ by*

$$P^k_{j\ell} = n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{j\ell}(g)} \pi(g).$$

Then
(1) $\pi(g) P^k_{j\ell} = \sum_{\nu=1}^{n_k} a^k_{\nu j}(g) P^k_{\nu\ell}$ and $P^k_{j\ell}\pi(g) = \sum_{\nu=1}^{n_k} a_{\ell\nu}(g) P^k_{j\nu}, \quad g \in G$.
(2) $P^k_{j\ell} P^{k'}_{\mu\nu} = P^k_{j\nu}$ if $k = k'$ and $\ell = \mu$, and equals 0 otherwise.
(3) $(P^k_{j\ell})^* = P^k_{\ell j}$.

Proof. For the first part of (1),

$$\begin{aligned}\pi(g) P^k_{j\ell} &= |G|^{-1} n_k \sum_{g' \in G} \overline{a^k_{j\ell}(g')} \pi(gg') = |G|^{-1} n_k \sum_{g' \in G} \overline{a^k_{j\ell}(g^{-1}g')} \pi(g') \\ &= |G|^{-1} n_k \sum_{g' \in G} \sum_{\nu=1}^{n_k} \overline{a^k_{j\nu}(g^{-1}) a^k_{\nu\ell}(g')} \pi(g') = \sum_{\nu=1}^{n_k} a^k_{\nu j}(g) P^k_{\nu\ell}\end{aligned}$$

The second part of (1) is proved similarly.

For (2),

$$\begin{aligned}P^k_{j\ell} P^{k'}_{\mu\nu} &= n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{j\ell}(g)} \pi(g) P^{k'}_{\mu\nu} = n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{j\ell}(g)} \sum_{t=1}^{n_k} a^{k'}_{t\mu}(g) P^{k'}_{t\nu} \\ &= n_k \sum_{t=1}^{n_k} \langle a^{k'}_{t\mu}, a^k_{j\ell} \rangle P^{k'}_{t\nu} = \delta_{kk'} \delta_{\ell\mu} P^k_{j\nu}.\end{aligned}$$

For (3),

$$(P^k_{j\ell})^* = n_k |G|^{-1} \sum_{g \in G} a^k_{j\ell}(g) \pi(g^{-1}) = n_k |G|^{-1} \sum_{g \in G} \overline{a^k_{\ell j}(g^{-1})} \pi(g^{-1}) = P^k_{\ell j}.$$

Set $V^k_j = P^k_{jj}(V)$. Note that , since $(P^k_{jj})^* = P^k_{jj} = (P^k_{jj})^2$, P^k_{jj} is the orthogonal projection of V on V^k_j. From property (2) of the above lemma, it follows that $V^k_j \perp V^{k'}_{j'}$ if $k \neq k'$ or $j \neq j'$.

Let $W = \oplus_{k=1}^r \oplus_{j=1}^{n_k} V^k_j$. Note that $P^k_{j\ell}(V) = P^k_{jj} P^k_{j\ell}(V) \subset V^k_j \subset W$. Fix $v_0 \in W^\perp$. Then

$$0 = \langle v_0, P^k_{j\ell}(v) \rangle = n_k |G|^{-1} \sum_{g \in G} a^k_{j\ell}(g) \langle v_0, \pi(g) v \rangle_V = n_k \langle a^k_{j\ell}, f \rangle_{\mathcal{A}(G)},$$

where $f(g) = \langle v_0, \pi(g) v \rangle$, $g \in G$. It follows that $f \in \mathcal{A}(G)^\perp$. Hence $f(g) = 0$ for all $g \in G$. Setting $v = v_0$ and $g = 1$, we have $\langle v_0, v_0 \rangle_V = 0$. Thus $v_0 = 0$. That is, $W = V$.

Next, note that $P^k_{j\ell}V^{k'}_t = 0$ if $k \neq k'$ or $t \neq \ell$, by part (2) of the above lemma.

Let $v \in V^k_\ell$. Then $v = P^k_{\ell\ell}(v')$ for some $v' \in V$. Now $v = P^k_{\ell\ell}(v') = P^k_{\ell\ell}(P^k_{\ell\ell}(v')) = P^k_{\ell\ell}(v)$, so we have

$$P^k_{j\ell}(v) = P^k_{j\ell}P^k_{\ell\ell}(v) = P^k_{jj}P^k_{j\ell}(v) \subset V^k_j.$$

Thus $P^k_{j\ell}(V^k_\ell) \subset V^k_j$. Now

$$V^k_\ell = P^k_{\ell\ell}V^k_\ell = P^k_{\ell j}P^k_{j\ell}V^k_\ell \subset P^k_{\ell j}V^k_j \subset V^k_\ell.$$

Hence we have $P^k_{\ell j}V^k_j = V^k_\ell$. Let v, $v' \in V^k_\ell$. Then

$$\langle P^k_{j\ell}(v), P^k_{j\ell}(v')\rangle = \langle (P^k_{j\ell})^* P^k_{j\ell}(v), v'\rangle = \langle P^k_{\ell j}P^k_{j\ell}(v), v'\rangle = \langle P^k_{\ell,\ell}(v), v'\rangle = \langle v, v'\rangle.$$

We have shown

Lemma. *$P^k_{j\ell}$ is an isometry of V^k_ℓ onto V^k_j.*

Choose an orthonormal basis $e^k_{11}, e^k_{21}, \ldots, e^k_{r_k,1}$ of $V^k_1 = P^k_{11}(V)$. Then $e^k_{j\ell} := P^k_{\ell 1}(e^k_{j1})$, $1 \leq j \leq r_k$ is an orthonormal basis of V^k_ℓ. It follows that the set

$$\{\, e^k_{j\ell} \mid 1 \leq j \leq r_k,\, 1 \leq \ell \leq n_k,\, 1 \leq k \leq r \,\}$$

is an orthonormal basis of V. Set $Y^k_j = \mathrm{Span}\{\, e^k_{j\ell} \mid 1 \leq \ell \leq n_k \,\}$. If $g \in G$, then

$$\pi(g)e^k_{j\ell} = \pi(g)P^k_{\ell 1}(e^k_{j1}) = \sum_{\nu=1}^{n_k} a^k_{\nu\ell}(g)P^k_{\nu 1}e^k_{j1} = \sum_{\nu=1}^{n_k} a^k_{\nu\ell}(g)e^k_{j\nu}.$$

This shows that Y^k_j is G-invariant and has the matrix $[a^k_{\nu\ell}(g)]_{\{1\leq\nu,\ell\leq n_k\}}$ relative to the given orthonormal basis of Y^k_j. This implies that $\pi\,|_{Y^k_j} \simeq \pi_k$, $1 \leq j \leq r_k$. Now $V = \oplus_{1\leq k\leq r} \oplus_{1\leq j\leq r_k} Y^k_j$, so we have decomposed π into a direct sum of irreducible representations. This decomposition is not unique.

Set $Y^k = \oplus_{j=1}^{r_k} Y^k_j$. Now $\{\, e^k_{j\ell} \mid 1 \leq j \leq r_k,\, 1 \leq \ell \leq n_k \,\}$ is an orthonormal basis of Y^k, and $\pi\,|_{Y^k} \simeq r_k\pi_k$. Because $P^k_{\ell\ell}$ is the orthogonal projection of V on V^k_ℓ and $Y^k = \oplus_{\ell=1}^{n_k} V^k_\ell$, it follows that $P^k := \sum_{\ell=1}^{n_k} P^k_{\ell\ell}$ is the orthogonal projection of V on Y^k. Looking at the definitions, we see that this orthogonal projection P^k is defined by

$$P^k = \sum_{\ell=1}^{n_k} n_k|G|^{-1}\sum_{g\in G}\overline{a^k_{\ell\ell}(g)}\pi(g) = n_k|G|^{-1}\sum_{g\in G}\overline{\chi_k(g)}\pi(g).$$

Suppose that W is a G-invariant subspace of V such that $\pi\,|_W$ is equivalent to a direct sum of π_k with itself some number of times. Let $\{\, v_1, \ldots, v_{n_k} \,\}$ be an orthonormal basis of an irreducible G-invariant subspace of W, chosen so that the matrix of the restriction of $\pi(g)$ to this subspace is $[a^k_{\ell m}(g)]$. Then

$$\begin{aligned}
P^k(v_j) &= n_k|G|^{-1}\sum_{g\in G}\overline{\chi_k(g)}\pi(g)v_j = n_k|G|^{-1}\sum_{g\in G}\sum_{\ell=1}^{n_k} a^k_{\ell\ell}(g)\sum_{\mu=1}^{n_k} a^k_{\mu j}(g)v_\mu \\
&= n_k|G|^{-1}\sum_{g\in G}\sum_{\ell,\mu=1}^{n_k}\overline{a^k_{\ell\ell}(g)}a^k_{\mu j}(g)v_\mu = n_k\sum_{\ell,\mu=1}^{n_k}\langle a^k_{\mu j}, a^k_{\ell\ell}\rangle v_\mu \\
&= n_k n_k^{-1} v_j = v_j
\end{aligned}$$

Therefore $P^k\,|\,W$ is the identity. Because P^k is the orthogonal projection of V on Y^k, we know that $P^k(v) = v$ if and only if $v \in Y^k$. It follows that $W \subset Y^k$. Now we may conclude that if we have a G-invariant subspace of V such that the restriction of π to that subspace is equivalent to $r_k\pi_k$, then that subspace must equal Y^k.

Lemma. *The subspaces Y^k are unique.*

The subspace Y^k is called the π_k-*isotypic* subspace of V. It is the (unique) largest subspace of V on which the restriction of π is a direct sum of representations equivalent to π_k. Of course, we will have $Y^k = \{0\}$ if no irreducible constituent of π is equivalent to π_k.

2.4. Induced representations

One method of producing representations of a finite group G is the process of induction: given a representation of a subgroup of G, we can define a related representation of G. Let (π, V) be a (finite-dimensional) representation of a subgroup H of G. Define

$$\mathcal{V} = \{\, f : G \to V \mid f(hg) = \pi(h)f(g),\ h \in H,\ g \in G\,\}.$$

We define the induced representation $i_H^G\pi = \mathrm{Ind}_H^G(\pi)$ by $(i_H^G\pi(g)f)(g_0) = f(g_0g)$, g, $g_0 \in G$. Observe that if $h \in H$, then

$$(i_H^G\pi(g)f)(hg_0) = f(hg_0g) = \pi(h)f(g_0g) = \pi(h)(i_H^G\pi(g)f)(g_0).$$

It follows from the definitions that the degree of $i_H^G\pi$ equals $|G||H|^{-1}$ times the degree of π. Let $\langle\cdot,\cdot\rangle_V$ be any inner product on V. Set $\langle f_1, f_2\rangle_{\mathcal{V}} = |G|^{-1}\sum_{g\in G}\langle f_1(g), f_2(g)\rangle_V$, f_1, $f_2 \in \mathcal{V}$. It is easy to check that this defines an inner product on $\mathcal{V}$ with respect to which $i_H^G\pi$ is unitary.

Example If $H = \{1\}$ and π is the trivial representation of H, then $i_H^G\pi$ is the right regular representation of G.

The Frobenius character formula expresses the character of $i_H^G\pi$ in terms of the character of π.

Theorem 10 (Frobenius character formula). *Let (π, V) be a representation of a subgroup H of G. Fix $g \in G$. Let $h_1, \dots, h_m$ be representatives for the conjugacy classes in H which lie inside the conjugacy class of g in G. Then*

$$\chi_{i_H^G \pi}(g) = |G||H|^{-1} \sum_{i=1}^{m} |cl_H(h_i)||cl_G(g)|^{-1} \chi_\pi(h_i).$$

Proof. Let $g \in G$. Define $T : \mathcal{V} \to \mathcal{V}$ by $T = \sum_{g' \in \mathrm{cl}(g)} i_H^G \pi(g')$. Note that $\mathrm{tr}\, T = |cl(g)| \chi_{i_H^G \pi}(g)$.

Let $\beta = \{ v_1, \dots, v_n \}$ be an orthonormal basis of V such that each $[\pi(h)]_\beta$, $h \in H$, is a unitary matrix. For each $j \in \{1, \dots, n\}$, define

$$f_j(g) = \begin{cases} |G|^{1/2}|H|^{-1/2}\pi(h)v_j, & \text{if } g = h \in H \\ 0, & \text{if } g \notin H. \end{cases}$$

Then $f_j(h_0 g) = |G|^{1/2}|H|^{-1/2}\pi(h_0 h)v = \pi(h_0) f_j(h) v$, if $g = h \in H$, and $f_j(h_0 g) = f_j(g) = 0$ if $g \notin H$. Thus $f_j \in \mathcal{V}$. Note that

$$\begin{aligned} \langle f_j, f_k \rangle_{\mathcal{V}} &= |G|^{-1} \sum_{g \in G} \langle f_j(g), f_k(g) \rangle_V = |H|^{-1} \sum_{h \in H} \langle \pi(h) v_j, \pi(h) v_k \rangle_V \\ &= |H|^{-1} \sum_{h \in H} \langle v_j, v_k \rangle_V = \langle v_j, v_k \rangle_V = \delta_{jk}. \end{aligned}$$

Therefore $\{ f_1, \dots, f_n \}$ is an orthonormal set in $\mathcal{V}$. Pick representatives $g_1, \dots, g_\ell$ of the cosets in $H \backslash G$ (that is, of the right H cosets in G). Then

$$\begin{aligned} \langle i_H^G \pi(g_i) f_j, i_H^G \pi(g_k)) f_s \rangle_{\mathcal{V}} &= |G|^{-1} \sum_{g \in G} \langle f_j(g g_i), f_s(g g_k) \rangle_V = |G|^{-1} \sum_{h \in H} \langle f_j(h), f_s(h g_i^{-1} g_k) \rangle_V \\ &= \delta_{ik} |G|^{-1} \sum_{h \in H} \langle \pi(h) f_j(1), \pi(h) f_s(1) \rangle_V \delta_{ik} \langle f_j, f_s \rangle_V = \delta_{ik} \delta_{js} \end{aligned}$$

Hence $\{ i_H^G \pi(g_i) f_j \}_{1 \le i \le \ell,\, 1 \le j \le n}$ is an orthonormal set. Since $\dim \mathcal{V} = |G||H|^{-1} \dim V = \ell n$, it is an orthonormal basis of $\mathcal{V}$ (with respect to which $i_H^G \pi$ is a unitary representation).

The kkth entry of T with respect to this basis is

$$\langle T(k\text{th basis element}), k\text{th basis element} \rangle_{\mathcal{V}}.$$

Therefore

$$\mathrm{tr}\, T = \sum_{i=1}^{\ell} \sum_{j=1}^{n} \langle T(i_H^G \pi(g_i) f_j), i_H^G \pi(g_i) f_j \rangle_{\mathcal{V}}.$$

As g' ranges over the conjugacy class $\mathrm{cl}(g)$, $g_i g' g_i^{-1}$ also ranges over $\mathrm{cl}(g)$. Hence

$$(i_H^G \pi)(g_i) T \, i_H^G \pi(g_i^{-1}) = T,$$

and the above expression for $\operatorname{tr} T$ becomes

$$\sum_{i=1}^{\ell}\sum_{j=1}^{n}\langle (i_H^G\pi)(g_i)Tf_j, (i_H^G\pi)(g_0)f_j\rangle_{\mathcal{V}} = \ell\sum_{j=1}^{n}\langle Tf_j, f_j\rangle_{\mathcal{V}}.$$

Now we rewrite each $\langle Tf_j, f_j\rangle_{\mathcal{V}}$ using the definitions of f_j and T.

$$\begin{aligned}\langle Tf_j, f_j\rangle_{\mathcal{V}} &= \sum_{g'\in \mathrm{cl}(g)} |G|^{-1}\sum_{g_0\in G}\langle f_j(g_0g'), f_j(g_0)\rangle_V = \sum_{g'\in\mathrm{cl}(g)\cap H}|G|^{-1}\sum_{h\in H}\langle f_j(hg'), f_j(h)\rangle_V\\ &= \sum_{s=1}^{m}|\mathrm{cl}_H(h_s)||G|^{-1}\sum_{h\in H}\langle f_j(hh_s), f_j(h)\rangle_V\\ &= |H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\sum_{h\in H}\langle \pi(h)\pi(h_s)v_j, \pi(h)v_j\rangle_V = \sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\langle\pi(h_s)v_j, v_j\rangle_V\end{aligned}$$

It follows that

$$\operatorname{tr} T = |G||H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\sum_{j=1}^{n}\langle \pi(h_s)v_j, v_j\rangle_V = |G||H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_H(h_s)|\chi_\pi(h_s).$$

Thus $\chi_{i_H^G\pi}(g) = |G||H|^{-1}\sum_{s=1}^{m}|\mathrm{cl}_G(g)|^{-1}|\mathrm{cl}_H(h_s)|\chi_\pi(h_s)$. qed

Example: Applying the Frobenius character formula with π the trivial representation of the trivial subgroup of G, we see that the character of the regular representation of G vanishes on all elements except for the identity element.

The inner product on $\mathcal{A}(G)$ restricts to an inner product on the space $\mathcal{C}(G)$ of class functions on G. When we wish to identify the fact that we are taking the inner product on $\mathcal{A}(G)$, we will sometimes write $\langle\cdot,\cdot\rangle_G$. Let H be a subgroup of G. We may view i_H^G as a map from $\mathcal{C}(H)$ to $\mathcal{C}(G)$, mapping χ_π to $i_H^G(\chi_\pi) := \chi_{i_H^G\pi}$, for π any irreducible representation of H. As the characters of the irreducible representations of H form a basis of $\mathcal{C}(H)$, the map extends by linearity to all of $\mathcal{C}(H)$. We can define a linear map r_G^H from $\mathcal{C}(G)$ to $\mathcal{C}(H)$ by restricting a class function on G to H. The next result, Frobenius Reciprocity, tells us that r_G^H is the adjoint of the map i_H^G.

If (π, V) is a representation of G and τ is an irreducible representation of G, the *multiplicity* of τ in π is defined to be the number of times that τ occurs in the decomposition of π as a direct sum of irreducible representations of G. This multiplicity is equal to $\langle\chi_\tau,\chi_\pi\rangle_G = \langle\chi_\pi,\chi_\tau\rangle_G$.

Theorem 11 (Frobenius Reciprocity). *Let (π, V) be an irreducible representation of H and let (τ, W) be an irreducible representation of G. Then $\langle\chi_\tau, \chi_{i_H^G\pi}\rangle_G = \langle r_G^H\chi_\tau, \chi_\pi\rangle_H$.*

Proof. Let π and τ be as in the statement of the theorem. Let $g \in G$ be such that $\mathrm{cl}_G(g) \cap H \neq \emptyset$. Choose $h_1, \dots, h_m$ as in the previous theorem. Then, using $\chi_\tau(g) = \chi_\tau(h_i)$, $1 \leq i \leq m$,

$$\begin{aligned}\chi_{i_H^G \pi}(g)\overline{\chi_\tau(g)} &= \left(|G||H|^{-1} \sum_{i=1}^{m} |\mathrm{cl}_H(h_i)||\mathrm{cl}_G(h_i)|^{-1} \chi_\pi(h_i)\right) \overline{\chi_\tau(g)} \\ &= |G||H|^{-1} \sum_{i=1}^{m} |\mathrm{cl}_H(h_i)||\mathrm{cl}_G(h_i)|^{-1} \chi_\pi(h_i)\overline{\chi_\tau(h_i)}\end{aligned}$$

Now when evaluating $\langle \chi_{i_H^G \pi}, \chi_\tau \rangle_G = |G|^{-1} \sum_{g \in G} \chi_{i_H^G \pi}(g)\overline{\chi_\tau(g)}$, we need only sum over those $g \in G$ such that $\mathrm{cl}(g) \cap H \neq \emptyset$. Then

$$\langle \chi_{i_H^G \pi}, \chi_\tau \rangle_G = |H|^{-1} \sum_{h \in H} \chi_\pi(h)\overline{\chi_\tau(h)} = \langle \chi_\pi, r_{G^H}(\chi_\tau) \rangle_H.$$

qed

Corollary (Transitivity of induction). *Suppose that $K \subset H$ are subgroups of G. Let (π, V) be a representation of K. Then $i_K^G \pi = i_H^G(i_K^H \pi)$.*

Proof. Note that it follows from the definitions that $r_G^K = r_H^K \circ r_G^H$. Taking adjoints, we have

$$i_K^G = (r_G^K)^* = (r_G^H)^* \circ (r_H^K)^* = i_H^G \circ i_K^H.$$

qed

Lemma. *Let (π, V) be a representation of a subgroup H of G. Fix $g \in G$. Let π' be the representation of gHg^{-1} defined by $\pi'(ghg^{-1}) = \pi(h)$, $h \in H$. Then $i_H^G \pi \simeq i_{gHg^{-1}}^G \pi'$.*

Proof. Let f be in the space of $i_H^G \pi$. Set $(Af)(g_0) = f(g^{-1}g_0)$, $g_0 \in G$. Let $h \in H$. Then

$$(Af)(ghg^{-1}g_0) = f(hg^{-1}g_0) = \pi(h)(Af)(g_0) = \pi'(ghg^{-1})(Af)(g_0).$$

Therefore Af belongs to the space of $i_{gHg^{-1}}^G \pi'$. It is clear that A is invertible. Note that

$$i_{gHg^{-1}}(g_1)(Af)(g_0) = (Af)(g_0 g_1) = f(g^{-1}g_0 g_1) = (A\, i_H^G \pi(g_1) f)(g_0).$$

qed

Let (π, V) be a finite-dimensional representation of a subgroup H of G. Let K be a subgroup of G, and let $g \in G$. Then $K \cap gHg^{-1}$ is a subgroup of G. Define a representation π^g of this subgroup by $\pi^g(k) = \pi(g^{-1}kg)$, $k \in K \cap gHg^{-1}$. Let $h \in H$. Then $K \cap ghH(gh)^{-1} = K \cap gHg^{-1}$ and $\pi^{gh}(k) = \pi(h^{-1}g^{-1}kgh) = \pi(h)^{-1}\pi^g(k)\pi(h)$. Hence $\pi^{gh} \simeq \pi^g$. This certainly implies that $i_{K \cap ghHhg^{-1}}^K \pi^{gh} \simeq i_{K \cap gHg^{-1}}^K \pi^g$.

