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# Morse landscapes of Riemannian functionals and related problems

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**Abstract.** The subject of this talk is Morse landscapes of natural functionals on infinite-dimensional moduli spaces appearing in Riemannian geometry.

First, we explain how recursion theory can be used to demonstrate that for many natural functionals on spaces of Riemannian structures, spaces of submanifolds, etc., their Morse landscapes are always more complicated than what follows from purely topological reasons. These Morse landscapes exhibit non-trivial "deep" local minima, cycles in sublevel sets that become nullhomologous only in sublevel sets corresponding to a much higher value of functional, etc.

Our second topic is Morse landscapes of the length functional on loop spaces. Here the main conclusion (obtained jointly with Regina Rotman) is that these Morse landscapes can be much more complicated than what follows from topological considerations only if the length functional has "many" "deep" local minima, and the values of the length at these local minima are not "very large".

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## 1. Introduction.

In this talk we will discuss Morse landscapes of functionals on infinite-dimensional moduli spaces naturally arizing in Differential Geometry.

Our first message is that in many cases these Morse landscapes are much more complicated than what would follow just from the Morse theory. The examples include some Riemannian functionals (i.e.functionals of the space of isometry classes of Riemannian metrics), functionals on spaces of submanifolds, and so on. Our approach initiated in [N1], [N2], [N3] and further developed in collaboration with Shmuel Weinberger ([NW1], [NW2], [NW3]) is based on recursion theory. In many cases we are able to prove disconnectedness of sublevel sets of functional of interest, and, moreover, the exponential growth of the number of connected components

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that merge only inside a much larger sublevel set. Sometimes this technique implies the existence of an infinite set of distinct local minima of the functional of interest, where only the existence of the global minimum was previously known. In other cases this recursion-theoretic approach is the only known method that can be used to establish the existence of critical points of a functional of interest.

In particular, methods using ideas from mathematical logic led to only known general results on the following problem posed by R. Thom "What is the best (or the nicest) metric on a given smooth manifold?" for compact manifolds of dimension $\geq 5$ (joint work with Shmuel Weinberger). They also constitute the only known tool to demonstrate that the theory of (high-dimensional) "thick" knots is drastically different from the "usual" knot theory.

In a different direction I will discuss Morse landscapes of the length functional on loop spaces $\Omega_p M^n$ and spaces $\Omega_{pq} M^n$ of paths between points $p, q$ on a closed simply-connected Riemannian manifold $M^n$. In [N5] I proved that if the length functional has a "very deep" non-trivial local minimum on $\Omega_p M^n$, then it has "many" "deep" local minima. [0] The proof used the idea of "effective universal coverings". A stronger form of this result can be proven using direct geometric methods recently invented by Regina Rotman. These methods also can be used to demonstrate that if the length functional has a critical point of a positive index of a "large" but finite depth, then it must have "many" "deep" local minima ([NR]).

## 2. "Thick" knots.

Knots are sometimes defined as submanifolds of $R^3$ (or $S^3$) diffeomorphic to $S^1$. More generally, one can consider higher-dimensional knots that are submanifolds of $R^{n+k}$ (or $S^{n+k}$) diffeomorphic to $S^n$, where $k$ is usually equal to two. Two knots have the same knot type, if they are isotopic. It makes sense to consider also "physical" or "thick" knots on a rope of small but non-zero thickness; two thick knots have the same type if they can be connected by an isotopy that preserves the thickness of the rope and does not increase its length. In the multidimensional case the isotopy should not increase the volume. Note that the thickness of the rope cannot exceed the injectivity radius of the normal exponential map. Therefore, the study of "thick" knot types is equivalent to the study of connected components of sublevel sets of the crumpledness functional $\kappa_v = \frac{vol^{\frac{1}{n}}}{r}$, where $vol$ denotes the volume, and $r$ denotes the injectivity radius of the normal exponential map. (In other words, $r$ is equal to the supremum of $x$ such that every two normals to the knot of length $\leq x$ starting at its different points do not intersect. Informally speaking, $r(\Sigma)$ is the largest radius of a nonself-intersecting tube arounf $\Sigma$.)