Changing notation slightly, we see that the above lemma tells us that $i_{K \cap gHg^{-1}}^K \pi^g \simeq i_{K \cap kgHg^{-1}k^{-1}} \pi^{kg}$. We now know that the equivalence class of $i_{K \cap gHg^{-1}}^K \pi^g$ is independent of the choice of element g inside its K-H-double coset (that is, we may replace g by kgh, $k \in K$, $h \in H$, without changing the equivalence class).

Theorem 12. *(Mackey) K and H be subgroups of G, and let (π, V) be a representation of H. Then*
$$(i_H^G\pi)_K = r_G^K(i_H^G\pi) \simeq \bigoplus_{g\in K\backslash G/H} i_{K\cap gHg^{-1}}^K(\pi^g).$$

Proof. Let $\rho = i_H^G\pi$. Let $\mathcal{V}$ be the space of ρ. Define a map $A : V \to \mathcal{V}$ by $(Av)(g) = \begin{cases} \pi(h)v, & \text{if } g = h \in H \\ 0, & \text{if } g \notin H. \end{cases}$, $v \in V$. Let $v_1 \in V$, and g_1, $g_2 \in G$. Then $\rho(g_1)Av_1 \in \rho(g_2)AV$ if and only if $\rho(g_2^{-1}g_1)Av_1 = Av_2$ for some $v_2 \in V$. Now $\rho(g_2^{-1}g_1)Av_1$ is supported in $Hg_1^{-1}g_2$ and Av_2 is supported in H. Hence the two functions are equal if and only if $g_1H = g_2H$. It follows that $\sum_{g\in G/H}\rho(g)AV = \oplus_{g\in G/H}\rho(g)AV$. Now $\rho(g)$ is invertible and A is one-to-one, so $\dim\rho(g)AV = \dim V$. Therefore the dimension of the latter direct sum equals $|G||H|^{-1}\dim V = \dim\mathcal{V}$. Thus $\mathcal{V} = \oplus_{g\in G/H}\rho(g)AV$.

Now we want to study $\mathcal{V}$ as a K-space and $\rho(g)AV$ is not K-stable. Given $g \in G$, the double coset KgH is a disjoint union of certain cosets $g'H$. So we group together those $\rho(g')AV$ such that $g'H \subset KgH$. Let $X(g) = \sum_{g'H\subset KgH}\rho(g')AV$. It should be understood that the above sum is taken over a set of representatives g' of the left H-cosets which lie in KgH. Now we have regrouped things and we have $\mathcal{V} = \oplus_{g\in K\backslash G/H}X(g)$. Now $\rho(k)X(g) = X(g)$ for all $k \in K$. We will prove that

$$(**) \qquad \rho_K\,|_{X(g)} = r_G^K\rho\,|_{X(g)} \simeq i_{K\cap gHg^{-1}}^K\pi^g.$$

The theorem is a consequence of (**) and the above direct sum decomposition of $\mathcal{V}$.

Now suppose that $g' = kgh$, $k \in K$, $h \in H$. Then $\rho(g')AV = \rho(kg)AV$. Let $k_0 \in K \cap gHg^{-1}$.

$$\rho(k_0g')AV = \rho(k_0g)AV = \rho(g(g^{-1}k_0g))AV = \rho(g)AV$$

. This implies that $X(g) = \sum_{K/(K\cap gHg^{-1})}\rho(k)\rho(g)AV$. Now we can easily check that if $k \in K$, then $\rho(k)\rho(g)AV = \rho(g)AV$ if and only if $\rho(g^{-1}kg)AV = AV$ if and only if $g^{-1}kg \in H$, that is $k \in K \cap gHg^{-1}$. So $X(g) = \oplus_{K/(K\cap gHg^{-1})}\rho(k)\rho(g)AV$.

Let $\mathcal{W}$ be the space of $i_{K\cap gHg^{-1}}^K\pi^g$. Now define $B : X(G) \to \mathcal{W}$ as follows. Let $v \in V$ and $k \in K$. Set $\varphi_v(k) = \pi^g(k)v$ if $k \in K\cap gHg^{-1}$ and $\varphi_v(k) = 0$ otherwise. Then $\varphi_v \in \mathcal{W}$. Given $v \in V$ and $k \in K$, set $B\rho(k)\rho(g)Av = i_{K\cap gHg^{-1}}^K\pi^g(k)\varphi_v$. It is a simple matter to check that B is invertible. Since $r_G^K\rho$ acts by right translation on $X(g)$ and $i_{K\cap gHg^{-1}}^K\pi^g$ acts by right translation on $\mathcal{W}$, we see that B intertwines these representations. Hence (**) holds. qed

Theorem 13. *Let H and K be subgroups of a finite group G. Let (π, V) and (ρ, W) be (finite-dimensional) representations of H and K, respectively. Then $\mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$ is isomorphic to*

$$\{\,\varphi : G \to \mathrm{End}_{\mathbb{C}}(V, W)\ |\varphi(kgh) = \rho(k)\circ\varphi(g)\circ\pi(h),\ k \in K,\, g \in G,\, h \in H\,\}.$$

Sketch of proof. Let $A \in \mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$. Define $\varphi_A : G \to \mathrm{End}_{\mathbb{C}}(V, W)$ by $\varphi_A(g)v = (Af_v)(g)$, $v \in V$, where $f_v(h) = \pi(h)v$, $h \in H$, and $f_v(g) = 0$ if $g \notin H$. Note that f_v is in the space of $i_H^G\pi$. Let $k \in K$, $g \in G$, $h \in H$, and $v \in V$. Then

$$\begin{aligned}\varphi_A(kgh)v &= (Af_v)(kgh) = \rho(k)(Af_v)(gh) = \rho(k)(\rho(h)Af_v)(g)\\ &= \rho(k)(Af_{\pi(h)v})(g) = \rho(k)\varphi_A(g)\pi(h)v\end{aligned}$$

where the second equality holds because Af_v is in the space of $i_K^G\rho$, the fourth equality holds because $A \in \mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$, and the fifth because $\pi(h)f_v = f_{\pi(h)v}$ for all $h \in H$. Hence φ_A has the desired properties relative to left translation by elements of K and right translation by elements of H.

Given $\varphi : G \to \mathrm{End}_{\mathbb{C}}(V, W) \mid \varphi(kgh) = \rho(k) \circ \varphi(g) \circ \pi(h)$, $g \in G$, $h \in H$, $k \in K$. Let f be in the space of $i_H^G\pi$. Define $A_\varphi f$ in the space of $i_K^G\rho$ by $(A_\varphi f)(g) = \sum_{g_0} \varphi(gg_0^{-1})f(g_0)$, where in the sum g_0 runs over a set of coset representatives for $K\backslash G$. Suppose that $k \in K$ and $g \in G$. Then

$$(A_\varphi f)(kg) = \sum_{g_0} \varphi(kgg_0^{-1})f(g_0) = \rho(k)\sum_{g_0} \varphi(gg_0^{-1})f(g_0) = \rho(k)(A\varphi f)(g).$$

Hence $A_\varphi f$ belongs to the space of $i_K^G\rho$.

Next, let g, $g_1 \in G$. Then

$$\begin{aligned}(A_\varphi i_H^G\pi(g_1)f)(g) &= \sum_{g_0} \varphi(gg_0^{-1})(i_H^G\pi(g_1)f)(g_0) = \sum_{g_0} \varphi(gg_0^{-1})f(g_0g_1)\\ &= \sum_{g_0} \varphi(gg_1g_0^{-1})f(g_0) = (A_\varphi f)(gg_1) = (i_K^G\rho(g_1)A_\varphi f)(g)\end{aligned}$$

Therefore $A_\varphi \in \mathrm{Hom}_G(i_H^G\pi, i_K^G\rho)$.

To finish the proof, check that $A \mapsto \varphi_A$ and $\varphi \mapsto A_\varphi$ are inverses of each other. The details are left as an exercise. qed

Corollary. *Let π be an irreducible representation of a subroup H of G. Then $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ is isomorphic to*

$$\mathcal{H}(G,\pi) := \{\, \varphi : G \to \mathrm{End}_{\mathbb{C}}(V) \mid \varphi(hgh') = \pi(h) \circ \varphi(g) \circ \pi(h'),\ g \in G,\ h, h' \in H \,\}.$$

Lemma. *The subspace of $\mathcal{H}(G,\pi)$ consisting of functions supported on the double coset HgH is isomorphic to $\mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi)$. where $H^g = H \cap gHg^{-1}$ and $\pi^g(h) = \pi(g^{-1}hg)$, $h \in H^g$.*

Proof. Fix $g \in G$. Given $\varphi \in \mathcal{H}(G,\pi)$ such that φ is supported on HgH, define a linear operator $B_\varphi : V \to V$ by $B_\varphi(v) = \varphi(g)v$, $v \in V$. Then, if $h \in H^g$, we have, using the fact that $g^{-1}hg \in H$ and $h \in H$, and properties of φ,

$$B_\varphi(\pi^g(h)v) = \varphi(g)(\pi(g^{-1}hg)v) = \varphi(hg)v = \pi(h)\varphi(g)v = \pi(h)B_\varphi(v).$$

Hence $B_\varphi \in \mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi)$.

Given $B \in \mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi)$, set $\varphi_B(h_1gh_2)v = \pi(h_1)B\pi(h_2)v$, for h_1, $h_2 \in H$ and $v \in V$, and $\varphi_B(g_1)v = 0$ if $g_1 \notin HgH$. Check that $\varphi_B \in \mathcal{H}(G,\pi)$, and also that the map $B \mapsto \varphi_B$ is the inverse of the map $\varphi \mapsto B_\varphi$. The details are left as an exercise. qed

Corollary. *Let (π, V) be a representation of a subgroup H of G. If $g \in G$, let $H^g = H \cap gHg^{-1}$ and set $\pi^g(h) = \pi(g^{-1}hg)$, $g \in H^g$. Then*

$$\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi) \simeq \bigoplus_{g \in H\backslash G/H} \mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi).$$

Corollary(Mackey irreducibility criterion). *Let (π, V) be an irreducible representation of a subgroup H of G. Then $i_H^G\pi$ is irreducible if and only if $\mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi) = 0$ for all $g \notin H$.*

Proof. Note that if $g = h \in H$, then $H^h = H$ and $r_H^{H^h}\pi = \pi$. Hence, by irreducibility of π, $\mathrm{Hom}_H(\pi, \pi^h) \simeq \mathbb{C}$. Therefore, since $i_H^G\pi$ is irreducible if and only if $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi) \simeq \mathbb{C}$, by the above proposition, $i_H^G\pi$ is irreducible if and only if $\mathrm{Hom}_{H^g}(\pi^g, r_H^{H^g}\pi) = 0$ whenever $g \notin H$. qed

Corollary. *If π is the trivial representation of a subgroup H of G, then $\dim \mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ equals the number of H-H-double cosets in G.*

Note that π^g and $r_H^{H^g}\pi$ are both the trivial representation of H^g (for any $g \in G$). Acccording to the above proposition, there is a one-dimensional contribution to $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ for each double coset HgH. qed

Given φ_1, $\varphi_2 \in \mathcal{H}(G,\pi)$, set

$$(\varphi_1 * \varphi_2)(g) = |G|^{-1} \sum_{g_0 \in G} \varphi_1(g_0) \circ \varphi_2(g_0^{-1}g).$$

This product makes $\mathcal{H}(G,\pi)$ into an algebra, known as a *Hecke algebra*.

Proposition. *The algebra $\mathrm{Hom}_G(i_H^G\pi, i_H^G\pi)$ is isomorphic to the Hecke algebra $\mathcal{H}(G,\pi)$.*

To prove the proposition, check that the vector space $A \mapsto \varphi_A$ of the the Theorem 13 is an algebra homomorphism: $\varphi_{A_1 \circ A_2} = \varphi_{A_1} * \varphi_{A_2}$. The details are left as an exercise.

Exercises:

1. Adapt the above arguments to prove: Let H and K be subgroups of G, let π be a representation of H, and let τ be a representation of K. Suppose that $i_H^G\pi$ and $i_K^G\tau$ are irreducible. Show that $i_H^G\pi \not\simeq i_K^G\tau$ if and only if for every $g \in G$

$$\mathrm{Hom}_{gHg^{-1}\cap K}(\pi^g, r_K^{gHg^{-1}\cap K}\tau) = 0.$$

2. Let A be an abelian subgroup of a group G. Show that each irreducible representation of G has degree at most $|G||A|^{-1}$.
3. Let π_j be a representation of a subgroup H_j of a group G_j, $j = 1, 2$. Prove that

$$i_{H_1 \times H_2}^{G_1 \times G_2} \pi_1 \otimes \pi_2 \simeq i_{H_1}^{G_1} \pi_1 \otimes i_{H_2}^{G_2} \pi_2.$$

4. Let π and ρ be representations of subgroups H and K of a finite group G. Let $g_1,\ g_2 \in G$. Define representations π^{g_1} and ρ^{g_2} of the group $g_1^{-1}Hg_1 \cap g_2^{-1}Kg_2$ by $\pi^{g_1}(x) = \pi(g_1 x g_1^{-1})$ and $\rho^{g_2}(x) = \rho(g_2 x g_2^{-1})$, $x \in g_1^{-1}Hg_1 \cap g_2 K g_2^{-1}$.
   (i) Prove that the equivalence class of $\tau^{(g_1,g_2)} := i^G_{g_1^{-1}Hg_1 \cap g_2 K g_2^{-1}}(\pi^{g_1} \otimes \rho^{g_2})$ depends only on the double coset $Hg_1g_2^{-1}K$.
   (ii) Prove that the internal tensor product $i_H^G \pi \otimes i_K^G \rho$ is equivalent to the direct sum of the representations $\tau^{(g_1,g_2)}$ as $g_1 g_2^{-1}$ ranges over a set of representatives for the H-K-double cosets in G. (*Hint*: Consider the restriction of the representation of $G \times G$ which is the outer tensor product $i_H^G \pi \otimes i_K^G \rho$ to the subgroup $\bar{G} = \{ (g, g) \mid g \in G \}$ and apply Theorem 11.)
5. Suppose that the finite group G is a the semidirect product of a subgroup H with an abelian normal subgroup A, that is, $G = H \ltimes A$. Let G act on the set $\hat{A}$ of irreducible (that is, one-dimensional) representations of A by $\sigma \mapsto \sigma^g$, where $\sigma^g(a) = \sigma(g^{-1}ag)$, $a \in A$, $\sigma \in \hat{A}$. Let $\{ \sigma_1, \ldots, \sigma_r \}$ be a set of representatives for the orbits of G on $\hat{A}$. Let $H_i = \{ \sigma_i^h = \sigma_i \}$.
   a) Let $G_i = H_i \ltimes A$. Show that σ_i extends to a representation of G_i via $\sigma_i(ha) = \sigma_i(a)$, $h \in H_i$, $a \in A$.
   b) Let π be an irreducible representation of H_i. Show that we may define an irreducible representation $\rho(\pi, \sigma_i)$ of G_i on the space V of π by: $\rho(\pi, \sigma_i)(ha) = \sigma_i(ha)\pi(h) = \sigma_i(a)\pi(h)$, $h \in H_i$, $a \in A$.
   c) Let π be an irreducible representation of H_i. Set $\theta_{i,\pi} = \mathrm{Ind}_{G_i}^G \rho(\pi, \sigma_i)$. Prove that $\theta_{i,\pi}$ is an irreducible representation of G.
   d) Let π and π' be irreducible representations of H_i and $H_{i'}$. Prove that $\theta_{i,\pi} \simeq \theta_{i',\pi'}$ implies $i = i'$ and $\pi \simeq \pi'$.
   e) Prove that $\{ \theta_{i,\pi} \}$ are all of the (equivalence classes of) irreducible representations of G. (Here, i ranges over $\{1, \ldots, r\}$ and, for i fixed, π ranges over all of the (equivalence classes of) irreducible representations of H_i).
6. Let H be a subgroup of a finite group G. Let $\mathcal{H}(G, 1)$ be the Hecke algebra associated with the trivial representation of the subgroup H.
   a) Show that if (π, V) is an irreducible representation of G, and V^H is the subspace of H-fixed vectors in V, then V^H becomes a representation of $\mathcal{H}(G, 1)$, that is, a

module over the ring $\mathcal{H}(G,1)$, with the action

$$f \cdot v = |G|^{-1} \sum_{g \in G} f(g)\pi(g)v, \qquad v \in V^H.$$

b) Show that if $V^H \neq \{0\}$, then V^H is an irreducible representation of $\mathcal{H}(G,1)$ (an irreducible $\mathcal{H}(G,1)$-module). (*Hint*: If W is a nonzero invariant subspace of V^H, and $v \in V^H$, use irreducibility of π to show that there exists a function f_1 on G such that $v = f_1 \cdot w$, where $w \in W$ and $f_1 \cdot w$ is defined as above even though $f_1 \notin \mathcal{H}(G,1)$. Next, show that if 1_H is the characteristic function of H, then $f := 1_H * f_1 * 1_H$, $f \in \mathcal{H}(G,1)$, and $f \cdot w = v$.)

c) Show that $(\pi, V) \to V^H$ is a bijection between the equivalence classes of irreducible representatations of G such that $V^H \neq \{0\}$ and the equivalence classes of irreducible representations of $\mathcal{H}(G,1)$.

Remarks - Representations of Hecke algebras: Suppose that (π, V) is an irreducible finite-dimensional representation of a subgroup H of G. A representation of the Hecke algebra $\mathcal{H}(G,\pi)$ is defined to be an algebra homomorphism from $\mathcal{H}(G,\pi)$ to $\mathrm{End}_{\mathbb{C}}(V')$ for some finite-dimensional complex vector space V'. (That is, V' is a finite-dimensional $\mathcal{H}(G,\pi)$-module).

Let (ρ, W) be a finite-dimensional representation of G. Then it is easy to check that the internal tensor product $(r_G^H \rho \otimes \pi, W \otimes V)$ contains the trivial representation of H if and only if $r_G^H \rho$ contains the representation $\pi^\vee$ of H that is dual to π.

Assume that the dual representaion $\pi^\vee$ is a subrepresentation of $r_G^H \rho$. Given $f \in \mathcal{H}(G,\pi)$, define a linear operator $\rho'(f)$ on $W \otimes V$ by

$$\rho'(f)(w \otimes v) = |G|^{-1} \sum_{g \in G} \rho(g)w \otimes f(g)v.$$

The fact that $f \in \mathcal{H}(G,\pi)$ can be used to prove that the subspace $(W \otimes V)^H$ of H-invariant vectors in $W \otimes V$ is $\rho'(f)$-invariant, and the map $f \mapsto \rho'(f) \,|\, (W \otimes V)^H$ defines a representation of the Hecke algebra $\mathcal{H}(G,\pi)$.

In this way, we obtain a map $\rho \mapsto \rho'$ from the set of representations of G whose restrictions to H contain $\pi^\vee$ and the set of nonzero representations of the Hecke algebra $\mathcal{H}(G,\pi)$. It can be shown that this map has the following properties:

(i) ρ is irreducible if and only if ρ' is irreducible.