The paper [N1] was one of the first papers on "thick" knots. [1] The most

[0] The depth of a local minimum $\mu$ of a functional $f$ can be defined as $\inf_\gamma \sup_t f(\gamma(t)) - f(\mu)$, where the infimum is taken over the set of all paths $\gamma$ starting at $\mu$ and ending at a point $\gamma(1)$ such that $f(\gamma(1)) < f(\mu)$. If $\mu$ is a global minimum, then, by definition, it has infinite depth.

[1] I am sure that the very natural idea to study knots of non-zero thickness occured independently to many other mathematicians, yet I found only one paper on "thick" knots preceding

basic question (in every dimension and codimension) is whether or not there exist "thick" knots that are trivial as usual knots but not trivial as "thick" knots. This question still remains open for the "classical" dimension one and codimension two, despite the fact that it is easy to sketch plausible candidates. The question becomes especially interesting for $n = 2$ and codimension one, where I cannot even guess what answer to expect. In [N1] I answered this question in affirmative for the dimension $n \geq 5$ and codimension one. Observe, that Smale's h-cobordism theorem implies that every embedded $n$-sphere of codimension one in $R^{n+1}$, $(n > 3)$, is isotopic to the standard sphere. In other words, there exists only the trivial knot type. Yet in [N1] (see also [N2]) I proved that:

**Theorem 2.1.** *For every $n \geq 5$ and for each sufficiently large $x$ the set of $n$-dimensional hypersurfaces $\Sigma^n \subset R^{n+1}$ diffeomorphic to $S^n$ and such that $\kappa_v(\Sigma^n) \leq x$ is not connected.*

For every knot $\Sigma^n \subset R^{n+1}$ denote the infimum of $y$ such that there exists an isotopy that passes through knots with $\kappa_v(\Sigma^n) \leq y$ and connects $\Sigma^n$ with the round sphere of radius one by $C(\Sigma^n)$. An easy compactness argument implies that for every positive $x$ there exists the supremum of $C(\Sigma^n)$ over the set of all knots $\Sigma^n$ such that $\kappa_v(\Sigma^n) \leq x$. Denote this supremum by $C_n(x)$. Note that the previous theorem follows from the assertion that for every $n \geq 5$ for all sufficiently large $x$ $C_n(x) > x$. We deduced this assertion (and, thus, the previous theorem) from the following much stronger assertion: [2]

**Theorem 2.2.** *Let $\phi$ be any computable [3] function. Then for every $n \geq 5$ and for all sufficiently large $x$ $C_n(x) > \phi(\lfloor x \rfloor)$).*

We will explain the proof of this theorem in the next section. The preceding discussion does not depend on a particular choice of the smoothness of considered knots as long as the considered knots are at least $C^{1,1}$-smooth, which is the minimal smoothness required for $r$ to be defined and positive. Now consider the space of all $C^{1,1}$-smooth $n$-dimesnional knots in $R^{n+1}$ with $C^1$ topology. Consider the following equivalence relation on this space: two knots are equivalent if they can be transormed one into the other by a similarity transformation of the ambient

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[N1], namely, [KV]. Although [N1] dealt mainly with high-dimensional "thick" knots of codimension one, some of its results, such as a $C^{1,1}$-compactness theorem remain valid for an arbitrary dimension/codimension. It also contained several basic problems about 1-dimensional "thick" knots in $R^3$ (Section 4, B,C,D in [N1]) that still remain unsolved. In recent years "thick" 1-dimensional knots in $R^3$ became the subject of a constantly growing number of publications -cf. [LSDR], [Dur] or [CFKSW].

[2] To be more precise in [N1] we proved that this inequality holds only for an infinite unbounded sequence of values of $x$. To prove that it holds for all sufficiently large values of $x$ one needs either to apply a trick from [N2] involving the busy beaver function or to use time-bounded Kolmogorov complexity as in [N3] and subsequent papers [NW1], [NW2], [NW3].