(ii) If ρ_1 and ρ_2 are irreducible, then ρ_1 and ρ_2 are equivalent if and only if ρ_1' and ρ_2' are equivalent.

(iii) For each nonzero irreducible representation (τ, U) of $\mathcal{H}(G,\pi)$, there exists an irreducible representation ρ of G such that ρ' is equivalent to τ.

The study of representations of reductive groups over finite fields (that is, finite groups of Lie type) is sometimes approached via the study of representations of Hecke algebras. In certain cases, $\mathcal{H}(G, \pi)$ may be isomorphic (as an algebra) to another Hecke algebra $\mathcal{H}(G', \pi')$, where G' is a different group (and π' is an irreducible representation of a subgroup H' of G'). In this case, the study of those irreducible representations of G whose restrictions to H contain $\pi^\vee$ reduces to the study of a similar set of representations of the group G'.

Representations of Hecke algebras also play a role in the study of admissible representations of reductive groups over p-adic fields. An example of such a group is $GL_n(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is the field of p-adic numbers. In this setting, the representation ρ of G will be infinite-dimensional (and admissible), the subgroup H will be compact, and open in G, π will be finite-dimensional (since H is compact) and the representation ρ' of the Hecke algebra will be finite-dimensional. In this setting, the definition of the Hecke algebras is slightly different from that for finite groups.

CHAPTER 3

Representations of $SL_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of order q. Then there exists a prime p and a positive integer ℓ such that $q = p^\ell$. For convenience, we assume that p is odd. We also assume that $q \neq 3$. Let $G = SL_2(\mathbb{F}_q)$ be the group of 2×2 matrices with entries in $\mathbb{F}_q$ and determinant equal to 1. In this chapter, we construct the (characters of the) irreducible representations of G.

Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\}$. Let A be the group of diagonal matrices in G, and let $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_q \right\}$. Note that $B = A \ltimes N \simeq \mathbb{F}_q^\times \ltimes \mathbb{F}_q$. Hence $|B| = q(q-1)$.

Let $g \in G$. Note that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin B$ if and only if $c \neq 0$. In that case, $b = c^{-1}(ad-1)$. Because a and d may be chosen freely in $\mathbb{F}_q$ and c may be chosen freely in $\mathbb{F}_q^\times$, it follows that the number of elements in the complement of B in G is equal to $q^2(q-1)$. Hence $|G| = q(q-1) + q^2(q-1) = q(q^2-1)$.

Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The double coset BwB is a disjoint union of right B cosets. Let $b_1, b_2 \in B$. Then $Bwb_1 = Bwb_2$ if and only if $b_1 b_2^{-1} \in w^{-1}Bw \cap B$. Note that

$$w^{-1} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} w = \begin{pmatrix} a^{-1} & 0 \\ -b & a \end{pmatrix}. \tag{3.1}$$

It follows that $wBw^{-1} \cap B = A$, and as g ranges over a set of (right) coset representatives for $A\backslash B$, then g ranges over a set of representatives for the right B cosets in BwB. Note that $A\backslash B \simeq N$. Hence we may (and do) view N as a set of coset representatives for $A\backslash B$ and for the right B cosets in BwB. Now $|N| = q$. Thus BwB contains exactly q right B cosets. Hence $|BwB| = q|B| = q^2(q-1)$. The subset $B \amalg BwB$ contains $q(q-1) + q^2(q-1) = q(q^2-1)$ elements, so must equal G.

Lemma. *G is generated by B and w, and there are two B-B double cosets in G, namely B and BwB. Also*

$$G = \amalg_{x \in \mathbb{F}_q} Bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \amalg B.$$

We will begin to study the representations of G by looking at representations of G which are induced from linear characters (one-dimensional representations) of B. Note that the derived group (commutator subgroup) of B is equal to N. Hence there is a bijection between the set of linear characters of $A \simeq B/N$ and the set of linear characters of B. Now $A \simeq \mathbb{F}_q^\times$ and $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$. Let $\zeta \in \mathbb{C}$ be a primitive root of unity of order $q-1$, and let α be a generator of $\mathbb{F}_q^\times$. For each m such that $0 \leq m \leq q-2$, the map $\alpha^j \mapsto \zeta^{jm}$ defines a linear character of $\mathbb{F}_q^\times$. It is clear that these characters are distinct. Since there are $q-1$ of them, this gives a complete list of the linear characters of $\mathbb{F}_q^\times$. Note that there are two characters of $\mathbb{F}_q^\times$ whose squares are trivial. One is the trivial representation of $\mathbb{F}_q^\times$, corresponding to $m=0$, and the other one, corresponds to $m=(q-1)/2$, and takes the value -1 on non-squares in $\mathbb{F}_q^\times$ and 1 on squares.

Let τ_0 be a character of $\mathbb{F}_q^\times$. The associated character τ of B is defined by $\tau\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} = \tau_0(a)$. Looking at equation (3.1), we see that the character τ^w of $B^w = wBw^{-1}\cap B = A$ (notation as in Chapter 2) is given by

(3.2)
$$\tau^w\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = \tau\left(w^{-1}\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} w\right) = \tau\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix} = \tau_0(a)^{-1}, \qquad a\in\mathbb{F}_q^\times.$$

Hence
$$\mathrm{Hom}_{B^w}(\tau^w, r_B^{B^w}\tau) = \mathrm{Hom}_A(\tau_0^{-1},\tau_0) = \begin{cases} \mathbb{C}, & \text{if } \tau_0^2 = 1 \\ 0, & \text{if } \tau_0^2\neq 1\end{cases}$$

Combining this with results from Chapter 2 concerning induced representations, we have

Lemma. *$i_B^G\tau$ is irreducible if and only if $\tau_0^2\neq 1$. If $\tau_0^2=1$, then* $\dim\mathrm{Hom}_G(i_B^G\tau, i_B^G\tau)=2$.

Given $a\in\mathbb{F}_q^\times$, set $s_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$. It is easy to check that if $a\neq\pm1$, then A is the centralizer of s_a in G. Hence if $a\neq\pm1$, the order $|\mathrm{cl}(s_a)|$ of the conjugacy class $\mathrm{cl}(s_a)$ in G is equal to $|G|/|A| = q(q+1)$. Note that if a, $b\in\mathbb{F}_q$, then $gs_ag^{-1}=s_b$ if and only if s_a and s_b have the same eigenvalues. If $a\neq\pm1$, this is equivalent to $b\in\{\,s_a, s_{a^{-1}}\,\}$. Thus, when $a\neq\pm1$, $\mathrm{cl}(s_a)=\mathrm{cl}(s_b)$ if and only if $b=a$ or $b=a^{-1}$. Now the centre of G is equal to $\{\,s_1,s_{-1}\,\}$, so there are two single-element conjugacy classes in G. According to the above there are $(q-3)/2$ non-central conjugacy classes which contain elements of A, each such class containing $q(q+1)$ elements. If $a\in\mathbb{F}_q^\times$ and $a\neq\pm1$, then it is easy to check that $\mathrm{cl}_B(s_a)=\{\,gs_ag^{-1}\mid g\in N\,\}=s_aN$. We can now conclude that

$$\mathrm{cl}_G(s_a)\cap B = s_aN \amalg s_{a^{-1}}N, \qquad a\in\mathbb{F}_q^\times,\ a\neq\pm1.$$

Note that $\mathrm{cl}_G(\pm I)=\pm I=\mathrm{cl}_G(\pm I)\cap B$. Applying the Frobenius Character Formula, we can compute $\chi_{i_B^G\tau}$ on A.

Lemma. *Let $a\in\mathbb{F}_q^\times$. Then*

$$\chi_{i_B^G\tau}(s_a) = |G||B|^{-1}|\mathrm{cl}_G(s_a)|^{-1}(|cl_B(s_a)|\tau(s_a)+|\mathrm{cl}_B(s_{a^{-1}})|\tau(s_{a^{-1}})) = \tau_0(a)+\tau_0(a^{-1}),\ a\neq\pm1.$$

and $\chi_{i_B^G\tau}(s_{\pm1}) = (q+1)\tau_0(\pm1)$.

Let ψ be a nontrivial character of (the additive group) $\mathbb{F}_q$. Let $t\in\mathbb{F}_q$. It is easy to see that the function $x\mapsto\psi_t(x):=\psi(tx)$ also defines a character of $\mathbb{F}_q$, and $\psi_s=\psi_t$ if and only if $s = t$. Hence $\{ \psi_t \mid t \in \mathbb{F}_q \}$ is the set of characters of $\mathbb{F}_q$. We can also view this set as the set of characters of N, since $N \simeq \mathbb{F}_q$.

Let $t \in \mathbb{F}_q$. Define a function $f_t^\tau : G \to \mathbb{C}$ by $f_t^\tau(b) = 0$ for $b \in B$ and

$$f_t^\tau \left(bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \tau(b)\psi_t(x), \qquad b \in B,\ x \in \mathbb{F}_q.$$

Clearly $f_t^\tau \neq 0$ and f_t^τ belongs to the space of $i_B^G\tau$. Let $\mathcal{V}_\tau^o$ be the subspace of the space $\mathcal{V}_\tau$ of $i_B^G\tau$ which consists of those functions which are supported in BwB. Then the set $\{ f_t^\tau \mid t \in \mathbb{F}_q \}$ is a basis of $\mathcal{V}_\tau^o$. Note that

$$\begin{aligned}\left(i_B^G\tau \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} f_t^\tau \right) \left(bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) &= f_t^\tau \left(bw \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \right) \\ &= \tau(b)\psi_t(x+y) = \psi_t(y) f_t^\tau \left(bw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right).\end{aligned}$$

Therefore (for t fixed) the span of f_t^τ is N-invariant and the restriction of $r_G^N(i_B^G\tau)$ to this one-dimensional space equivalent to the character ψ_t of N. It follows that the restriction of $r_G^N(i_B^G\tau)$ to V_τ^o is equivalent to the regular representation of N, since it is the direct sum of all of the characters of N, each occurring exactly once.

Let f_B^τ be the function which is zero on BwB and satisfies $f_B^\tau(b) = \tau(b)$, $b \in B$. The space $\mathcal{V}_\tau'$ of $\mathcal{V}_\tau$ consisting of functions supported on B is one-dimensional, and is spanned by the function f_B^τ. It is clear that the restriction of $r_G^N(i_B^G\tau)$ to $\mathcal{V}_\tau'$ is equivalent to the trivial representation of N.

Lemma.
(1) $r_G^N(i_B^G\tau) \simeq 2\psi_0 \oplus \bigoplus_{t \in \mathbb{F}_q^\times} \psi_t$.
(2) $\chi_{i_B^G\tau} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{cases} q+1, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}.$

Lemma. *If $g \in G$ has no eigenvalues in $\mathbb{F}_q^\times$, then $\chi_{i_B^G\tau}(g) = 0$.*

Proof. If $\chi_{i_B^G\tau}(g) \neq 0$, then by the Frobenius Character Formula, $\mathrm{cl}_G(g) \cap B \neq \emptyset$. Any element of B has eigenvalues in $\mathbb{F}_q^\times$. qed

Lemma. *Let τ_0 and τ_0' be characters of $\mathbb{F}_q^\times$, with extensions τ and τ' to B. Then $\chi_{i_B^G\tau} = \chi_{i_B^G\tau'}$ if and only if $\tau_0 = (\tau_0')^{\pm 1}$.*

Proof. Note that $\tau_0(-1) = \tau_0^{-1}(-1)$. Hence $\tau_0(-1) = \tau_0'(-1)$ if and only if $(\tau_0 + \tau_0^{-1})(-1) = (\tau_0' + \tau_0'^{-1})(-1)$. Given the above results about $\chi_{i_B^G\tau}$ and $\chi_{i_B^G\tau'}$, we see that $\chi_{i_B^G\tau} = \chi_{i_B^G\tau'}$ if and only if

$$(\tau_0 + \tau_0^{-1})(a) = (\tau_0' + \tau_0'^{-1})(a), \qquad \forall\ a \in \mathbb{F}_q^\times.$$

By linear independence of characters of $\mathbb{F}_q^\times$, this is equivalent to $\tau_0 = \tau_0'^{\pm 1}$. qed

Corollary. *Let τ and τ_0' be as above Assume that $\tau_0^2 \neq 1$. Then*

$$\mathrm{Hom}_G(i_B^G\tau', i_B^G\tau) = \begin{cases} \mathbb{C}, & \text{if } \tau_0' = \tau_0^{\pm 1} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $(\tau_0')^2 = 1$. Then $i_B^G\tau$ is irreducible of degree $q+1$ and $i_B^G\tau'$ is reducible of degree $q+1$. Therefore the two representations do not have any irreducible constituents in common.

Now suppose that $(\tau_0')^2 \neq 1$ Then $i_B^G\tau$ and $i_B^G\tau'$ are irreducible and by the previous result are equivalent if and only if $\tau_0 = \tau_0'^{\pm 1}$. qed

From above, we can conclude that there are have $(q-3)/2$ equivalence classes of irreducible representations of the form $i_B^G\tau$ (of course with $\tau^2 \neq 1$).

The next step is to decompose $i_B^G\tau$ in the two cases with $\tau^2 = 1$.

Lemma. *The trivial representation of G occurs as a subrepresentation of the representation $i_B^G 1$ induced from the trivial representation of B. The other irreducible constituent of $i_B^G 1$ has degree q and its character is equal to $\chi_{i_B^G 1}$ minus the characteristic function of G.*

Proof. We already know that $i_B^G 1$ has two irreducible constituents and has degree $q+1$. Therefore it suffices to prove that the trivial representation of G occurs as a subrepresentation of $i_B^G 1$. The characteristic function of G belongs to the space of $i_B^G 1$ and it is clearly invariant under right translation by elements of G. (Although it is not needed for the proof of the lemma, it is worth noting that the space of the other irreducible constituent of $i_B^G 1$ consists of functions which are left B-invariant and which satisfy $f(1) + \sum_{u \in N} f(wu) = 0$). qed

Note that, since we have assumed that p is odd, $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ has order 2. Fix a nonsquare $\varepsilon \in \mathbb{F}_q^\times$. Then 1 and ε are coset representatives for $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. Let λ be the unique character of $\mathbb{F}_q^\times$ of order 2. Let $f_t = f_t^\tau$ (for τ corresponding to λ), $t \in \mathbb{F}_q$, and let $f_B = f_B^\tau$. Let $\rho = i_B^G\tau$. We know that G is generated by N, A and w. Hence a subspace of the space $\mathcal{V}_\lambda$ is G-invariant if and only if it is N, A, and w-invariant. We have already described the N-invariant subspaces. Next, we describe some B-invariant subspaces. Let

$$\mathcal{W}_1 = \mathrm{Span}\{\, f_t \mid t \in (\mathbb{F}_q^\times)^2 \,\} \quad \text{and} \quad \mathcal{W}_\varepsilon = \mathrm{Span}\{\, f_t \mid t \in \varepsilon(\mathbb{F}_q^\times)^2 \,\}.$$

If $x \in \mathbb{F}_q$, let $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Lemma. *Let $a \in \mathbb{F}_q^\times$.*

(1) If $t \in \mathbb{F}_q^\times$, then $\rho(s_a)f_t = \lambda(a)f_{a^{-2}t}$.

(2) $\rho(s_a)f_0 = \lambda(a)f_0$

(3) $\rho(s_a)f_B = \lambda(a)f_B$.

(4) $\mathcal{W}_1$, $\mathcal{W}_\varepsilon$, $\mathrm{Span}\{f_0\}$, and $\mathrm{Span}\{f_B\}$ are B-invariant.

Proof. Part (4) follows from parts (1)–(3). For parts (1) and (2), let $a \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q$ and $b \in B$. Then

$$\begin{aligned}(\rho(s_a)f_t)(bwu_x) &= f_t(bwu_x s_a) = f_t(bs_{a^{-1}}wu_{a^{-2}x}) = \lambda(a^{-1})\tau(b)f_t(wu_{a^{-2}x})\\ &= \lambda(a)\tau(b)\psi_t(a^{-2}x) = \lambda(a)\tau(b)\psi_{a^{-2}t}(x) = \lambda(a)f_{a^{-2}t}(bwu_x).\end{aligned}$$

Above we have used the fact that $bs_a^{-1} \in s_{a^{-1}}bN$ and τ is trivial on N, and $\lambda = \lambda^{-1}$. Part (3) is left as an exercise. qed

Exercise: Prove that there are 4 inequivalent irreducible representations of B that are not one-dimensional, and they all have degree $(q-1)/2$.

Now we consider the action of w. It is easy to see that, as representations of B, $\mathcal{W}_1$ and $\mathcal{W}_\varepsilon$ are irreducible. It follows from results above that they are inequivalent, since as representations of N they have no constituents in common. As we see below, no subspace of $\mathrm{Span}\{\, f_B, f_0 \,\}$ is w-invariant. Hence $\mathcal{W}_1$ and $\mathcal{W}_\varepsilon$ cannot belong to the same G-subrepresentation of $i_B^G\tau$.

We will use

(3.2) $$wu_x w = s_{-x^{-1}}wu_{-x^{-1}}, \qquad x \in \mathbb{F}_q^\times$$

Let $\rho = i_B^G\tau$, where τ corresponds to λ.

Lemma. *Let $t \in \mathbb{F}_q$.*

(1) Then $(\rho(w)f_t)(bwu_x) = f_t(bwu_x w) = \lambda(-x)f_t(bwu_{-x^{-1}}) = \lambda(-x)\psi_t(-x^{-1})$, $x \in \mathbb{F}_q^\times$, $b \in B$,

(2) $(\rho(w)f_t)(bw) = 0$, $b \in B$.

(3) $(\rho(w)f_t)(b) = \tau(b)$, $b \in B$.

Let

$$\langle \varphi_1, \varphi_2 \rangle = \varphi_1(1)\overline{\varphi_2(1)} + \sum_{x \in \mathbb{F}_q} \varphi_1(wu_x)\overline{\varphi_2(wu_x)}, \qquad \varphi_1,\, \varphi_2 \in \mathcal{V}_\tau.$$

Lemma.

(1) $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{V}_\tau$ with respect to which ρ is a unitary representation of G.

(2) $\{\, f_B,\, f_t \mid t \in \mathbb{F}_q \,\}$ is an orthogonal basis of $\mathcal{V}_\tau$ (relative to the given inner product).

Proof. Part (1) is easily verified using the decomposition of G given in the first lemma. For part (2), note that if $t \in \mathbb{F}_q$, the support of f_t does not intersect B. Hence $\langle f_t, f_B \rangle = 0$. Let s, $t \in \mathbb{F}_q$. Then

$$\langle f_t, f_s \rangle = \sum_{x \in \mathbb{F}_q} \psi_t(x)\overline{\psi_s(x)} = q\,\delta_{st},$$

using orthogonality relations of characters of $\mathbb{F}_q$. qed

Let $\Gamma = \sum_{x \in \mathbb{F}_q^\times} \lambda(x)\psi(x)$.

Lemma.
(1) $\rho(w)f_B = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q} f_t$.
(2) $\rho(w)f_0 = f_B + q^{-1}\Gamma\sum_{t\in\mathbb{F}_q^\times} \lambda(t)f_t$.

Proof. For (1), note that

$$q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q} f_t(w) = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q}\psi_t(0) = \lambda(-1) = f_B(w^2) = (\rho(w)f_B)(w)$$

and, for $x\in\mathbb{F}_q^\times$,

$$q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q} f_t(wu_x) = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q}\psi_t(x) = q^{-1}\lambda(-1)\sum_{t\in\mathbb{F}_q}\psi_x(t) = 0 = (\rho(w)f_B)(wu_x),$$

since ψ_x is a nontrivial character of $\mathbb{F}_q$, and $wu_xw^{-1}\notin B$ if $x\in\mathbb{F}_q^\times$.

From the previous lemma, we have $\rho(w)f_0 = f_B + \sum_{t\in\mathbb{F}_q} c_t f_t$ for some scalars c_t. And because $\{\, f_B,\ f_t \mid t\in\mathbb{F}_q\,\}$ is an orthogonal basis of $\mathcal{V}_\tau$, we have $c_t = \langle\rho(w)f_0, f_t\rangle/\langle f_t, f_t\rangle = q^{-1}\langle\rho(w)f_0, f_t\rangle$, $t\in\mathbb{F}_q$. Because $f_t(1) = (\rho(w)f_0)(w) = f_0(-1) = 0$, we have

$$\begin{aligned}\langle\rho(w)f_0, f_t\rangle &= \sum_{x\in\mathbb{F}_q^\times}\rho(w)f_0(wu_x)\overline{f_t(wu_x)} = \sum_{x\in\mathbb{F}_q^\times}\lambda(-x)\psi_0(-x^{-1})\overline{\psi_t(x)} = \sum_{x\in\mathbb{F}_q^\times}\lambda(-x)\psi_t(-x)\\ &= \begin{cases}\sum_{x\in\mathbb{F}_q^\times}\lambda(t^{-1}x)\psi(x) = \lambda(t)\sum_{x\in\mathbb{F}_q^\times}\lambda(x)\psi(x) = \lambda(t)\Gamma, & \text{if } t\in\mathbb{F}_q^\times\\ \sum_{x\in\mathbb{F}_q^\times}\lambda(x) = 0, & \text{if } t=0\end{cases}\end{aligned}$$

Lemma. $\Gamma^2 = q\lambda(-1)$.

Proof. Let $\Phi(t) = \sum_{x\in\mathbb{F}_q^\times}\psi(tx)\lambda(x)$, $t\in\mathbb{F}_q^\times$. Then

$$\Phi(t) = \sum_{y\in\mathbb{F}_q^\times}\psi(y)\lambda(t)^{-1}\lambda(y) = \lambda(t)\Gamma.$$

Then, if $x\in\mathbb{F}_q^\times$,

$$\sum_{t\in\mathbb{F}_q^\times}\Phi(t)\psi(tx) = \Gamma\sum_{t\in\mathbb{F}_q^\times}\lambda(t)\psi(tx) = \Gamma\Phi(x) = \Gamma^2\lambda(x).$$

But we also have

$$\begin{aligned}\sum_{t\in\mathbb{F}_q^\times}\Phi(t)\psi(tx) &= \sum_{t\in\mathbb{F}_q^\times}\sum_{u\in\mathbb{F}_q^\times}\psi(tu+tx)\lambda(u) = \sum_{u\in\mathbb{F}_q^\times}\left(\sum_{t\in\mathbb{F}_q^\times}\psi(t(u+x))\right)\lambda(u)\\ &= \lambda(-x)(q-1) - \sum_{u\in\mathbb{F}_q^\times,\, u\neq -x}\lambda(u) = \lambda(-x)(q-1) - (0-\lambda(-x)) = \lambda(-x)q\end{aligned}.$$

Therefore $\Gamma^2\lambda(x) = \lambda(-x)q$. This implies $\Gamma^2 = \lambda(-1)q$. qed

Lemma.
(1) $\rho(w)(\Gamma f_0 \pm q f_B) = \pm q^{-1}\Gamma(\Gamma f_0 \pm q f_B) + \lambda(-1)\sum_{t\in\mathbb{F}_q^\times}(\lambda(t) \pm 1)f_t$.
(2) $\rho(w)(\Gamma f_0 + q f_B) \in \mathrm{Span}(\Gamma f_0 + q f_B) + \mathcal{W}_1$.
(3) $\rho(w)(\Gamma f_0 - q f_B) \in \mathrm{Span}(\Gamma f_0 - q f_B) + \mathcal{W}_\varepsilon$.

Set $\mathcal{V}^+ = \mathrm{Span}(\Gamma f_0 + q f_B) + \mathcal{W}_1$ and $\mathcal{V}^- = \mathrm{Span}(\Gamma f_0 - q f_B) + \mathcal{W}_\varepsilon$. We can see that $\mathcal{V}^+ = (\mathcal{V}^-)^\perp$ and $\mathcal{V}^- = (\mathcal{V}^+)^\perp$. The above lemma suggests that $\mathcal{V}^+$ and $\mathcal{V}^-$ may be G-invariant. We need to show that $\rho(w)\mathcal{W}_1 \subset \mathcal{V}^+$ and $\rho(w)\mathcal{W}_\varepsilon \subset \mathcal{V}^-$.

Lemma.
(1) *Let* $s, t \in \mathbb{F}_q^\times$. *Then* $\langle \rho(w)f_t, f_s\rangle = 0$ *if* $s \notin t(\mathbb{F}_q^\times)^2$.
(2) *If* $t \in (\mathbb{F}_q^\times)^2$, *then* $\langle \rho(w)f_t, \Gamma f_0 - q f_B\rangle = 0$.
(3) *If* $t \in \varepsilon(\mathbb{F}_q^\times)^2$, *then* $\langle \rho(w)f_t, \Gamma f_0 + q f_B\rangle = 0$.

Proof. Let $s, t \in \mathbb{F}_q^\times$. Then

$$\langle \rho(w)f_t, f_s\rangle = \sum_{x\in\mathbb{F}_q^\times} \rho(w)f_t(wu_x)\overline{f_s(wu_x)} = \sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi_t(x^{-1})\psi_s(x) = \sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi(tx^{-1}+sx).$$

Now suppose that $s = \varepsilon t u^2$ for some $u \in \mathbb{F}_q^\times$. Then

$$\sum_{x\in\mathbb{F}_q^\times} \lambda(x)\psi(t^{-1}x + sx) = \sum_{x\in(\mathbb{F}_q^\times)^2} \psi_t(x^{-1} + x\varepsilon u^2) - \sum_{x\in(\mathbb{F}_q^\times)^2} \psi_t(\varepsilon x^{-1} + xu^2).$$

The map $x \mapsto xu^2$ is a bijection from $\{\, x^{-1} + x\varepsilon u^2 \mid x \in (\mathbb{F}_q^\times)^2 \,\}$ and $\{\, \varepsilon x^{-1} + xu^2 \mid x \in (\mathbb{F}_q^\times)^2 \,\}$. It follows that $\langle \rho(w)f_t, f_{\varepsilon t u^2}\rangle = 0$ for all $u \in \mathbb{F}_q^\times$.

For (2), let $t \in (\mathbb{F}_q^\times)^2$. Then

$$\begin{aligned}\langle \rho(w)f_t, \Gamma f_0 - q f_B\rangle &= \langle f_t, \rho(-w)(\Gamma f_0 - q f_B)\rangle \\ &= \lambda(-1)\langle f_t, -\Gamma q^{-1}(\Gamma f_0 - q f_B) + \lambda(-1)\sum_{s\in\varepsilon(\mathbb{F}_q^\times)^2} -2f_s\rangle = 0.\end{aligned}$$

The proof of part (3) is similar to that of part (2) and is left as an exercise. qed

Corollary. *$\mathcal{V}^+$ and $\mathcal{V}^-$ are G-invariant.*

Let $\pi_\lambda^+ = \rho\,|_{\mathcal{V}^+}$ and $\pi_\lambda^- = \rho\,|_{\mathcal{V}^-}$.

For the proof of the following lemma, see the discussion on conjugacy classes which appears on pages 31.

Lemma. *Let* $x \in \mathbb{F}_q^\times$.
(1) $|\mathrm{cl}_G(u_x)| = (q^2-1)/2$.
(2) $\mathrm{cl}_G(u_x) = \mathrm{cl}_G(u_1)$ *if* $x \in (\mathbb{F}_q^\times)^2$.

(3) $\mathrm{cl}_G(u_x) = \mathrm{cl}_G(u_\varepsilon)$ *if* $x \in \varepsilon(\mathbb{F}_q^\times)^2$.
(4) $\mathrm{cl}_G(u_1) \neq \mathrm{cl}_G(u_\varepsilon)$.

Theorem. *$\rho = \pi_\lambda^+ \oplus \pi_\lambda^-$, π_λ^+ and π_λ^- are irreducible and inequivalent. Furthermore,*
(1) $\chi_{\pi_\lambda^+}(s_{\pm 1}) = \lambda(-1)(q+1)/2$.
(2) $\chi_{\pi_\lambda^+}(s_a) = \lambda(a)$, $a \in \mathbb{F}_q^\times$, $a^2 \neq 1$.
(3) $\chi_{\pi_\lambda^+}(\pm u_1) = \lambda(-1)(1+\Gamma)/2$.
(4) $\chi_{\pi_\lambda^+}(\pm u_\varepsilon) = \lambda(-1)(1-\Gamma)/2$.
(5) $\chi_{\pi_\lambda^+}(g) = 0$ *if g has no eigenvalues in* $\mathbb{F}_q^\times$.
(6) $\chi_{\pi_\lambda^-} = \chi_\rho - \chi_{\pi_\lambda^+} = \chi_{i_B^G\tau} - \chi_{\pi_\lambda^+}$ *(where τ is the character of B corresponding to λ).*

Proof. Because the restrictions of π_λ^+ and π_λ^- to N are inequivalent, the representations are inequivalent. For part (1), note that $\dim \mathcal{V}^+ = (q+1)/2$ and $\rho(-1) = \lambda(-1)I$.

For part (2), recall that $\rho(s_a)f_t = \lambda(a)f_{a^{-2}t}$, $t \in (\mathbb{F}_q^\times)^2$. Therefore $\mathrm{tr}\,(\rho(s_a)\,|_{\mathcal{W}_1}) = 0$ if $a^2 \neq 1$. Also, $\rho(s_a)(\Gamma f_0 + q f_B) = \lambda(a)(\Gamma f_0 + q f_B)$, $a \in \mathbb{F}_q^\times$.

Let $x \in \mathbb{F}_q$. Set

$$h_1(x) = \sum_{t \in (\mathbb{F}_q^\times)^2} \psi_t(x) = \sum_{t \in (\mathbb{F}_q^\times)^2} \psi_x(t) \quad \text{and} \quad h_\varepsilon(x) = \sum_{t \in \varepsilon(\mathbb{F}_q^\times)^2} \psi_t(x) = \sum_{t \in \varepsilon(\mathbb{F}_q^\times)^2} \psi_x(t).$$

If $x \in \mathbb{F}_q^\times$, then $h_1(x) + h_\varepsilon(x) = -1$ because ψ_x is a nontrivial character of $\mathbb{F}_q$. Also, if $x \in \mathbb{F}_q^\times$,

$$h_1(x) - h_\varepsilon(x) = \sum_{t \in \mathbb{F}_q^\times} \lambda(t)\psi_x(t) = \sum_{s \in \mathbb{F}_q^\times} \lambda(x^{-1}s)\psi(s) = \lambda(x)\Gamma.$$

Solving for $h_1(x)$ and $h_\varepsilon(x)$, we get $h_1(x) = (-1+\lambda(x)\Gamma)/2$ and $h_\varepsilon(x) = (-1-\lambda(x)\Gamma)/2$, $x \in \mathbb{F}_q^\times$. Now $\mathrm{tr}\,(\rho(u_x)\,|_{\mathcal{W}_1}) = h_1(x)$ and $\rho(u_x)(\Gamma f_0 + q f_B) = \Gamma f_0 + q f_B$. Therefore $\chi_{\pi_\lambda^+}(u_1) = h_1(1) + 1 = (1+\Gamma)/2$. Similarly, $\chi_{\pi_\lambda^+}(u_\varepsilon) = h_1(\varepsilon) + 1 = (1-\Gamma)/2$. Parts (3) and (4) follow.

Let G_{spl} be the set of elements of G which have eigenvalues in $\mathbb{F}_q^\times$. Any element in G_{spl} is conjugate to one of s_a, $a \in \mathbb{F}_q^\times$ or one of $\pm u_1$ and $\pm u_\varepsilon$. Therefore

$$\begin{aligned}\sum_{g \in G_{spl}} |\chi_{\pi_\lambda^+}(g)|^2 &= 2((q+1)/2)^2 + q(q+1)(q-3)/2 + 2((1+\Gamma+\bar{\Gamma}+\Gamma\bar{\Gamma})/4)(q^2-1)/2 \\ &\quad + 2((1-\Gamma-\bar{\Gamma}+\Gamma\bar{\Gamma})/4)(q^2-1)/2 \\ &= (q+1)^2/2 + q(q+1)(q-3)/2 + (1+\Gamma\bar{\Gamma})(q^2-1)/2\end{aligned}$$

Now

$$\bar{\Gamma} = \sum_{x \in \mathbb{F}_q^\times} \lambda(x)\psi(-x) = \lambda(-1) \sum_{x \in \mathbb{F}_q^\times} \lambda(x)\psi(x) = \lambda(-1)\Gamma.$$

Hence $\Gamma\bar{\Gamma} = \lambda(-1)\Gamma^2 = \lambda(-1)^2 q = q$. Substituting above results in $\sum_{g\in G_{spl}} |\chi_{\pi_\lambda^+}(g)|^2 = q(q^2-1) = |G|$. Because π_λ^+ is irreducible, we have $\sum_{g\in G} |\chi_{\pi_\lambda^+}(g)|^2 = |G|$. Therefore $\chi_{\pi_\lambda^+}(g) = 0$ if $g \notin G_{spl}$ qed

Before moving on to finding the other irreducible characters of G, we discuss conjugacy classes further. Let $g \in G$ be unipotent (that is, $(g-1)^2 = 0$). Then both eigenvalues of g equal 1. Suppose that $g \neq 1$ and g is unipotent. Then, because u_1 is the Jordan canonical form of g, there exists $g_0 \in GL_2(\mathbb{F}_q)$ such that $g_0 g g_0^{-1} = u_1$. That is, there is one noncentral unipotent conjugacy class in $GL_2(\mathbb{F}_q)$. This class breaks up into several conjugacy classes in G. The centralizer of u_1 in G is $\pm N$, so $|\mathrm{cl}_G(u_1)| = (q^2-1)/2$. From $s_a u_x s_a = u_{a^2x}$, $x \in \mathbb{F}_q^\times$, we see that $u_{a^2} \in \mathrm{cl}_G(u_1)$. Let $d(\varepsilon)$ be the diagonal matrix in $GL_2(\mathbb{F}_q)$ having diagonal entries ε and 1, respectively. If $u_\varepsilon \in \mathrm{cl}_G(u_1)$, then, choosing $g \in G$ such that $gu_1g^{-1} = i_\varepsilon$, we have $d(\varepsilon)u_1 d(\varepsilon)^{-1} = gu_1g^{-1}$. This implies $g^{-1}d(\varepsilon)$ centralizes u_1. Therefore $g^{-1}d(\varepsilon) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for some $a \in F_q^\times$ and some $b \in \mathbb{F}_q$. Taking determinants, we have $\varepsilon = a^2$, which is impossible, since ε is a non-square. Hence $\mathrm{cl}_G(u_1) \neq \mathrm{cl}_G(u_\varepsilon)$.

Now suppose that $g \in G_{ell}$. Then the eigenvalues of g, being roots of the characteristic polynomial of g are roots of a quadratic polynomial which is irreducible over $\mathbb{F}_q$. Hence the eigenvalues lie in a quadratic extension of $\mathbb{F}_q$. Up to isomorphism, there is exactly one quadratic extension of $\mathbb{F}_q$. Since ε is a non-square in $\mathbb{F}_q^\times$, $\mathbb{F}_q(\sqrt{\varepsilon})$ is a quadratic extension of $\mathbb{F}_q$. Hence the eigenvalues of g lie in $\mathbb{F}_q(\sqrt{\varepsilon}) - \mathbb{F}_q$. Now $g \in G$, so the product of the eigenvalues equals 1. It follows that the eigenvalues of g are of the form $a + b\sqrt{\varepsilon}$ and $a - b\sqrt{\varepsilon}$ where $a \in \mathbb{F}_q$, $b \in \mathbb{F}_q^\times$, and $a^2 - b^2\varepsilon = 1$. An example of such a g is the matrix $\begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix}$, with $b \neq 0$ and $a^2 - b^2\varepsilon = 1$. Let

$$T = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{F}_q,\ a^2 - b^2\varepsilon = 1 \right\}.$$

Then T is a subgroup of G and any element of $T - \{\pm I\}$ belongs to G_{ell}. Note that $\{1, \varepsilon\}$ is a basis of $\mathbb{F}_q(\sqrt{\varepsilon})$, and the matrix of an elment $a + b\sqrt{\varepsilon}$ with respect to this basis is equal to $\begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix}$. The map $\mathcal{N} : \mathbb{F}_q(\sqrt{\varepsilon})^\times \to \mathbb{F}_q^\times$ is a surjective homomorphism. Hence the kernel of $\mathcal{N}$ has order $(q^2-1)/(q-1) = q+1$. The map $a + b\sqrt{\varepsilon} \mapsto \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix}$ from $\mathbb{F}_q(\sqrt{\varepsilon})^\times$ to $GL_2(\mathbb{F}_q)$ restricts to an isomorphism between $\mathcal{N}$ and T. Hence $|T| = q+1$. Also, T is cyclic, since it is a subgroup of the cyclic group $\mathbb{F}_q(\sqrt{\varepsilon})^\times \simeq \mathbb{F}_{q^2}^\times$. It is a simple matter to check that if $\gamma \in T - \{\pm I\}$, then the centralizer of γ in G is equal to T. Hence $|\mathrm{cl}_G(\gamma)| = q(q-1)$. Now let $\gamma = \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \in T$ be such that $\gamma^2 \neq 1$. Any element of T which is conjugate to γ must have the same eigenvalues as γ, namely $a+b\sqrt{\varepsilon}$ and $a-b\sqrt{\varepsilon}=(a+b\sqrt{\varepsilon})^{-1}$. Checking that γ and γ^{-1} are conjugate in G, we have $\mathrm{cl}_G(\gamma)\cap T=\{\gamma,\gamma^{-1}\}$. There are $q-1$ noncentral elements in T. Thus there are exactly $(q-1)/2$ noncentral conjugacy classes which intersect T, each containing $q(q-1)$ elements.

Proposition.

(1) The centre of G equals $\{\pm I\}$.

(2) If $a\in\mathbb{F}_q^\times$ and $a\neq\pm1$, then $|cl_G(s_a)|=q(q+1)$. There are exactly $(q-3)/2$ noncentral conjugacy classes which intersect A.

(3) $\mathrm{cl}(u_1)\neq\mathrm{cl}(u_\varepsilon)$, $\mathrm{cl}(-u_1)\neq\mathrm{cl}(-u_\varepsilon)$, and $|\mathrm{cl}(\pm u_\varepsilon)|=|\mathrm{cl}(\pm u_1)|=(q^2-1)/2$.

(4) If $\gamma\in T$ and $\gamma\neq\pm1$, then $|cl(\gamma)|=q(q-1)$. There are exactly $(q-1)/2$ noncentral conjugacy classes which intersect T.

(5) There are $q+4$ conjugacy classes in G.

Proof. The only part which has not already been proved is part (5). First, counting the conjugacy classes mentioned in parts (1)–(4), we have a total of

$$2+(q-3)/2+4+(q-1)/2=q+4.$$

Counting the number of elements in the unions of all of these conjugacy classes, we get

$$2+q(q+1)(q-3)/2+4(q^2-1)/2+q(q-1)(q-1)/2=q(q^2-1)=|G|.$$

The number of distinct irreducible characters of G constructed so far (from the representations $i_B^G\tau$ and their irreducible constituents) is equal to $(q-3)/2+4=(q+5)/2$, $(q-3)/2$ of degree $q+1$, one of degree 1, one of degree q, and two of degree $(q+1)/2$. We must find another $(q+3)/2$ irreducible characters. We have found the characters of those irreducible representations of G whose restrictions to N contain the trivial representations of N (equivalently whose restrictions to B contain a one-dimensional representation of B). An irreducible representation of G is said to be *cuspidal* if its restriction to N does not contain the trivial representation of N (equivalently, its restriction to B contains no one-dimensional representations of B).

As we will see, the nontrivial characters of the group T will determine the other irreducible characters of G, but not in exactly the same way that the characters of A determined the characters already constructed.

Exercise: For each $x\in\mathbb{F}_q$, define

$$\psi^+(\pm u_x)=\psi(x),\quad \psi^-(\pm u_x)=\pm\psi(x)$$
$$\psi_\varepsilon^+(\pm u_x)=\psi_\varepsilon(x),\quad \psi_\varepsilon^-(\pm u_x)=\pm\psi_\varepsilon(x).$$

Let $\sigma_1^\pm=i_{\pm N}^B\psi^\pm$, and $\sigma_\varepsilon^\pm=i_{\pm N}^B\psi_\varepsilon^\pm$. Prove that

(1) $\sigma_1^\pm$ and $\sigma_\varepsilon^\pm$ are inequivalent irreducible representations of B, and any irreducible representation of B of degree greater than 1 is equivalent to one of them.
(2) $\chi_{\sigma_1^\pm}(\pm u_1) = \chi_{\sigma_\varepsilon^\pm}(\pm u_\varepsilon) = \pm(\Gamma - 1)/2$
(3) $\chi_{\sigma_1^\pm}(\pm u_\varepsilon) = \chi_{\sigma_\varepsilon^\pm}(\pm u_1) = \pm(-\Gamma - 1)/2$.
(4) All four of the above characters vanish on $B - \pm N$.