[3] Formally speaking, here "computable" means "Turing computable" or, equivalently, "recursive". Equivalently, a reader can take any computer programming language, strip it of all restrictions on the size of data (if there are any), and strip it of all data types but the integer numbers. A function is computable if and only if it can be described by a computer program in this language.

Euclidean space. Clearly each equivalence class is connected, and the crumpledness functional is constant on every equivalence class. Denote the space of the equivalence classes by $Knots_{n,1}$. In [N1] we proved that sublevel sets of $\kappa_v$ are compact subsets of $Knots_{n,1}$. Therefore $\kappa_v$ attains its local minimum on every connected component of each of its sublevel sets. These local minima will be automatically local minima of $\kappa_v$ on the whole space $Knots_{n,1}$. The disconnectedness of $\kappa_v^{-1}((0,x])$ for arbitrarily large values of $x$ implies that the set of local minima of $\kappa_v$ is unbounded, and the set of values of $\kappa_v$ at its local minima is infinite. Combining this observation with the previous theorem we see that there exists an infinite set of local minima of $\kappa_v$, where the depth is much higher than the value of $\kappa_v$.

**Theorem 2.3.** *For every $n \geq 5$ and every computable function $\phi : N \longrightarrow N$ there exists an infinite sequence $\Sigma_i^n$ of local minima of $\kappa_v$ on $Knots_{n,1}$ such that the set of values of $\kappa_v$ at these local minima is unbounded, and the depth of each of these local minima $\Sigma_i^n$ is greater than $\phi(\lfloor \kappa_v(\Sigma_i^n) \rfloor)$.*

This theorem holds for other versions of the crumpledness functional , e.g. $\kappa_d = \frac{diam}{r}$ as well as for many other functionals (see [N1]). The theorem can be also generalized to spaces of trivial knots of arbitrary codimension (of dimension $n > 4$), as well as for the spaces of trivial knots of dimension 3 or 4 and codimension 2. (The last generalization follows from the results of [NW0].) The theorem and its generalizations for other crumpledness functionals obviously hold for the space of trivial $C^{1,1}$-smooth $n$-dimensional knots in $R^{n+k}$, where one does not take the quotient with respect to the action of the group of similarities of the ambient Euclidean space. In this form the theorem can be generalized for the cases, when 1) The submanifold can be diffeomorphic to an arbitrary closed manifold $M^n$ , $(n > 4)$, instead of $S^n$; and 2) The ambient manifold can be not $R^n$ but an arbitrary closed Riemannian manifold, as well as a complete non-compact Riemannian manifold from a wide class. (Of course, the considered space of submanifolds needs to be non-empty; if it is not connected, the theorem holds for each of its connected components.)

## 3. Methods I: Algorithmic unsolvability of the diffeomorphism problem and its applications.

The following theorem was first proven by Sergei Novikov (see its proof in the Appendix of [N4]):

**Theorem 3.1.** *For every $n \geq 5$ there is no algorithm deciding whether or not a given manifold $M^n$ is diffeomorphic to the $n$-sphere.*

To make this theorem precise one needs to explain how $M^n$ is presented in a finite form. In [N4] we observed that this theorem is true even in the case when $M^n$ is a non-singular real algebraic hypersurface $\{x \in R^{n+1} | p(x) = 0\}$, where $p$

is a polynomial with rational coefficients. In this case $M^n$ can be presented by the vector of coefficients of $p$. The other ways to present $M^n$ in a finite form include: 1) $C^\infty$-semialgebraic atlases (also known as Nash atlases; see [BHP]); 2) Smooth triangulations; 3) Smooth real algebraic subvarieties of Euclidean spaces of a higher codimension defined over the field of algebraic numbers.