A class function φ on a finite group G' is a *virtual character* of G' if φ is an integral linear combination of characters of irreducible representations of G. Because the sum and product of the characters of representations of G' are also characters of representations (the product being the character of a tensor product), the set of virtual characters of G' is a ring (with pointwise addition and multiplication of functions). Let $\chi_1, \ldots, \chi_r$ be the characters of a complete set of irreducible representations of G'. Suppose that $\varphi = \sum_{j=1}^r \ell_j \chi_j$. Then $\langle \varphi, \varphi \rangle_{\mathcal{A}(G')} = \sum_{j=1}^r \ell_j^2$. It follows that if $\langle \varphi, \varphi \rangle = 1$, then $\varphi = \pm\chi_j$ for some j. If, in addition, $\varphi(1) > 0$, then $\varphi = \chi_j$. Note that we cannot take this approach with arbitrary class functions on G because we can find complex numbers $c_1, \ldots, c_r$ with $\sum_{j=1}^r |c_j|^2 = 1$ without forcing $\sum_{j=1}^r c_j \chi_j$ to be a multiple of one χ_j. One approach to determining irreducible characters of a finite group G' is to generate as many virtual characters φ as possible, and then look for those which satisfy $\langle \varphi, \varphi \rangle_{\mathcal{A}(G)} = 1$.

Let π be an irreducible cuspidal representation of G. Because the restriction of π to the subgroup B is the direct sum of irreducible representations of B of degree greater than 1, and every irreducible representation of B has degree $(q-1)/2$, $\chi_\pi(1)$ is divisible by $(q-1)/2$. By Frobenius reciprocity, π occurs as a subrepresentatation of $i_B^G \sigma_1^\pm = i_{\pm N}^G \psi^\pm$ or $i_B^G \sigma_\varepsilon^\pm = i_{\pm N}^G \psi_\varepsilon^\pm$. The proof of the following lemma is left as an exercise.

Lemma.
(1) $\chi_{i_{\pm N}^G \psi^\pm}(\pm 1) = \chi_{i_{\pm N}^G \psi_\varepsilon^\pm}(\pm 1) = \pm(q^2 - 1)/2.$
(2) $\chi_{i_{\pm N}^G \psi^\pm}(\pm u_1) = \pm(\Gamma - 1)/2$
(3) $\chi_{i_{\pm N}^G \psi^\pm}(\pm u_\varepsilon) = \pm(-\Gamma - 1)/2$
(4) $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}(\pm u_1) = \pm(-\Gamma - 1)/2$
(5) $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}(\pm u_\varepsilon) = \pm(\Gamma - 1)/2$
(6) $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}$ *and* $\chi_{i_{\pm N}^G \psi^\pm}$ *vanish on elements* $s_a \in A$ *with* $a^2 \neq 1$ *and* $\gamma \in T$ *with* $\gamma^2 \neq 1$.

The irreducible characters constructed so far as all constant on G_{ell}. Therefore the restrictions of the characters of the cuspidal representations of G must separate the conjugacy classes $\mathrm{cl}(\gamma)$, $\gamma \in T$ such that $\gamma^2 \neq 1$. From the above lemma, we see that the characters $\chi_{i_{\pm N}^G \psi^\pm}$ and $\chi_{i_{\pm N}^G \psi_\varepsilon^\pm}$ vanish on G_{ell}, so the characters of the elliptic reprsentations cannot all be expressed as linear combinations of these characters. The nontrivial characters of T do separate these conjugacy classes in G_{ell}. So perhaps the characters of the representations $i_T^G \theta$ for θ ranging over nontrivial characters of T will be related to the characters of the cuspidal representations of G. The proof of the following lemma is left as an exercise.

Lemma. *Let θ be a character of T. Then*

(1) $\chi_{i_T^G\theta}(\pm 1) = \theta(\pm 1)q(q-1)$

(2) $\chi_{i_T^G\theta}(\gamma) = \theta(\gamma) + \theta(\gamma^{-1})$ *if* $\gamma \in T$ *and* $\gamma^2 \neq 1$.

(3) Let $a \in \mathbb{F}_q^\times$ *be such that* $a^2 \neq 1$. *Then* $\chi_{i_T^G\theta}(s_a) = \chi_{i_T^G}(\pm u_1) = \chi_{i_T^G}(\pm u_\varepsilon) = 0$.

Since the functions $\chi_{i^G_{\pm N}\psi^\pm}$, $\chi_{i^G_{\pm N}\psi^\pm_\varepsilon}$ and $\chi_{i_T^G\theta}$ are virtual characters of G, so is any integral linear combination of these functions. We will look for virtual characters φ of this form which satisfy $\langle \varphi, \varphi\rangle = 1$.

Suppose that π is an irreducible representation of G. Since $-I$ belongs to the centre of G, it follows from Schur's Lemma that $\pi(-I)$ is a scalar multiple of the identity operator. And $(-I)^2 = I$ forces the scalar multiple to equal ± 1. Looking back at the irreducible representations of the form $i_B^G\tau$, on the unipotent set, the character formula involves the trivial character of N (appearing twice) and sum $\chi_{i^G_{\pm N}\psi^+} + \chi_{i^G_{\pm N}\psi^+_\varepsilon}$ or $\chi_{i^G_{\pm N}\psi^-} + \chi_{i^G_{\pm N}\psi^-_\varepsilon}$, with $+$ if the above scalar is -1 and $-$ if the scalar is -1 (note that this scalar equals $\tau(-1)$). On noncentral elements of of A, the sum $\tau + \tau^{-1}$ appears. With this in mind, knowing that the trivial representation of N will not occur in the character of a cuspidal representation, suppose that θ is a character of T, and set

$$\varphi_\theta = \begin{cases} \ell\chi_{i_T^G\theta} + m(\chi_{i^G_{\pm N}\psi^+} + \chi_{i^G_{\pm N}\psi^+_\varepsilon}, & \text{if } \theta(-1) = 1 \\ \ell\chi_{i_T^G\theta} + m(\chi_{i^G_{\pm N}\psi^-} + \chi_{i^G_{\pm N}\psi^-_\varepsilon}, & \text{if } \theta(-1) = -1. \end{cases}$$

where m, $\ell \in \{\pm 1\}$. Now if φ_θ is the character of a cuspidal representation, we must have $\varphi_\theta(1) = \ell q(q-1) + m(q^2-1) = (q-1)(\ell q + m(q+1)) > 0$, equal to a multiple of $(q-1)/2$, and dividing $q(q^2-1) = |G|$. Checking the possibilities, we must have $m = 1$ and $\ell = -1$, so $\varphi_\theta(\pm 1) = \theta(-1)(q-1)$.

The values of φ_θ on $\pm u_\varepsilon$, $\pm u_1$, $\gamma \in T$ and s_a, $a \in \mathbb{F}_q^\times$ may be obtained from character values given in the previous two lemmas.

Lemma.

(1) $\varphi_\theta(\pm 1) = \theta(-1)(q-1)$

(2) $\varphi_\theta(\pm u_1) = \varphi_\theta(\pm u_\varepsilon) = -\theta(\pm 1)$.

(3) If $\gamma \in T$ *and* $\gamma^2 \neq 1$, *then* $\varphi_\theta(\gamma) = -\theta(\gamma) - \theta(\gamma^{-1})$.

(4) If $a \in \mathbb{F}_q^\times$ *and* $a^2 \neq 1$, *then* $\varphi_\theta(s_a) = 0$.

Lemma.

$$\langle \varphi_\theta, \varphi_\theta\rangle_{\mathcal{A}(G)} = \begin{cases} 1, & \textit{if } \theta^2 \neq 1 \\ 2, & \textit{if } \theta^2 = 1 \end{cases}.$$

Proof. According to the above lemma, $|\varphi_\theta(\pm u_1)|^2 = |\varphi_\theta(\pm u_\varepsilon)|^2 = 1$. Thus

$$\begin{aligned}
q(q^2-1)\langle\varphi_\theta,\varphi_\theta\rangle_{\mathcal{A}(G)} &= \sum_{g\in G}\varphi_\theta(g)\overline{\varphi_\theta(g)} \\
&= 2(q-1)^2 + 4(q^2-1)/2 + q(q-1)(1/2)\sum_{\gamma\in T,\,\gamma\neq\pm1}(\theta(\gamma)+\theta(\gamma^{-1}))^2 \\
&= 4q(q-1) + q(q-1)/2\sum_{\gamma\in T\,\gamma^2\neq\pm1}(\theta(\gamma)^2+2+\theta(\gamma^{-1})^2) \\
&= 4q(q-1) + q(q-1)\left((\sum_{\gamma\in T}\theta(\gamma)^2)-2+(q-1)\right) \\
&= 4q(q-1)+q(q-1)\begin{cases} q-3, & \text{if } \theta^2\neq 1 \\ 2(q-1), & \text{if } \theta^2 = 1.\end{cases} \\
&= \begin{cases} q(q^2-1), & \text{if } \theta^2\neq 1 \\ 2q(q^2-1), & \text{if } \theta^2 = 1.\end{cases}
\end{aligned}$$

Theorem. *Let θ be a character of T.*

(1) If $\theta^2\neq 1$, there exists an irreducible representation π_θ of G such that $\chi_{\pi_\theta} = \varphi_\theta$. Also, If θ' is a character of T with nontrivial square, then $\pi_\theta\simeq\pi_{\theta'}$ if and only if $\theta' = \theta^{\pm1}$. Also π_θ is not equivalent to any of the irreducible representations obtained as irreducible constituents of representations induced from one-dimensional representations of B.

(2) Let θ be a character of T such that $\theta^2 = 1$. Then there exist irreducible representations π_θ^+ and π_θ^- of G such that $\varphi_\theta = \chi_{\pi_\theta^+}\pm\chi_{\pi_\theta^-}$.

Proof. Apply the previous result, together with comments on properties of virtual characters, to obtain the proofs of the assertions regarding irreducibility of π_θ when $\theta^2\neq 1$.

From the values of $\chi_{\pi_\theta} = \varphi_\theta$, we can see that $\chi_{\pi_\theta} = \chi_{\pi_{\theta'}}$ if and only if the functions $\theta+\theta^{-1}$ and $\theta' + (\theta')^{-1}$ agree on T. Hence, by linear independence of characters of T, we have $\pi_\theta\simeq\pi_{\theta'}$ if and only if $\theta' = \theta^{\pm1}$. Clearly the character π_θ is distinct from the characters of any of the irreducible constituents of the representations $i_B^G\tau$.

For (2), assume that $\theta^2 = 1$. We know that φ_θ is a virtual character. It follows from $\langle\varphi_\theta,\varphi_\theta\rangle = 2$ that 2 equals the sums of squares of the integers occuring as coefficients in the expression of φ_θ as a linear combination of irreducible characters. Hence exactly 2 of the coefficients are nonzero, each one in the set $\{\pm1\}$. Now $\varphi_\theta(1) > 0$. So it is clear that π_θ^+ and π_θ^- can be chosen so that the signs must be as stated, qed

At this point, ignoring the representations in part (2) above (whose characters we have not yet computed), we have produced $(q-1)/2$ irreducible characters of the form $i_B^G\tau$, 2 irreducible characters of constitutents of $i_B^G 1$, $\chi_{\pi_\lambda^+}$ and $\chi_{\pi_\lambda^-}$, and $(q-1)/2$ inequivalent irreducible characters χ_{π_θ}. That is, we have produced $q+2$ distinct irreducible characters of G. There are $(q+4)-(q+2) = 2$ remaining to find (and both are cuspidal).

Computing the sums of the squares of the degrees of the irreducible characters already produced, we obtain $|G|-(q-1)^2/2$. Therefore, if d_1 and d_2 are the degrees of the remaining irreducible characters, we have $d_1^2 + d_2^2 = (q-1)^2/2$. We already know that d_1 and d_2 are divisible by $(q-1)/2$ (the degree of any irreducible representation of B that is not one-dimensional). Therefore $d_1 = d_2 = (q-1)/2$.

Suppose that θ is a character of T such that $\theta^2 = 1$. Let π_θ^+ and π_θ^- be as above. Suppose that one of them is not cuspidal (that is, one of them contains a one-dimensional representation of B). Then, because $r_G^B \varphi_\theta$ is a combination of the characters of B of degree $(q-1)/2$, the other one must also be non-cuspidal, and $\varphi_\theta = \chi_{\pi_\theta^+} - \chi_{\pi_\theta^-}$. Furthermore, at least one of π_θ^+ and π_θ^- must have a character which is nonvanishing on G_{ell}, because φ_θ is not nonzero on G_{ell}. As we have seen, there are exactly two equivalence classes of non-cuspidal irreducible representations of G whose characters are not identically zero on G_{ell}. They are the two constitutents of $i_B^G 1$. One is the trivial representation of G, and the other has degree q. Every other non-cuspidal irreducible representation has degree $q+1$. Now $\varphi_\theta(1) = \chi_{\pi_\theta^+}(1) = \chi_{\pi_\theta^-}(1) = q-1$. It follows that π_θ^+ has degree q and π_θ^- is the trivial representation of G. Looking at the values of the characters of these representations, and of φ_θ, we see that we must have θ trivial.

Lemma. *Let θ be the trivial character of T. Then φ_θ is the difference of the irreducible character of degree q and the trivial representation of G.*

Now let ν be the unique character of T of order 2. So ν takes the value -1 on non-squares in T, and 1 on squares in T. According to the comments above, we know that both π_ν^+ and π_ν^- are cuspidal. Now they must have degree a multiple of $(q-1)/2$. From the form of φ_ν, we see that it may be the case that $\varphi_\nu = \chi_{\pi_\nu^+} - \chi_{\pi_\nu^-}$ where $\chi_{\pi_\nu^+}(1) = q-1$ and $\chi_{\pi_\nu^-}(1) = (q-1)/2$. In that case, since we know that the only cuspidal representations which are not equivalent some π_θ with $\theta^2 \neq 1$ all have degree $(q-1)/2$, it follows that $\pi_\nu^+ \simeq \pi_\theta$ for some θ with $\theta^2 \neq 1$. But we can check that $\langle \varphi_\nu, \varphi_\theta \rangle_{\mathcal{A}(G)} = 0$ if $\theta^2 \neq 1$. Therefore it is not possible for $\chi_{\pi_\nu^+}$ to have degree $q-1$.

Proposition. *Let ν be the character of T which has order 2. Then there exist two irreducible inequivalent cuspidal representations π_ν^+ and π_ν^+, both of degree $(q-1)/2$, such that $\varphi_\nu = \chi_{\pi_\nu^+} + \chi_{\pi_\nu^-}$. Furthermore,*

(1) $\chi_{\pi_\nu^+}(\pm u_1) = \chi_{\pi_\nu^-}(\pm u_\varepsilon) = \nu(\pm 1)(-1+\Gamma)/2$.

(2) $\chi_{\pi_\nu^-}(\pm u_1) = \chi_{\pi_\nu^+}(u_\varepsilon) = \nu(\pm 1)(-\Gamma-1)/2$.

(3) If $\gamma \in T$ and $\gamma^2 \neq 1$, then $\chi_{\pi_\nu^+}(\gamma) = \chi_{\pi_\nu^-}(\gamma) = \nu(\gamma)$.

(4) If $a \in \mathbb{F}_q^\times$ and $a^2 \neq 1$, then $\chi_{\pi_\nu^+}(s_a) = \chi_{\pi_\nu^-}(s_a) = 0$.

Proof. We know that $r_G^B(\chi_{\pi_\nu^+})$ and $r_G^B(\chi_{\pi_\nu^-})$ are irreducible characters of B having the property that $r_G^B(\chi_{\pi_\nu^+} + \chi_{\pi_\nu^+})$ equals $\chi_{\sigma_1^+} + \chi_{\sigma_\varepsilon^+}$ if $\nu(-1) = 1$ and $\chi_{\sigma_1^-} + \chi_{\sigma_\varepsilon^-}$ if $\nu(-1) = -1$. This is enought to obtain parts (1) and (2).

Part (3) can be proved using orthogonality relations.

For part (4), since π_ν^+ and π_ν^- are cuspidal, their restrictions to B don't contain any degree one representation of B. As we have already mentioned, the character of an irreducible representation of B of degree $(q-1)/2$ vanishes on elements of the form s_a, $a^2 \neq 1$. qed

Theorem. *Let ψ be a nontrivial character of F. Let λ and ν be the unique characters of $\mathbb{F}_q^\times$ and T of order 2, respectively. Set $\Gamma = \sum_{x \in \mathbb{F}_q^\times} \psi(x)\lambda(x)$. Let τ range over the characters of $\mathbb{F}_q^\times$ whose squares are nontrivial. Let θ range over the characters of T whose squares are nontrivial. Then the irreducible characters of $G = SL_2(\mathbb{F}_q)$ (for q odd and $q \neq 3$) are given below. Note that $\pi_\tau \simeq \pi_{\tau^{-1}}$ and $\pi_\theta \simeq \pi_{\theta^{-1}}$. In order, the rows give the values of the characters $\chi_{i_B^G \tau}$, the trivial character, the unique irreducible character of degree q, $\chi_{\pi_\lambda^+}$, $\chi_{\pi_\lambda^-}$, χ_{π_θ}, $\chi_{\pi_\nu^+}$, and $\chi_{\pi_\nu^-}$.*

± 1	s_a, $a^2 \neq 1$	$\pm u_1$	$\pm u_\varepsilon$	$\gamma \in T$, $\gamma^2 \neq 1$
$\tau(\pm 1)(q+1)$	$\tau(a) + \tau(a)^{-1}$	$\tau(\pm 1)$	$\tau(\pm 1)$	0
1	1	1	1	1
q	1	0	0	-1
$\lambda(\pm 1)(q+1)/2$	$\lambda(a)$	$\lambda(\pm 1)(1+\Gamma)/2$	$\lambda(\pm 1)(1-\Gamma)/2$	0
$\lambda(\pm 1)(q+1)/2$	$\lambda(a)$	$\lambda(\pm 1)(1-\Gamma)/2$	$\lambda(\pm 1)(1+\Gamma)/2$	0
$\theta(\pm 1)(q-1)$	0	$-\theta(\pm 1)$	$-\theta(\pm 1)$	$-\theta(\gamma) - \theta(\gamma)^{-1}$
$\nu(\pm 1)(q-1)/2$	0	$\nu(\pm 1)(-1+\Gamma)/2$	$\nu(\pm 1)(-1-\Gamma)/2$	$\nu(\gamma)$
$\nu(\pm 1)(q-1)/2$	0	$\nu(\pm 1)(-1-\Gamma)/2$	$\nu(\pm 1)(-1+\Gamma)/2$	$\nu(\gamma)$

In the next chapter we will describe (without proofs) how the above parametrization of the irreducible characters of $SL_2(\mathbb{F}_q)$ fits in with general results on the characters of finite groups of Lie type.

CHAPTER 4

Representations of finite groups of Lie type

Let $\mathbb{F}_q$ be a finite field of order q and characteristic p. Let G be a finite group of Lie type, that is, G is the $\mathbb{F}_q$-rational points of a connected reductive group $\mathbb{G}$ defined over $\mathbb{F}_q$. For example, if n is a positive integer $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ are finite groups of Lie type. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. Let

$$Sp_{2n}(\mathbb{F}_q) = \{\, g \in GL_{2n}(\mathbb{F}_q) \mid {}^t g J g = J \,\}.$$

Then $Sp_{2n}(\mathbb{F}_q)$ is a symplectic group of rank n and is a finite group of Lie type.

For $G = GL_n(\mathbb{F}_q)$ or $SL_n(\mathbb{F}_q)$ (and some other examples), the *standard Borel subgroup* B of G is the subgroup of G consisting of the upper triangular elements in G. A *standard parabolic subgroup* of G is a subgroup of G which contains the standard Borel subgroup B. If P is a standard parabolic subgroup of $GL_n(\mathbb{F}_q)$, then there exists a partition $(n_1, \ldots, n_r)$ of n (a set of positive integers n_j such that $n_1 + \cdots + n_r = n$) such that $P = P_{(n_1,\ldots,n_r)} = M \ltimes N$, where $M \simeq GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q)$ has the form

$$M = \left\{ \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_r \end{pmatrix} \;\middle|\; A_j \in GL_{n_j}(\mathbb{F}_q),\ 1 \le j \le r \right\}.$$

and

$$N = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{n_r} \end{pmatrix} \right\},$$

where $*$ denotes arbitary entries in $\mathbb{F}_q$. The subgroup M is called a (standard) Levi subgroup of P, and N is called the unipotent radical of P. Note that the partition $(1, 1, \ldots, 1)$ corresponds to B and (n) corresponds to G. A standard parabolic subgroup of $SL_n(\mathbb{F}_q)$ is equal to $P_{(n_1,\ldots,n_r)} \cap SL_n(\mathbb{F}_q)$ for some partition $(n_1, \ldots, n_r)$ of n. A *parabolic subgroup* of G is a subgroup which is conjugate to a standard parabolic subgroup. By replacing $\mathbb{F}_q$ by a field F in the above definitions, we can define parabolic subgroups of $GL_n(F)$. For more information on parabolic subgroups of general linear groups see the book of Alperin and Bell.

For convenience, assume that $G = GL_n(\mathbb{F}_q)$. Let $P = P_{(n_1,\ldots,n_r)}$ be a proper standard parabolic subgroup of G. Let π_j be an irreducible representation of $GL_{n_j}(\mathbb{F}_q)$, $1 \le j \le r$.

Then $\pi := \pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible representation of M. Extend π to a representation of P by letting elements of N act via the identity (on the space of π). Let $i_P^G \pi = \mathrm{Ind}_P^G \pi$ be the representation of G induced from π. This process of going from a representation of a Levi subgroup of a proper parabolic subgroup to a representation of G is known as parabolic induction (or Harish-Chandra induction). It is possible to show that if P' is another parabolic subgroup of G having M as a Levi subgroup, and unipotent radical N', then $i_P^G \pi \simeq i_{P'}^G \pi$. For this reason, the notation $i_M^G \pi$ is sometimes used in place of $i_P^G \pi$.

Now we can define a standard parabolic subgroup P_M of M to be a group of the form $P_M = P_1 \times \cdots \times P_r$ where P_j is a standard parabolic subgroup of $GL_{n_j}(\mathbb{F}_q)$. Write $P_j = M_j \ltimes N_j$, where M_j is the standard Levi subgroup of P_j and N_j is the unipotent radical of P_j. Let $N_M = N_1 \times \cdots N_r$. Then $P_M N$ is a standard parabolic subgroup of $GL_{n_j}(\mathbb{F}_q)$, with standard Levi factor $M_1 \times \cdots \times M_r$ and unipotent radical $N_M N$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ where σ_j is an irreducible representation of M_j. Then we can check that $i_{P_M N}^G \sigma \simeq i_P^G(i_{P_M}^M \sigma)$, where on the left side, σ is extended to $P_M N$ by letting elements of $N_M N$ act trivially, and on the right side, σ is extended to P_M by letting N_M act trivially. This property is called transitivity of parabolic induction.

Let N be the unipotent radical of a proper parabolic subgroup of G and let π be a representation of G. Then the restriction $r_G^N \pi = \pi_N$ of π to N is a direct sum of irreducible representations of N. Via Frobenius reciprocity type arguments, we can prove that if an irreducible representation π of G contains the trivial representation of the unipotent radical of proper parabolic subgroup of G, then π occurs as a subrepresentation of a parabolically induced representation of G. An irreducible representation π of G is *cuspidal* if $r_G^N \pi$ does not contain the trivial representation of N for all choices of unipotent radicals of proper parabolic subgroups of G. If π is cuspidal, then by Frobenius reciprocity, $\mathrm{Hom}_G(\pi, i_P^G \pi') = 0$ if P is a proper parabolic subgroup of G and π' is an irreducible representation of a Levi factor of P. The following theorem is valid for irreducible representations of finite groups of Lie type, not just for general linear groups.

Theorem. *(Proposition 9.13 of Carter's book) Let π be an irreducible representation of G. Then one of the following holds:*

(1) π is cuspidal

(2) There exists a proper parabolic subgroup $P = M \ltimes N$ of G and a cuspidal representation π' of M such that $\mathrm{Hom}_G(\pi, i_P^G \pi') \neq 0$.

As a consequence of the above theorem, one approach to finding the irreducible representations of G involves two steps:

Step 1 : Find all cuspidal representations of those groups occurring as Levi factors of parabolic subgroups of G (including G itself).

Step 2 : Find all of the irreducible constituents of the representations $i_P^G \pi'$ where P is a proper parabolic subgroup of G and π' is a cuspidal representation of a Levi factor of P.

In $SL_2(\mathbb{F}_q)$, there is one conjugacy class of proper parabolic subgroups, namely the conjugacy class of the standard Borel subgroup. And the subgroup $A \simeq \mathbb{F}_q^\times$ of B is a standard Levi factor. Every irreducible representation (one-dimensional representation) of A is cuspidal, since A has no proper parabolic subgroups. The above theorem tells us that the non-cuspidal representations of $GL_2(\mathbb{F}_q)$ are exactly the irreducible subrepresentations of the representations $i_B^G\tau$ as τ ranges over the set of one-dimensional representations of A. As we saw in Chapter 3, letting ε be a fixed nonsquare in $\mathbb{F}_q^\times$, the cuspidal representations of $SL_n(\mathbb{F}_q)$ are associated to the nontrivial characters of the group

$$T = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{F}_q,\ a^2 - b^2\varepsilon = 1 \right\}.$$

In the case of $SL_2(\mathbb{F}_q)$ the groups A and T are representatives for the conjugacy classes of maximal tori in $SL_2(\mathbb{F}_q)$.

In 1976, for G a finite group of Lie type, Deligne and Lusztig published a paper showing that it is possible to associate a virtual character of G to each character of a maximal torus (see references).

Next we describe the maximal tori in $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$. Let k be a positive integer. The $\mathbb{F}_{q^k}$ is a vector space over $\mathbb{F}_q$ of dimension k. Choose a basis $\beta = \{ x_1, \dots, x_k \}$ of $\mathbb{F}_{q^k}$ over $\mathbb{F}_q$. Given $y \in \mathbb{F}_{q^k}$, let $g_y \in GL_k(\mathbb{F}_q)$ be the matrix relative to β of the linear transformation on $\mathbb{F}_{q^k}$ given by left multiplication by y. Then $\{ g_y \mid y \in \mathbb{F}_{q^k}^\times \}$ is a subgroup of $GL_k(\mathbb{F}_q)$ which is isomorphic, via $y \mapsto g_y$, to $\mathbb{F}_{q^k}^\times$. This subgroup depends on the choice of basis β. Any two such subgroups of $GL_k(\mathbb{F}_q)$ are conjugate, via the relevant change of basis matrix.

Suppose that $n_1, \dots, n_r$ are integers such that $n_1 \geq n_2 \geq \cdots \geq n_r > 0$ and $n_1 + \cdots + n_r = n$. Fix a basis of $\mathbb{F}_{q^{n_j}}$ over $\mathbb{F}_q$, $1 \leq j \leq r$. Then the group $\mathbb{F}_{q^{n_1}}^\times \times \cdots \times \mathbb{F}_{q^{n_r}}^\times$, is isomorphic to a subgroup $T_{(n_1,\dots,n_r)}$ of $GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q) \subset GL_n(\mathbb{F}_q)$. A *maximal torus* T in $GL_n(\mathbb{F}_q)$ is a subgroup which is conjugate to $T_{(n_1,\dots,n_r)}$ for some $(n_1, \dots, n_r)$ as above. A maximal torus in $SL_n(\mathbb{F}_q)$ is conjugate to some $T_{(n_1,\dots,n_r)} \cap SL_n(\mathbb{F}_q)$. For $SL_2(\mathbb{F}_q)$ there are 2 conjugacy classes of maximal tori, represented by $T_{(1,1)} \cap SL_2(\mathbb{F}_q) = A$, and $T_{(2)} \cap SL_2(\mathbb{F}_q)$, the group T mentioned above.

Let T be a maximal torus in G and let θ be a character of T. Let $R_{T,\theta}$ be the virtual character of G associated to the pair (T, θ) by Deligne and Lusztig. The value of $R_{T,\theta}$ on an element g of G is given by an alternating sum of the trace of the action of g on certain ℓ-adic cohomology groups. For more information on this, see the paper of Deligne and Lusztig, or the book of Carter.

A matrix $u \in GL_n(\mathbb{F}_q)$ is *unipotent* if $u - 1$ is nilpotent. A matrix $\gamma \in GL_n(\mathbb{F}_q)$ is *semisimple* if γ is diagonalizable over some finite extension of $\mathbb{F}_q$. Let $g \in GL_n(\mathbb{F}_q)$. The characteristic polynomial of g, being a polynomial of degree n, splits over a finite extension $\mathbb{F}_{q^k}$ of $\mathbb{F}_q$, for some $k \leq n$. Results from linear algebra tell us that there exist matrices γ and $u \in GL_n(\mathbb{F}_{q^k})$ such that γ is diagonalizable and u is unipotent, with $\gamma u = u\gamma = g$. It can be shown that γ, $u \in GL_n(\mathbb{F}_q)$. Hence any element in $GL_n(\mathbb{F}_q)$ can be expressed in a unique way in the form $g = \gamma u$, with $\gamma \in GL_n(\mathbb{F}_q)$ semisimple, $u \in GL_n(\mathbb{F}_q)$ unipotent, and $\gamma u = u\gamma$. This is called the (multiplicative) *Jordan decomposition* of g. If G is a finite group of Lie type, then G is a subgroup of $GL_n(\mathbb{F}_q)$ for some n, and if $g = \gamma u$ is the Jordan decomposition of g in $GL_n(\mathbb{F}_q)$, the elements γ and u lie in G. So there is a Jordan decomposition for elements of a finite group of Lie type.

Attached to any a maximal torus T in G, there exists a particular class function Q_T^G on G, called the *Green function* corresponding to T. The Green function Q_T^G is supported on the unipotent set. If $G = GL_n(\mathbb{F}_q)$, the values of the Green functions are known. In fact, if u is unipotent the value $Q_T^G(u)$ is obtained as the value of a certain polynomial (depending on T and $\mathrm{cl}_G(u)$) in the variable q.

Theorem (Character formula for $R_{T,\theta}$). *Let T be a maximal torus in $G = GL_n(\mathbb{F}_q)$ or $SL_n(\mathbb{F}_q)$ and let θ be a character of T. Let $g \in GL_n(\mathbb{F}_q)$ have Jordan decomposition $g = \gamma u$, with semisimple part γ and unipotent part u. Let H be the centralizer of γ in G. Then*

$$R_{T,\theta}(g) = |H|^{-1} \sum_{x \in G,\, x^{-1}\gamma x \in T} \theta(x^{-1}\gamma x) Q^H_{xTx^{-1}}(u).$$

Exercises: Let $\gamma \in GL_n(\mathbb{F}_q)$ be semisimple.

(1) Prove that some conjugate of γ lies in $T_{(n_1,\ldots,n_r)}$ for some (not necessarily unique) $(n_1, \ldots, n_r)$.

(2) Prove that the centralizer H of γ in G is isomorphic to a direct product of the form $GL_{r_1}(\mathbb{F}_{q^{s_1}}) \times \cdots \times GL_{r_t}(\mathbb{F}_{q^{s_t}})$, where $r_1 s_1 + \cdots + r_t s_t = n$.

The character formula for $R_{T,\theta}$ resembles the Frobenius character formula in certain ways. If the conjugacy class of the semisimple part γ of g does not intersect T, then $R_{T,\theta}(g) = 0$. And when the conjugacy class of γ does intersect T, the expression for $R_{T,\theta}(g)$ involves values of θ on certain conjugates of γ in T. In the special case where $T = T_{(1,\ldots,1)}$ or $T_{(1,\ldots,1)} \cap SL_n(\mathbb{F}_q)$, then it can be shown that $R_{T,\theta} = \pm \chi_{i_B^G \theta}$.

If T and T' are maximal tori in G, let

$$N(T, T') = \{\, g \in G \mid gTg^{-1} = T' \,\}$$
$$W(T, T') = \{\, Tg \mid g \in N(T, T') \,\}$$

Theorem (Orthogonality relations for $R_{T,\theta}$'s). *Let θ and θ' be characters of T and T', respectively. Then*

$$\langle R_{T,\theta}, R_{T',\theta} \rangle = \{\, \omega \in W(T, T') \mid {}^{\omega}\theta' = \theta \,\},$$

where ${}^{\omega}\theta'(\gamma) := \theta'(g\gamma g^{-1})$, *for* $\gamma \in T$ *and* $\omega = Tg$.

Corollary. *If* T *and* T' *are not* G*-conjugate, then* $\langle R_{T,\theta}, R_{T',\theta'}\rangle = 0$.

Note that this corollary does not tell us that the set of irreducible characters appearing in $R_{T,\theta}$ is disjoint from those appearing in $R_{T',\theta'}$ when T and T' are not conjugate. Consider the example of $G = SL_2(\mathbb{F}_q)$. Let τ be a character of the maximal torus A in $G = SL_2(\mathbb{F}_q)$. According to comments above, $R_{A,\tau} = i_B^G\tau$. Let T be as in Chapter 3. Then it is possible to show that if θ is a character of T and φ_θ is as in Chapter 3, then $R_{T,\theta} = \pm\varphi_\theta$. Now suppose that $\tau = \tau_0$ is the trivial character of A and $\theta = \theta_0$ is the trivial character of T. Let χ_q be the character of the irreducible representation of $SL_2(\mathbb{F}_q)$ of degree q. Let χ_0 be the character of the trivial representation. As we saw in Chapter 3, $\chi_{i_B^G\tau_0} = R_{A,\tau_0} = \chi_0 + \chi_q$ and $\varphi_\theta = -\chi_0 + \chi_q$.

Let $N_G(T) = \{\, g \in G \mid gTg^{-1} = T \,\}$ be the normalizer of T in G. Then $W(T) := N_G(T)/T$ is a finite group. A character θ of T is said to be in *general position* if $\theta^w \neq \theta$ for all nontrivial elements of $W(T)$.

Theorem (corollary of above theorem). *If* θ *is in general position, then* $\pm R_{T,\theta}$ *is an irreducible character of* G.

Theorem. *If* π *is an irreducible representation of* G*, then there exists a maximal torus* T *of* G *and a character* θ *of* T *such that* $\langle \chi_\pi, R_{T,\theta}\rangle \neq 0$.

Suppose that $(n_1, \ldots, n_r) \neq (n)$. Then $T_{(n_1,\ldots,n_r)}$ is a subgroup of the Levi factor $M = GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q)$ of a proper parabolic subgroup P of G. Any class function f on M can be extended to a class function on $P = M \ltimes N$ by setting $f(mu) = f(m)$, $m \in M$, $u \in N$. Hence we can view i_P^G as a map from class functions on M to class functions on G. Deligne and Lusztig proved that if $T = T_{(n_1,\ldots,n_r)}$ and θ is a character of T, then $R_{T,\theta} = i_P^G(R^M_{T,\theta})$, where $R^M_{T,\theta}$ is the virtual character of M corresponding to the pair (T,θ). (Note that T is a maximal torus in M). It follows that $R_{T,\theta}$ is in the span of the irreducible characters of G which occur as constituents of representations $i_P^G\sigma$, for various representations σ of M. Thus $\langle \chi_\pi, R_{T,\theta}\rangle = 0$ for all irreducible cuspidal representations π.

Proposition. *Suppose that* π *is an (irreducible) cuspidal representation of* $GL_n(\mathbb{F}_q)$*,* $n \geq 2$*. Then*

(1) $\langle \chi_\pi, R_{T_{(n)},\theta}\rangle \neq 0$ *for some character* θ *of* $T_{(n)}$.

(2) $\chi_\pi = \pm R_{T_{(n)},\theta}$ *for some character* θ *of* $T_{(n)}$ *that is in general position.*

(3) If T *is a maximal torus which is not conjugate to* $T_{(n)}$ *and* θ *is a character of* T*, then* $\langle \chi_\pi, R_{T,\theta}\rangle = 0$.

Parts (1) and (3) have analogues for cuspidal representations for other finite groups of Lie type. In those cases, the maximal torus $T_{(n)}$ must be replaced by a set of representatives for the conjugacy classes of maximal tori in G that do no intersect the Levi factor of any proper parabolic subgroup of G. The analogue of Part (2) does not hold for all cuspidal representations of other finite groups of Lie type. The two irreducible characters of of $SL_2(\mathbb{F}_q)$ of degree $(q-1)/2$ which were produced in Chapter 3 correspond to cuspidal representations whose characters are not of the form $\pm R_{T,\theta}$ (for any choice of T and θ). All of the irreducible characters of $SL_2(\mathbb{F}_q)$ of degree $q-1$ correspond to cuspidal representations whose characters equal $\pm R_{T,\theta}$ (with $\theta^2 \neq 1$, and T as in Chapter 3).

Hecke algebras are often used in the study of the representations of finite groups of Lie type. Suppose that P and P' are parabolic subgroups of G and (π, V) and (π', V') are irreducible representations of the Levi factors M and M' of standard parabolic subgroups P and P', respectively. Then, according to results from Chapter 2, the space $\mathrm{Hom}_G(i_P^G\pi, i_{P'}^G\pi')$ is isomorphic to the space of functions $\varphi : G \to \mathrm{End}_{\mathbb{C}}(V, V')$ such that $\varphi(xgx') = \pi(x) \circ \varphi(g) \circ \pi'(x')$ for all $g \in G$, $x \in P$, and $x' \in P'$. In view of results discussed above, the case where π and π' are cuspidal is of particular interest. The following theorem is proved by studying properties of the above isomorphism - in particular, by analyzing the properties of the functions supported on each double coset PgP' and satisfying the above conditions.

Theorem. *Let π and π' be cuspidal representations of Levi factors M and M' of standard parabolic subgroups P and P' of $GL_n(\mathbb{F}_q)$, respectively. Then*
(1) If M and M' are not conjugate, then $\mathrm{Hom}_G(i_P^G\pi, i_{P'}^G\pi') = 0$.
(2) If M and M' are conjugate, then either $\mathrm{Hom}_G(i_P^G\pi, i_{P'}^G\pi') = 0$ or $i_P^G\pi \simeq i_{P'}^G\pi'$.

Suppose that $P' = P$ and $\pi' = \pi$. Then $\mathcal{H}(G, \pi)$ is a Hecke algebra which is isomorphic (as an algebra) to $\mathrm{Hom}_G(i_P^G\pi, i_P^G\pi)$. It is possible to prove that decomposition of representations of the form $i_P^G\pi$ into direct sums of irreducible representations can be reduced to cases where $P = P_{(m,\ldots,m)}$ and $m \neq n$ is a divisor of n (with m occurring n/m times). In those cases, $\pi = \sigma \otimes \cdots \otimes \sigma$ for some cuspidal representation σ of $GL_m(\mathbb{F}_q)$ (where σ occurs n/m times in the tensor product). Let $G' = GL_{n/m}(\mathbb{F}_{q^m})$. Let B' be the standard Borel subgroup of G'. Then, letting π' be the trivial representation of G', we know that the Hecke algebra $\mathcal{H}(G', \pi')$ is isomorphic to $\mathrm{Hom}_{G'}(i_{B'}^{G'}\pi', i_{B'}^{G'}\pi')$.

Theorem. *(Notation as above). The algebra $\mathcal{H}(G, \pi)$ is isomorphic (in a canonical way) to the Hecke algebra $\mathcal{H}(G', \pi')$.*

Corollary. *There is a canonical bijection $\tau \leftrightarrow \tau'$ between the set of irreducible constituents τ of $i_P^G\pi$ and the set of irreducible constituents of τ' of $i_{B'}^{G'}\pi'$. The bijection satisifes:*
(1) The multiplicity of π in $r_G^P\tau$ equals the multiplicity of the trivial representation π' of B' in $r_{G'}^{B'}\tau$.

(2) The degree of τ divided by the degree of τ' equals the degree of π times $|G||P|^{-1}|G'|^{-1}|B'|$.

CHAPTER 5

Topological Groups, Representations, and Haar Measure

5.1. Topological spaces

If X is a set, a family $\mathcal{U}$ of subsets of X defines a *topology* on X if

(i) $\emptyset \in \mathcal{U}$, $X \in \mathcal{U}$.
(ii) The union of any family of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.
(iii) THe intersection of a finite number of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.

If $\mathcal{U}$ defines a topology on X, we say that X is a *topological space.* The sets in $\mathcal{U}$ are called *open sets.* The sets of the form $X \setminus U$, $U \in \mathcal{U}$, are called *closed sets.* If Y is a subset of X the *closure* of Y is the smallest closet set in X that contains Y.

Let Y be a subset of a topological space X. Then we may define a topology $\mathcal{U}_Y$ on Y, called the *subspace* or *relative* topology, or the topology on Y *induced* by the topology on X, by taking $\mathcal{U}_Y = \{\, Y \cap U \mid U \in \mathcal{U}\,\}$.

A system $\mathcal{B}$ of subsets of X is called a *basis* (or *base*) for the topology $\mathcal{U}$ if every open set is the union of certain sets in $\mathcal{B}$. Equivalently, for each open set U, given any point $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Example: The set of all bounded open intervals in the real line $\mathbb{R}$ forms a basis for the usual topology on $\mathbb{R}$.

Let $x \in X$. A *neighbourhood* of x is an open set containing x. Let $\mathcal{U}_x$ be the set of all neighbourhoods of x. A subfamily $\mathcal{B}_x$ of $\mathcal{U}_x$ is a *basis* or *base* at x, a *neighbourhood basis* at x, or a *fundamental system of neighbourhoods* of x, if for each $U \in \mathcal{U}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subset U$. A topology on X may be specified by giving a neighbourhood basis at every $x \in X$.