Here is a very brief sketch of the proof of this theorem. According to the classical theorem independently proven by S. Adyan and M. Rabin there exists an infinite sequence of finite presentations of groups $G_i, i = 1, 2, \ldots$ such that there is no algorithm deciding for every given $i$ whether or not $G_i$ is isomorphic to the trivial group. (In other words, the set $I$ of all $i$ such that $G_i$ is trivial is non-recursive.) The standard proof of this theorem (cf. [Mil]) produces $G_i$ that are perfect, that is $H_1(G_i) = G_i/[G_i, G_i]$ is trivial. S. Novikov observed that one can alter finite presentations of these groups in a certain explicit way to obtain a new sequence of finite presentations of *superperfect* groups $\bar{G}_i$ so that $\bar{G}_i$ is trivial if and only if $G_i$ is trivial. (Thus, there is no algorithm deciding for a given value of $i$ whether or not $\bar{G}_i$ is trivial. Superperfectness of a group means the vanishing of the first two homology groups of a group. Also, note that groups $\bar{G}_i$ are universal central extensions of groups $G_i$.) According to [Ke] the superperfectness of a finitely presented group $G$ is the necessary and sufficient condition of the realizibility of $G$ as the fundamental group of a smooth homology $n$-sphere, that is a smooth closed manifold with the same homology groups as $S^n$, for every $n \geq 5$. Thus, we can effectively realize groups $\bar{G}_i$ as fundamental groups of homology spheres $\Sigma_i^n$. Moreover, the proof of the quoted result from [Ke] implies that this construction can be carried in $R^{n+1}$ so that $\Sigma_i^n$ will be a smooth hypersurface in $R^{n+1}$. Smale's h-cobrdism theorem implies that a homology sphere $\Sigma_i^n$ embedded as a hypersurface in $R^{n+1}$ is diffeomorphic to $S^n$ if and only if it is simply-connected, and, therefore, if and only if $\bar{G}_i$ is trivial. This completes the proof of the theorem.

This theorem (or rather its proof outlined above) has the following immediate corollary:

**Corollary 3.2.** *For every closed smooth manifold $M_0^n$ of dimension $n > 4$ there is no algorithm that decides whether or not a given manifold $M^n$ is diffeomorphic to $M_0^n$.*

Indeed we can just construct a sequence $M_i^n$ by forming connected sums of a copy of $M_0^n$ with smooth homology spheres $\Sigma_i^n$ from the outline of a proof of the previous theorem. The manifold $M_i^n$ is diffeomorphic to $M_0^n$ if and only if the fundamental group of $\Sigma_i^n$ is trivial.

Note that it is not known whether or not this theorem remains true in dimension four. However, A. Markov proved that this theorem is true for manifolds $M_0^4$ diffeomorphic to the connected sum of a sufficient number $N_0$ of copies of $S^2 \times S^2$ with an arbitrary closed 4-manifoldi (cf. [BHP], [Sh]). Here one can take $N_0 = 14$([Sh]). This theorem enables us to extend some of our techniques that we are going to describe below to such four-dimensional manifolds.

Now the general idea behind the proof of Theorem 2.2 as well as of some results stated in the next sections can be described as follows. Consider a class $C$

of diffeomorphism types of compact smooth $n$-dimensional manifold, where $n > 4$. The class $C$ can be the class of all $n$-manifolds, or, for example, the class of all manifolds embeddable in $R^{n+1}$. We require that the class $C$ is large enough to ensure that S. Novikov's theorem will be true in this class: For every manifold $M^n$ from $C$ there is no algorithm deciding whether or not a given manifold from $C$ is diffeomorphic to $M^n$.

We consider situations, when for every manifold in $M^n \in C$ there is a natural "moduli space" $Moduli(M^n)$. associated with this manifold. (In the situation of Theorem 2.2 $Moduli(M^n) = Knots_{n,1}$. To prove theorems stated in section 5 below we will be choosing $Moduli(M^n)$ as certain subsets of the space of Riemannian structures on $M^n$.) Let $\phi$ be a non-negative functional on a moduli space $Moduli_n$ defined as the disjoint union of connected spaces $Moduli(M^n)$ associated with all manifolds $M^n \in C$. We are assuming that $Moduli_n$ is endowed with a metric, $\rho$. First, we are going to make the following assumptions about $\phi$, $\rho$ and the class $C$:

0) There exists a countable dense set $D \in Moduli_n$. Elements of $D$ are representable in a finite form. For any $M^n \in C$ there is no algorithm deciding whether or not a given element $\mu \in D$ is in $Moduli(M^n)$ (that is, represents $M^n$).