If X and Y are topological spaces, there is a natural topology on the Cartesian product $X \times Y$ that is defined in terms of the topologies on X and Y, called the *product topology.* Let $x \in X$ and $y \in Y$. The sets $U_x \times V_y$, as U_x ranges over all neighbourhoods of x, and V_y ranges over all neighbourhoods of y forms a neighbourhood basis at the point $(x, y) \in X \times Y$ (for the product topology).

If X and Y are topological spaces, a function $f : X \to Y$ is *continuous* if whenever U is an open set in Y, the set $f^{-1}(U) = \{\, x \in X \mid f(x) \in U\,\}$ is an open set in X. A function $f : X \to Y$ is a *homeomorphism* (of X onto Y) if f is bijective and both f and f^{-1} are continuous functions.

An *open covering* of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family. A *subcover* of an open covering is a an open covering of X which consists of sets belonging to the open covering. A topological space X is *compact* if every open covering of X contains a finite subcover.

A subset Y of a topological space X is *compact* if it is compact if Y is compact in the subspace topology. A topological space X is *locally compact* if for each $x \in X$ there exists a neighbourhood of x whose closure is compact.

A topological space X is Hausdorff (or T_2) if given distinct points x and $y \in X$, there exist neighbourhoods U of x and V of y such that $U \cap V = \emptyset$. A closed subset of a locally compact Hausdorff space is locally compact.

5.2. Topological groups

A *topological group* G is a group that is also a topological space, having the property the maps $(g_1, g_2) \mapsto g_1 g_2$ from $G \times G \to G$ and $g \mapsto g^{-1}$ from G to G are continuous maps. In this definition, $G \times G$ has the product topology.

Lemma. *Let G be a topological group. Then*

(1) The map $g \mapsto g^{-1}$ is a homeomorphism of G onto itself.

(2) Fix $g_0 \in G$. The maps $g \mapsto g_0 g$, $g \mapsto g g_0$, and $g \mapsto g_0 g g_0^{-1}$ are homeomorphisms of G onto itself.

A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G, and let $p : G \to G/H$ be the canonical mapping of G onto G/H. We define a topology $\mathcal{U}_{G/H}$ on G/H, called the *quotient topology*, by $\mathcal{U}_{G/H} = \{\, p(U) \mid U \in \mathcal{U}_G \,\}$. (Here, $\mathcal{U}_G$ is the topology on G). The canonical map p is open (by definition) and continuous. If H is a closed subgroup of G, then the topological space G/H is Hausdorff. If H is a normal subgroup of G, then G/H is a topological group.

If G and G' are topological groups, a map $f : G \to G'$ is a *continuous homomorphism* of G into G' if f is a homomorphism of groups and f is a continuous function. If H is a closed normal subgroup of a topological group G, then the canonical mapping of G onto G/H is an open continuous homomorphism of G onto G/H.

A topological group G is a *locally compact group* if G is locally compact as a topological space.

Proposition. *Let G be a locally compact group and let H be a closed subgroup of G. Then*

(1) H is a locally compact group (in the subspace topology).

(2) If H is normal in G, then G/H is a locally compact group.

(3) If G' is a locally compact group, then $G \times G'$ is a locally compact group (in the product topology).

5.3. General linear groups and matrix groups

Let F be a field that is a topological group (relative to addition). Assume that points in F are closed sets in the topology on F. For example, we could take $F = \mathbb{R}$,

$\mathbb{C}$ or the p-adic numbers $\mathbb{Q}_p$, p prime. Let n be a positive integer. The space $M_{n\times n}(F)$ of $n \times n$ matrices with entries in F is a topological group relative to addition, when $M_{n\times n}(F) \simeq F^{n^2}$ is given the product topology. The multiplicative group $GL_n(F)$, being a subset (though not a subgroup) of $M_{n\times n}(F)$, is a topological space in the subspace topology. The determinant map det from $M_{n\times n}(F)$ to F, being a polynomial in matrix entries, is a continuous function. Now $F^\times = F\backslash\{0\}$ is an open subset of F (since points are closed in F). Therefore, by continuity of det, $GL_n(F) = \det^{-1}(F^\times)$ is an open subset of $M_{n\times n}(F)$. It is easy to show that matrix multiplication, as a map (not a homomorphism) from $M_{n\times n}(F) \times M_{n\times n}(F)$ to $M_{n\times n}(F)$ is continuous. It follows that the restriction to $GL_n(F) \times GL_n(F)$ is also continuous. Let $g \in GL_n(F)$. Recall that Cramer's rule gives a formula for the ijth entry of g^{-1} as the determinant of the matrix given by deleting the ith row and jth column of g, divided by $\det g$. Using this, we can prove that $g \mapsto g^{-1}$ is a continuous map from $GL_n(F)$ to $GL_n(F)$. Therefore $GL_n(F)$ is a topological group. We can also see that if F is a locally compact group (for example if $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$), then $GL_n(F)$ is a locally compact group.

The group $SL_n(F)$, being the kernel of the continuous homomorphism $\det : GL_n(F) \to F^\times$, is a closed subgroup of $SL_n(F)$, so is a locally compact group whenever F is a locally compact group. If I_n is the $n \times n$ identity matrix and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then $Sp_{2n}(F) = \{\, g \in GL_{2n}(F) \mid {}^tgJg = J \,\}$ is a closed subgroup of $GL_{2n}(F)$. If $S \in GL_n(F)$ is a symmetric matrix, that is ${}^tS = S$, the group $O_n(S) = \{\, g \in GL_n(F) \mid {}^tgSg = S \,\}$ is an *orthogonal group*, and is a closed subgroup of $GL_n(F)$. Depending on the field F, different choices of S can give rise to non-isomorphic orthogonal groups. If E is a quadratic extension of F and $X \in M_{n\times n}(E)$, let $\bar{X}$ be the matrix obtained from X by letting the nontrivial element of the Galois group $Gal(E/F)$ act on each the entries of X. Suppose that $h \in GL_n(E)$ is a matrix such that ${}^t\bar{h} = h$ (h is *hermitian*). Then the group $U(h) = \{\, g \in GL_{2n}(E) \mid {}^t\bar{g}hg = h \,\}$ is called a *unitary group* and is a closed subgroup of $GL_n(E)$. If $(n_1, \ldots, n_r)$ is a partition of n then the corresponding standard parabolic subgroup $P = P_{(n_1,\ldots,n_r)}$ of $GL_n(F)$ is a closed subgroup of $GL_n(F)$, as are any Levi factor of P, and the unipotent radical of P.

5.4. Matrix Lie groups

A *Lie group* is a topological group that is a differentiable manifold with a group structure in which the multiplication and inversion maps from $G\times G$ to G and from G to G are smooth maps. Without referring to the differentiable manifolds, we may define a *matrix Lie group*, or a *closed Lie subgroup of* $GL_n(\mathbb{C})$ to be a closed subgroup of the topological group $GL_n(\mathbb{C})$. (This latter definition is reasonable because $GL_n(\mathbb{C})$ is a Lie group, and it can be shown that a closed subgroup of a Lie group is also a Lie group). A connected matrix Lie group is *reductive* if it is stable under conjugate transpose, and *semisimple* if it is reductive and has finite centre. The book of Hall [Hall] gives an introduction to matrix Lie groups, their structure, and their finite-dimensional representations. For other references on Lie groups and their representations, see [B], [K1] and [K2].

5.5. Finite-dimensional representations of topological groups and matrix Lie groups

Let G be a topological group. A (complex) finite-dimensional *continuous* representation of G is a finite-dimensional (complex) representation (π, V) of G having the property that the map $g \mapsto [\pi(g)]_\beta$ from G to $GL_n(\mathbb{C})$ is a continuous homomorphism for some (hence any) basis β of V. The continuity property is equivalent to saying that every matrix coefficient of π is a continuous function from G to $\mathbb{C}$. Hence to prove the following lemma, we need only observe that the character of π is a finite sum of matrix coefficients of π.

Lemma. *Let π be a continuous finite-dimensional representation of G. Then the character χ_π of π is a continuous function on G.*

Example: Let π be a continuous one-dimensional representation of the locally compact group $\mathbb{R}$. Then π is a continuous function from $\mathbb{R}$ to $\mathbb{C}$ such that $\pi(0) = 1$ and $\pi(t_1+t_2) = \pi(t_1)\pi(t_2)$ for all t_1, $t_2 \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{C}$ is continuously differentiable and the support of f is contained in a compact subset of $\mathbb{R}$, then $\int_{-\infty}^{\infty} f(t)\pi(t)\,dt$ converges. Choose f so that $c = \int_{-\infty}^{\infty} f(t)\pi(t)\,dt \neq 0$. Multiplying $\pi(t_1 + t_2)$ by $f(t_2)$ and integrating, we have

$$\int_{-\infty}^{\infty} f(t_2)\pi(t_1 + t_2)\,dt_2 = \pi(t_1)\int_{-\infty}^{\infty} f(t_2)\pi(t_2)\,dt_2 = c\pi(t_1), \qquad t_1 \in \mathbb{R}.$$

Then

$$\pi(t_1) = c^{-1}\int_{-\infty}^{\infty} f(t_2)\pi(t_1 + t_2)\,dt_2 = c^{-1}\int_{-\infty}^{\infty} \pi(t)f(t - t_1)\,dt, \qquad t_1 \in \mathbb{R}.$$

Because $t_1 \mapsto \int_{-\infty}^{\infty} \pi(t)f(t - t_1)\,dt$ is a differentiable function of t_1, we see that π is a differentiable function. Differentiating both sides of $\pi(t_1 + t_2) = \pi(t_1)\pi(t_2)$ with respect to t_1 and then setting $t_1 = 0$ and $t = t_2$, we obtain $\pi'(t) = \pi'(0)\pi(t)$. Setting $k = \pi'(0)$, we have $\pi'(t) = k\pi(t)$, $t \in \mathbb{R}$. Solving this differential equation yields $\pi(t) = ae^{kt}$ for some $a \in \mathbb{C}$. And $\pi(0) = 1$ forces $a = 1$. Hence $\pi(t) = e^{kt}$. Now if we take a $z \in \mathbb{C}$, it is clear that $t \mapsto e^{zt}$ is a one-dimensional continuous representation of $\mathbb{R}$.

Lemma. *Let $z \in \mathbb{C}$. Then $\pi_z(t) = e^{zt}$ defines a one-dimensional continuous representation of $\mathbb{R}$. The representation π_z is unitary if and only if the real part of z equals 0. Each one-dimensional continuous representation of $\mathbb{R}$ is of the form π_z for some $z \in \mathbb{C}$, and* *any one-dimensional continuous representation of $\mathbb{R}$ is a smooth (infinitely differentiable) function of t.*

Theorem. *A continuous homomorphism from a Lie group G to a Lie group G' is smooth.*

Let G be a Lie group (for example, a matrix Lie group). Then, because $GL_n(\mathbb{C})$ is a Lie group, via a choice of basis for the space of the representation, a finite-dimensional representation of G is a continuous homomorphism from G to $GL_n(\mathbb{C})$. According to the theorem, the representation must be a smooth map from G to $GL_n(\mathbb{C})$.

Corollary. *A continuous finite-dimensional representation of a Lie group is smooth.*

Definition. A *Lie algebra* over a field F is a vector space $\mathfrak{g}$ over F endowed with a bilinear map, the *Lie bracket*, denoted $(X,Y) \mapsto [X,Y] \in \mathfrak{g}$ satisfying $[X,Y] = -[Y,X]$ and the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \forall\ X,\ Y,\ Z \in \mathfrak{g}.$$

If G is a Lie group, the Lie algebra $\mathfrak{g}$ of G is defined to be the set of left-G-invariant smooth vector fields on G. A vector field is a smoothly varying family of tangent vectors, one for each $g \in G$, and it can be shown that if X is identified with the corresponding tangent vector at the identity element, then the Lie algebra $\mathfrak{g}$ is identified with the tangent space at the identity.

If we work with matrix Lie groups, we can take a different approach. If $X \in M_{n\times n}(\mathbb{C})$, then the matrix exponential $e^X = \sum_{k=0}^{\infty} X^k/k!$ is an element of $GL_n(\mathbb{C})$.

Proposition. *Let G be a matrix Lie group. Then the Lie algebra $\mathfrak{g}$ is equal to*

$$\mathfrak{g} = \{\, X \in M_{n\times n}(\mathbb{C}) \mid e^{tX} \in G\ \forall\ t \in \mathbb{R}\,\}.$$

and the bracket $[X,Y]$ of two elements of $\mathfrak{g}$ is equal to the element $XY - YX$ of $M_{n\times n}(\mathbb{C})$.

Using the fact that $\det(e^X) = e^{\operatorname{tr} X}$, we can see that the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, resp. $\mathfrak{sl}_n(\mathbb{R})$, of $SL_n(\mathbb{C})$, resp. $SL_n(\mathbb{R})$, is just the set of matrices in $M_{n\times n}(\mathbb{C})$, resp. $M_{n\times n}(\mathbb{R})$, that have trace equal to 0. Now suppose that $G = Sp_{2n}(F)$ with $F = \mathbb{R}$ or $\mathbb{C}$. Note that $e^{tX} \in G$ if and only if $Je^{tX^t}J^{-1} = e^{-tX}$. From

$$X = \frac{d}{dt}(e^{tX})\,|_{t=0} = \lim_{t\to 0}(e^{tX} - 1)/t.$$

we can see that $Je^{tX^t}J^{-1} = e^{-tX}$ for all $t \in \mathbb{R}$ implies $JX^tJ^{-1} = -X$. The converse is easy to see. Therefore the Lie algebra $\mathfrak{sp}_{2n}(F)$ of $Sp_{2n}(F)$ is given by

$$\mathfrak{sp}_{2n}(F) = \{\, X \in M_{n\times n}(F) \mid JX^tJ^{-1} = -X\,\} = \{\, X \in M_{n\times n}(F) \mid JX^t + XJ = 0\,\}.$$

The same type of approach can be used to find the Lie algebras of orthogonal and unitary matrix Lie groups.

If V is a finite-dimensional complex vector space, $\mathrm{End}_{\mathbb{C}}(V)$ is a Lie algebra relative to the bracket $[X,Y] = X \circ Y - Y \circ X$. This Lie algebra is denoted by $\mathfrak{gl}(V)$. A linear map $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a *Lie algebra homomorphism* if

$$\phi([X,Y]) = \phi(X) \circ \phi(Y) - \phi(Y) \circ \phi(X), \qquad X, Y \in \mathfrak{g}.$$

A finite-dimensional representation of $\mathfrak{g}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ for some finite-dimensional complex vector space V.

Proposition. *Let G be a matrix Lie group, and let (π, V) be a continuous finite-dimensional representation of G. Then there is a unique representation $d\pi$ of of the Lie algebra $\mathfrak{g}$ of G (acting on the space V) such that*

$$\pi(e^X) = e^{d\pi(X)}, \qquad X \in \mathfrak{g}.$$

Furthermore $d\pi(X) = \frac{d}{dt}\pi(e^{tX})\,|_{t=0}$, $X \in \mathfrak{g}$, and π is irreducible if and only if $d\pi$ is irreducible.

If the matrix Lie group G is simply connected, that is, the topological space G is simply connected, then any finite-dimensional representation of $\mathfrak{g}$ lifts to a finite-dimensional representation of G, and the representations of G and $\mathfrak{g}$ are related as in the above proposition.

5.6. Groups of t.d. type

A Hausdorff topological group is a t.d. group if G has a countable neighbourhood basis at the identity consisting of compact open subgroups, and G/K is a countable set for every open subgroup K of G. Some t.d. groups are matrix groups over p-adic fields.

Let p be a prime. Let $x \in \mathbb{Q}^\times$. Then there exist unique integers m, n and r such that m and n are nonzero and relatively prime, p does not divide m or n, and $x = p^r m/n$. Set $|x|_p = p^{-r}$. This defines a function on $\mathbb{Q}^\times$, which we extend to a function from $\mathbb{Q}$ to the set of nonnegative real numbers by setting $|0|_p = 0$. The function $|\cdot|_p$ is called the *p-adic absolute value* on $\mathbb{Q}$. It is a valuation on $\mathbb{Q}$ - that is, it has the properties

(i) $|x|_p = 0$ if and only if $x = 0$

(ii) $|xy|_p = |x|_p|y|_p$

(iii) $|x+y|_p \le |x|_p + |y|_p$.

The usual absolute value on the real numbers is another example of a valuation on $\mathbb{Q}$. The p-adic abolute value satisfies the *ultrametric* inequality, that is, $|x+y|_p \le \max\{|x|_p, |y|_p\}$. Note that the ultrametric inequality implies property (iii) above. A valuation that satisfies the ultrametric inequality is called a *nonarchimedean* valuation.

Note that the set $\{ |x|_p \mid x \in \mathbb{Q}^\times \}$ is a discrete subgroup of $\mathbb{R}^\times$. Hence we say that $|\cdot|_p$ is a *discrete* valuation. The usual absolute value on $\mathbb{Q}$ is an example of an *archimedean* valuation. Clearly it is not a discrete valuation. Two valuations on a field F are said to be *equivalent* is one is a positive power of the other.

Theorem. *(Ostrowski) A nontrivial valuation on $\mathbb{Q}$ is equivalent to the usual absolute value or to $|\cdot|_p$ for some prime p.*

If F is a field and $|\cdot|$ is a valuation on F, the *topology on F induced by* $|\cdot|$ has as a basis the sets of the form $U(x,\epsilon) = \{ y \in F \mid |x-y| < \epsilon \}$, as x varies over F, and ϵ varies over all positive real numbers. A field F' with valuation $|\cdot|'$ is a *completion* of the field F with valuation $|\cdot|$ if $F \subset F'$, $|x|' = |x|$ for all $x \in F$, F' is complete with respect to $|\cdot|'$ (every Cauchy sequence with respect to $|\cdot|'$ has a limit in F') and F' is the closure of F with respect to $|\cdot|$. So F' is the smallest field containing F such that F' is complete with respect to $|\cdot|'$.

The real numbers is the completion of $\mathbb{Q}$ with respect to the usual absolute value on $\mathbb{Q}$.

The *p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. (We denote the extension of $|\cdot|_p$ to $\mathbb{Q}_p$ by $|\cdot|_p$ also). The p-adic integers $\mathbb{Z}_p$ is the set $\{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \}$. Note that $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$ (this follows from the ultrametric inequality and the mulitiplicative property of $|\cdot|_p$), and $\mathbb{Z}_p$ contains $\mathbb{Z}$. The set $p\mathbb{Z}_p$ (the ideal of $\mathbb{Q}_p$ generated by the element p) is a maximal ideal of $\mathbb{Z}_p$ and $\mathbb{Z}_p/p\mathbb{Z}_p$ is therefore a field.

Let $x \in \mathbb{Q}^\times$. Write $x = p^r m/n$ with $r \in \mathbb{Z}$ and m and n nonzero integers such that m and n are relatively prime and not divisible by p. Because m and n are relatively prime and not divisible by p, the equation $nX \equiv m (\operatorname{mod} p)$ has a unique solution $a_r \in \{ 1, \ldots, p-1 \}$. That is, there is a unique integer $a_r \in \{1, \ldots, p-1\}$ such that p divides $m - na_r$. Since $|n|_p = 1$, p divides $m - na_r$ is equivalent to $|(m/n) - a_r|_p < 1$, and also to $|x - a_r p^r|_p < |x|_p = p^{-r}$. Expressing $x - a_r p^r$ in the form $p^s m'/n'$ with $s > r$ and m' and n' relatively prime integers, we repeat the above argument to produce an integer $a_s \in \{1, \ldots, p-1\}$ such that

$$|x - a_r p^r - a_s p^s|_p < |x - a_r p^r|_p = p^{-s}.$$

If $s > r+1$, set $a_{r+1} = a_{r+2} = \cdots a_{s-1} = 0$, to get

$$|x - \sum_{n=r}^{s} a_n p^n|_p < p^{-s}.$$

Continuing in this manner, we see that there exists a sequence $\{ a_n \mid n \geq r \}$ such that $a_n \in \{ 0, 1, \ldots, p-1 \}$ and, given any integer $M \geq r$,

$$|x - \sum_{n=r}^{M} a_n p^n|_p < p^{-M}.$$

It follows that $\sum_{n=r}^{\infty} a_n p^n$ converges in the p-adic topology to the rational number x.

On the other hand, it is quite easy to show that if $a_n \in \{0, 1, \ldots, p-1\}$ and r is an integer, then $\sum_{n=r}^{\infty} a_n p^n$ converges to an element of $\mathbb{Q}_p$ (though not necessarily to a rational number).

Lemma. *A nonzero element x of $\mathbb{Q}_p$ is uniquely of the form $\sum_{n=r}^{\infty} a_n p^n$, with $a_n \in \{0, 1, 2, \ldots, p-1\}$, for some integer r with $a_r \neq 0$. Furthermore, $|x|_p = p^{-r}$. (Hence $x \in \mathbb{Z}_p$ if and only if $r \geq 0$).*

Lemma. *$\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$.*

Proof. Let $a \in \mathbb{Z}_p$. According to the above lemma, $a = \sum_{n=r}^{\infty} a_n p^n$ for some sequence $\{ a_n \,|\, n \geq r \}$, where $|a|_p = p^{-r} \leq 1$ implies that $r \geq 0$. If $r > 0$, then $a \in p\mathbb{Z}_p$. For convenience, set $a_0 = 0$ when $|a|_p < 1$. If $r = 0$, then $a_0 \in \{ 1, \ldots, p-1 \}$. Define a map from $\mathbb{Z}_p$ to $\mathbb{Z}/p\mathbb{Z}$ by $a \mapsto a_0$. This is a surjective ring homomorphism whose kernel is equal to $p\mathbb{Z}_p$. qed

A *local field* F is a (nondiscrete) field F which is locally compact and complete with respect to a nontrivial valuation. The fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$, p prime, are local fields. If $|\cdot|$ is a nontrivial nonarchimedean valuation on a field F, then $\{ x \in F \mid |x| < 1\}$ is a maximal ideal in the ring $\{ x \in F \mid |x| \leq 1 \}$, so the quotient is a field, called the *residue class field* of F. The following lemma can be used to check that $\mathbb{Q}_p$ is a local field.

Lemma. *Let $|\cdot|$ be a nonarchimedean valuation on a field F. Then F is locally compact with respect to $|\cdot|$ if and only if*

(1) F is complete (with respect to $|\cdot|$)

(2) $|\cdot|$ is discrete

(3) The residue class field of F is finite.

For every integer N, $p^N\mathbb{Z}_p$ is a compact open (and closed) subgroup of $\mathbb{Q}_p$. It is not hard to see that $\{ p^N\mathbb{Z}_p \mid N \geq 0 \}$ forms a countable neighbourhood basis at the identity element 0. From the above lemma, we have that $\mathbb{Q}_p^\times \simeq \langle p \rangle \times \mathbb{Z}_p^\times$. Hence $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete. It can be shown that any open subgroup of $\mathbb{Q}_p$ is of the form $p^N\mathbb{Z}_p$ for some integer N. Thus $\mathbb{Q}_p/K$ is discrete for every open subgroup K of $\mathbb{Q}_p$. So the group $\mathbb{Q}_p$ is a t.d. group. For more information on valuations, the p-adic numbers, and p-adic fields, see the beginning of [M] (course notes for Mat 1197).

As discussed in the section 5.3, because $\mathbb{Q}_p$ is locally compact, the topological group $GL_n(\mathbb{Q}_p)$ is also locally compact. In fact $GL_n(\mathbb{Q}_p)$ is a t.d. group. If j is a positive integer, let K_j be the set of $g \in GL_n(\mathbb{Q}_p)$ such that every entry of $g-1$ belongs to $p^j\mathbb{Z}_p$. Then K_j is a compact open subgroup, and $\{ K_j \mid j \geq 1 \}$ forms a countable neighbourhood basis at the identity element 1.

Let K be an open subgroup of $GL_n(\mathbb{Q}_p)$. Then $K_j \subset K$ for some $j \geq 1$. Hence to prove that G/K is countable, it suffices to prove that G/K_j is countable for every j. For a discussion of the proof that G/K_j is countable, see [M].

Closed subgroups (and open subgroups) of t.d. groups are t.d. So any closed subgroup of $GL_n(\mathbb{Q}_p)$ is a t.d. group. These groups are often called p-adic groups. We remark that groups like $GL_n(\mathbb{Q}_p)$, $SL_n(\mathbb{Q}_p)$, $Sp_{2n}(\mathbb{Q}_p)$, etc., are the groups of $\mathbb{Q}_p$-rational points of reductive linear algebraic groups that are defined over $\mathbb{Q}_p$. Such groups have another topology, the Zariski topology (coming from the variety that is the algebraic group). The structure of these groups is often studied via algebraic geometry, in contrast with the structure of Lie groups, which is studied via differential geometry.

As with Lie groups, there is a notion of smoothness for representations. A (complex) representation (π, V) is *smooth* if for each $v \in G$, the subgroup $\{ g \in G \mid \pi(g)v = v \}$ is an open subgroup of G. This definition is also valid if V is infinite-dimensional. This notion of smoothness is very different from that for Lie groups - in fact, connected Lie groups don't have any proper open subgroups. Because of the abundance of compact open subgroups in t.d. groups, and the fact that the general theory of representations of compact groups is well understood (see Chapter 6), properties of representations of t.d. groups are often studied via their restrictions to compact open subgroups.

Lemma. *Suppose that (π, V) is a smooth finite-dimensional representation of a compact t.d. group G. Then there exists an open compact normal subgroup K of G and a representation ρ of the finite group G/K such that $\rho(gK)v = \pi(g)v$ for all $g \in G$ and $v \in K$.*

Proof. By smoothness of π and finite-dimensionality of π, there exists an open compact subgroup K' of G such that $\pi(k')v = v$ for all $k' \in K'$ and $v \in V$. Choose a set $\{ g_1, \ldots, g_r \}$ of coset representatives for G/K'. The subgroup $K := \cap_{j=1}^r k_j K' k_j^{-1}$ is an open compact normal subgroup of G and $\pi(k)v = v$ for all $k \in K$ and $v \in V$. It follows that there exists a representation (ρ, V) of the finite group G/K such that $\rho(gK)v = \pi(g)v$, $v \in V$, and $g \in G$. qed

We remark that the subgroup K' (hence the representation ρ) in the above lemma are not unique. Now suppose that (π, V) is a smooth (not necessarily finite-dimensional) representation of a (not necessarily compact) t.d. group G. Let K be a compact open subgroup of G. The restriction $\pi_K = r_G^K \pi$ of π to K is a (possibly infinite) direct sum of irreducible smooth representations of K. As we will see in Chapter 6, irreducible unitary representations of compact groups are finite-dimensional. Applying the above lemma, we can see that each of the irreducible representations of K which occurs in π_K is attached to a representation of some finite group. One difficulty in studying the representations of non-compact t.d. groups involves determining which compact open subgroups K and which irreducible constituents of π_K can be used to effectively study properties of π. For more information on representations of p-adic groups, see the notes [C] or the course notes for Mat 1197 ([M])

5.7. Haar measure on locally compact groups

If X is a topological space, a *σ-ring* in X is a nonempty family of subsets of X having the property that arbitrary unions of elements in the family belong to the family, and if A and B belong to the family, then so does $\{\, x \in A \mid x \notin B \,\}$. If X is a locally compact topological space, the *Borel ring* in X is the smallest σ-ring in X that contains the open sets. The elements of the Borel ring are called *Borel sets.* A function $f : X \to \mathbb{R}$ is *(Borel) measurable* if for every $t > 0$, the set $\{\, x \in X \mid |f(x)| < t \,\}$ is a Borel set.

Let G be a locally compact topological group. A *left Haar measure* on G is a nonzero regular measure μ_ℓ on the Borel σ-ring in G that is left G-invariant: $\mu_\ell(gS) = \mu_\ell(S)$ for measurable set S and $g \in G$. Regularity means that

$$\mu_\ell(S) = \inf\{\, \mu_\ell(U) \mid U \supset S,\ U \text{ open} \,\} \quad \text{and} \quad \mu_\ell(S) = \sup\{\, \mu_\ell(C) \mid C \subset S,\ C \text{ compact} \,\}.$$

Such a measure has the properties that any compact set has finite measure and any nonempty open set has positive measure. Left invariance of μ_ℓ amounts to the property

$$\int_G f(g_0 g)\, d\mu_\ell(g) = \int_G f(g) d\mu_\ell(g), \qquad \forall\ g_0 \in G,$$

for any Haar integrable function f on G.

Theorem. *([Halmos], [HR], [L]) If G is a locally compact group, there is a left Haar measure on G, and it is unique up to positive real multiples.*

There is also a right Haar measure μ_r, unique up to positive constant multiples, on G. Right and left Haar measures do not usually coincide.

Exercise. Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \ \mid\ x \in \mathbb{R}^\times,\, y \in \mathbb{R} \right\}.$$

Show that $|x|^{-2} dx\, dy$ is a left Haar measure on G and $|x|^{-1} dx\, dy$ is a right Haar measure on G.

The (locally compact topological) group G is called *unimodular* if each left Haar measure is also a right Haar measure. Clearly, G is unimodular if G is abelian. Conjugation by a fixed $g_0 \in G$ is a homemomorphism of G onto itself, so the measure $S \mapsto \mu_\ell(g_0 S g_0^{-1}) = \mu_\ell(S g_0^{-1})$ (S measurable) is also a left Haar measure. By uniqueness of left Haar measure, there exists a constant $\delta(g_0) > 0$

$$\int_G f(g_0 g g_0^{-1})\, d\mu_\ell(g) = \delta(g_0) \int_G f(g)\, d\mu_\ell(g), \qquad f \text{ integrable}$$

A *quasicharacter* of G is a continuous homomorphism from G to $\mathbb{C}^\times$.

Proposition.

(1) *The function* $\delta : G \to \mathbb{R}_+^\times$ *is a quasicharacter*

(2) $\delta(g)d\mu_\ell(g)$ *is a right Haar measure.*

Proof. The fact that conjugation is an action of G on itself implies that $\delta : G \to \mathbb{R}_+^\times$ is a homomorphism. The proof of continuity is omitted. Note that

$$\delta(g_0)\int_G f(g)\,d\mu_\ell(g) = \int_G f(g_0 \cdot g_0^{-1}gg_0)\,d\mu_\ell(g) = \int_G f(gg_0)\,d\mu_\ell(g).$$

Replacing f by $f\delta$ and dividing both sides by $\delta(g_0)$, we obtain

$$\int_G f(g)\delta(g)\,d\mu_\ell(g) = \int_G f(gg_0)\delta(g)\,d\mu_\ell(g).$$

This shows that $\delta(g)d\mu_\ell(g)$ is right invariant. qed

In view of the above, we may write $d\mu_r(g) = \delta(g)d\mu_\ell(g)$. The function δ is called the *modular quasicharacter* of G. Clearly G is unimodular if and only if the modular quasicharacter is trivial. If G is unimodular, we simply refer to Haar measure on G.

Exercises:

(1) Let dX denote Lebesgue measure on $M_{n\times n}(\mathbb{R})$. This is a Haar measure on $M_{n\times n}(\mathbb{R})$. Show that $|det(g)|^{-n}dg$ is both a left and a right Haar measure on $GL_n(\mathbb{R})$. Hence $GL_n(\mathbb{R})$ is unimodular.

(2) Let n_1 and n_2 be positive integers such that $n_1 + n_2 = n$. $P = P_{(n_1,n_2)}$ be the standard parabolic subgroup of $GL_n(\mathbb{R})$ corresponding to the partition (n_1, n_2) (see Chapter 4 for the definition of standard parabolic subgroup of a general linear group). Let $g = \begin{pmatrix} g_1 & X \\ 0 & g_2 \end{pmatrix} \in P$, with $g_j \in GL_{n_j}(\mathbb{R})$ and $X \in M_{n_1\times n_2}(\mathbb{R})$. Let dg_j be Haar measure on $GL_{n_j}(\mathbb{R})$, and let dX be Haar measure on $M_{n_1\times n_2}(\mathbb{R})$. SHow that $d_\ell g = |\det g_1|^{-n_2}dg_1\,dg_2\,dX$ and $d_r g = |\det g_2|^{-n_1}dg_1\,dg_2\,dX$ are left and right Haar measures on P (respectively). Hence the modular quasicharacter of P is equal to $\delta(g) = |\det g_1|^{n_1}|\det g_2|^{-n_2}$.

(3) Show that the homeomorphism $g \mapsto g^{-1}$ turns μ_ℓ into a right Haar measure. Conclude that if G is unimodular, then $\int_G f(g)\,d\mu_\ell(g) = \int_G f(g^{-1})d\mu_\ell(g)$ for all measurable functions f.

Proposition. *If G is compact, then G is unimodular and $\mu_\ell(G) < \infty$.*

Proof. Since δ is a continuous homomorphism and G is compact, $\delta(G)$ is a compact subgroup of $\mathbb{R}_+^\times$. But $\{1\}$ is the only compact subgroup of $\mathbb{R}_+^\times$. Haar measure on any locally compact group has the property that any compact subset has finite measure. Hence $\mu_\ell(G) < \infty$ whenever G is compact. qed

If G is compact, *normalized* Haar meaure on G is the unique Haar measure μ on G such that $\mu(G) = 1$. When working with compact groups, we will always work relative to normalized Haar measure and we will write write $\int_G f(g)\, dg$ for $\int_G f(g)\, d\mu(g)$.

5.8. Discrete series representations

Let G be a locally compact unimodular topological group. A unitary representation π of G on a Hilbert space V (with inner product $\langle \cdot, \cdot \rangle$) is *continuous* if for every v, $w \in V$, the function $g \mapsto \langle \pi(g)v, w\rangle$ is a continuous function on G. That is, matrix coefficients of π are continuous functions on G. Note that such a representation may be infinite-dimensional. (In particular, if G is a noncompact semisimple Lie group, then all nontrivial irreducible continuous unitary representations of G are infinite-dimensional.)

Suppose that (π, V) is an irreducible continuous unitary representation of G. Let Z be the centre of G. A generalization of Schur's Lemma to this setting shows that if $z \in Z$, then there exists $\omega(z) \in \mathbb{C}^\times$ such that $\pi(z) = \omega(z)I$. Because π is a continuous unitary representation, the function $z \mapsto \omega(z)$ is a continuous linear character of the group Z. In particular, $|\omega(z)| = 1$ for all $z \in Z$. The representation π is said to be *square-integrable* mod Z, or to be a *discrete series* representation, if there exist nonzero vectors v and $w \in V$ such that

$$\int_{G/Z} |\langle v, \pi(g)w\rangle|^2 \, dg^\times < \infty,$$

where $dg^\times$ is Haar measure on the locally compact group G/Z. Thus π is a discrete series representation if some nonzero matrix coefficient of π is square-integrable modulo Z.

Fix an ω as above. Let $C_c(G, \omega)$ be the space of continuous functions from f G to $\mathbb{C}$ that satisfy $f(zg) = \omega(z)f(g)$ for all $g \in G$ and $z \in Z$, and are compactly supported modulo Z (there exists a compact subset C_f of G such that the support of f lies inside the set $C_f Z$). Define an inner product on $C_c(G, \omega)$ by $(f_1, f_2) = \int_{G/Z} f_1(g)\overline{f_2(g)}\, dg^\times$. Let $L^2(G, \omega)$ be the completion of $C_c(G, \omega)$ relative to the norm $\|f\| = (f, f)^{1/2}$, $f \in C_c(G, \omega)$. The group G acts by right translation on $L^2(G, \omega)$, and this defines a continuous unitary representation of G on the Hilbert space $L^2(G, \omega)$.

Theorem. *(Schur orthogonality relations). Let (π, V) and (π', V') be irreducible continuous unitary representations of G such that $\omega = \omega'$.*

(1) The following are equivalent:

(i) π is square-integrable mod Z.

(ii) $\int_{G/Z} |\langle v, \pi(g)w\rangle|^2 \, dg^\times < \infty$ for all v, $w \in V$.

(iii) π is equivalent to a subrepresentation of the right regular representation of G on $L^2(G, \omega)$.

(2) *If the conditions of (1) hold, then there exists a number $d(\pi) > 0$, called the formal degree of π (depending only on the normalization of Haar measure on G/Z), such that*

$$\int_{G/Z} \overline{\langle v_1, \pi(g)w_1\rangle}\langle v_2, \pi(g)w_2\rangle \, dg^\times = d(\pi)^{-1}\overline{\langle v_1, v_2\rangle}\langle w_1, w_2\rangle, \qquad \forall\, v_1, v_2, w_1, w_2 \in V.$$

(3) *If π is not equivalent to π', then*

$$\int_{G/Z} \overline{\langle v, \pi(g)w\rangle}\langle v', \pi'(g)w'\rangle \, dg^\times = 0 \qquad \forall\, v, w \in V,\ v', w' \in V'.$$

5.9. Parabolic subgroups and representations of reductive groups

The description of the parabolic subgroups of general linear groups and special linear groups over finite fields given in Chapter 4 is valid for general linear and special linear groups over any field F - simply replace the matrix entries in the finite field by matrix entries in the field F. General linear and special linear groups are examples of reductive groups. We do not give the definition of parabolic subgroup for arbitrary reductive groups.

Suppose that G is the F-rational points of a connected reductive linear algebraic group, where $F = \mathbb{R}$, $F = \mathbb{C}$, F is a p-adic field (for example, $F = \mathbb{Q}_p$), or F is a finite field.

The "Philosophy of Cusp Forms" says that the collection of representations of a reductive group G should be partitioned into disjoint subsets in such a way that each subset is attached to an associativity class of parabolic subgroups of G. Two parabolic subgroups $P = M \ltimes N$ and $P' = M' \ltimes N'$ of G are *associate* if and only if the Levi factors M and M' are conjugate in G. The representations attached to the group G itself are called *cuspidal* representations, and their matrix coefficients are called *cusp forms*. If P is a proper parabolic subgroup of G, the representations attached to P are associated to (Weyl group orbits of) cuspidal representations of a Levi factor M of P. (Note that M is itself a reductive group). Furthermore, the representations of G associated to a given cuspidal representation σ of M occur as subquotients of the induced representation $\mathrm{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$, where δ_P is the modular quasicharacter of P (see § 5.7) and σ is extended to a representation of $P = M \ltimes N$ by letting it be trivial on N.

The problem of understanding the representations of the group G can be approached via the Philosophy of Cusp Forms, and is therefore divided into two parts. The first part is to determine the cuspidal representations of the Levi subgroups M of G, and the second part is to analyze representations parabolically induced from such cuspidal representations.

In certain contexts, a cuspidal representation is simply a discrete series representation (see § 5.8 for the definition of discrete series representation). If G is a connected reductive Lie group (for example $G = SL_n(\mathbb{R})$ or $G = Sp_{2n}(\mathbb{R})$, then there are two cases to consider.

An element of a matrix Lie group is *semisimple* if it is semsimple as a matrix, that is, it can be diagonalized over the field of complex numbers. A *Cartan subgroup* of G is a closed subgroup that is a maximal abelian subgroup consisting of semisimple elements. In the first case, G contains no Cartan subgroups that are compact modulo the centre of G (for example, this is the case if G is semisimple and $F = \mathbb{C}$, of if $SL_n(\mathbb{R})$ and $n \geq 3$), and hence G has no discrete series representations. In the second case, up to conjugacy G contains one Cartan subgroup T that is compact modulo the centre of G, and the discrete series of G are parametrized in a natural way by the so-called *regular* characters of T.

An irreducible unitary representation of G is *tempered* if it occurs in the decomposition of the regular representation of G on the Hilbert space $L^2(G)$ of square-integrable functions on G. If π is a tempered representation of G, then there exists a parabolic subgroup $P = M \ltimes N$ and a discrete series representation of M such that π occurs as a constituent of the induced representation $\mathrm{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$.

If G is a reductive p-adic group (that is, F is a p-adic field), a continuous complex-valued function f on G is a *supercusp form* if the support of f is compact modulo the centre of G and $\int_N f(gn)\,dn = 0$ for all $g \in G$ and all unipotent radicals N of proper parabolic subgroups of G. An irreducible smooth representation (where a smooth representation is as defined in §5.6) of G is *supercuspidal* if the matrix coefficients of the representation are supercusp forms. Given an irreducible smooth representation π of G, there exists a parabolic subgroup $P = M \ltimes N$ of G and a supercuspidal representation σ of M such that π is a subquotient of $\mathrm{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$. Hence in this context, it is suitable to interpret "cuspidal representation" as supercuspidal representation. (Recall that a similar result was described in Chapter 4 in the case that F is a finite field).

If π is a supercuspidal representation of G, then there exists a quasicharacter ω of the centre Z of G such that $\pi(z) = \omega(z)I$, $z \in Z$. It is easy to see that if ω is unitary (that is, $|\omega(z)| = 1$ for all $z \in Z$), then π is a discrete series representation. A reductive p-adic group has many supercuspidal representations and hence many discrete series representations. However, there exist discrete series representations that are not supercuspidal.

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