1) There exists an algorithm computing the distance $\rho$ between every pair of elements of $D$ within to any prescribed (rational) accuracy.

2) The function $\phi$ can be effectively majorized: There exists an algorithm that for a given element $\mu \in D$ computes an upper bound for $\phi(\mu)$.

3) For every $x$ the sublevel set $\phi^{-1}([0, x]) \subset Moduli_n$ is precompact. Moreover, there exists an algorithm that for every given positive rational $x$ and $\epsilon$ constructs a finite $\epsilon$-net in $\phi^{-1}([0, x])$. All elements of this $\epsilon$-net are in the countable set $D$.

4) There exists a computable decreasing positive function $\delta_n(x)$ such that every two $\delta_n(x)$-close points from $\phi^{-1}([0, x])$ are points from $Moduli(M^n)$ for the same manifold $M^n$.

Now we are going to demonstrate that for every $M^n \in C$ there exists an unbounded increasing sequence of values of $x$ such that sublevel sets $S_x = \phi^{-1}([0, x]) \bigcap Moduli(M^n)$ are disconnected. Moreover, for these values of $x$ $S_x$ is a union of two non-empty subsets $S_{1x}$, $S_{2x}$ such that the distance between each pair of points $\mu_1 \in S_{1x}$, $\mu_2 \in S_{2x}$ is at least $\delta_n(x)$.

Indeed, assume the opposite. Then we will construct an algorithm deciding whether or not a given manifold $N^n \in C$ is diffeomorphic to $M^n$, thus obtaining a contradiction with our assumptions. We start from calculating an upper bound $y$ for the value of $\phi$ at the given manifold (which is presented as an element from $D$). We can always make it large enoung to ensure that $\phi^{-1}([0, y]) \bigcap Moduli(M^n)$ is connected. Then we construct $\delta_n(y)/10$-net in $\phi^{-1}([0, y]) \subset Moduli_n$. The next step is to construct a graph such that the points of the constructed net will be its vertices, and two vertices are connected by an edge, if the corresponding points are approximately $\delta_n(y)/2$-close. Here we are allow ourselves an error in these calculations that does not exceed $\delta_n(y)/4$. Now our connectedness assumption implies that exactly one component of the constructed graph contains elements of $Moduli(M^n)$. We can assume that our algorithm knows one vertex $v_0$ from this connected component. (This vertex will be in this connected component for all sufficiently large values of $y$). Now our algorithm needs to determine a vertex $w$ of the net which is $\delta_n(y)/10$-close to the given element of $Moduli_n$, and to determine whether or not $w$ and $v_0$ are in the same component of the constructed graph. The given element of $Moduli_n$ represents a manifold diffeomorphic to $M^n$ if and only if $w$ and $v_0$ are in the same component.

The obtained contradiction demonstrates that the sets $S_x$ must be disconnected for some arbitrarily large values of $x$. An argument from [N2] (that involves Rado's busy beaver function) can be used to prove this assertion for all sufficiently large values of $x$. Yet there is another method using the notion of Kolmogorov complexity that can be used to prove not only the disconnectedness of sets $S_x$ but lower bounds for their number. This idea will be described in more details in the next section.

If sublevel sets of $\phi$ are not only precompact, but compact, then we can find distinct local minima of $\phi$ at the bottom of different connected components of its sublevel sets. In some applications of this method sublevel sets of $\phi$ are compact, when the manifold belongs to a class $C' \subset C$ but not necessarily in the general case. Then we establish the compactness of *some* of the connected components of sublevel sets of $\phi$ by using the fact that there is no algorithm that distinguishes a manifold $M^n \in C$ of interest for us from manifolds known to be in the subclass $C'$. (Of course, this fact should be true for this idea to work.)

To prove the theorems stated in the previous section one uses this idea in the situation when $Moduli_n$ is the space of equivalence classes of codimension one closed submanifolds of $R^{n+1}$. (Two submanifolds are equivalent if they can be transformed one into the other by a similarity of $R^{n+1}$.) Further, $C$ is the class of closed $n$-manifolds embeddable into $R^{n+1}$, $M^n = S^n$, and $\phi = \kappa_v$ or $\kappa_d$.

However, note that in most of the situations, when we would like to apply this method, the assumptions 3) or 4) either do not hold, or are difficult to establish. Nevertheless, the method sometimes can be salvaged using new ideas some of which will be explained in sections 6 and 7.

# 4. Methods II: Kolmogorov complexity and time-bounded Kolmogorov complexity.

In this section we will explain how to modify the method sketched in the previous section so as to obtain not merely disconnectedness of sublevel sets $\phi^{-1}([0,x])$ of a functional of interest on $Moduli(M^n)$, but a lower bound for the number of connected components that grows exponentially with $x$.

A *decision problem* consists of a countable set $A$ and its subset $B$. Elements of $A$ are presentable in a finite form, and there is a computable complexity function $A \longrightarrow Z^+$. For each $L$ there exist only finitely many elements of $A$ of complexity $\leq L$. One is interested in existence/non-existence of an algorithm deciding whether or not a given element of $A$ is an element of $B$. Assume that there is no such algorithm. Then one can ask for such an algorithm that uses arbitrary *oracle information.* The amount of oracle information is allowed to grow with the complexity of instances of the problem. We are assuming that the information is presented as a sequence of 0s and 1s. The "amount of information" is just the length of this sequence. Of course, one can ask for the list of all answers for all instances of the problem of complexity $\leq L$. Yet one is interested in the *minimal* amount of oracle information sufficient to solve the problem. The minimal number of bits of oracle information sufficient to solve the problem for all instances of complexity $\leq L$ is called *Kolmogorov complexity* of the decision problem. Of course, one can "hide" a constant number of bits of oracle information in the algorithm, so the Kolmogorov complexity is a function of $L$ defined only up to adding a constant summand. For example, let $G$ be a finitely presented group with unsolvable word problem, $A$ the set of all words in the considered finite presentation, $B$ the set of all words representing trivial elements, and assume that we define the complexity of words as their length. The resulting decision problem is the word problem for $G$; it can be solved in a computable time using the following oracle information: For each $L$ we request (the binary representation of) the number $w(L)$ of all trivial words with $\leq L$ letters. To use this information we start generating trivial words of length $\leq L$ using longer and longer products of conjugates of relations, and stop when the length of the list reaches $w(L)$. One can be sure that all the remaining words correspond to non-trivial elements of $G$. So, the Kolmogorov complexity of the word problem for words of length $\leq L$ grows not faster than a linear function of $L$. However, it is not difficult to note that the time of work of this "algorithm" grows faster than any computable function. Assume that we impose an additional constraint: the time of work of the algorithm that uses the oracle information should not exceed a given computable function $\lambda$. The resulting notion is called time-bounded Kolomogorov complexity of the considered decision problem (cf. [LV] for an introduction to its properties). A theorem of Barzdin ([B]) can be used to show that, in general, one cannot now do much better than to ask for the list of all answers for the word problem: There exists a finitely presented group $G$ such that for every computable $\lambda$ the time-bounded Kolmogorov complexity of the word problem is not less than $\frac{Const^L}{c(\lambda)} - const$ for some $Const > 1$, $c(\lambda) > 0$. In [N3] we prove that for every closed smooth manifold $M_0^n$ of dimension $n > 4$ and computable time $\lambda$ the time-bounded Kolomogorov complexity of the decision problem "Is a given smooth manifold diffeomorphic to $M_0^n$?" is also not less than $\frac{Const(n)^L}{c(\lambda)} - const$ for some universal $Const(n) > 1$. Here the complexity $L$ can be, for example, the number of simplices in a smooth triangulation of the given manifold. To relate this result to geometry of sublevel sets of $\phi$ note that the mentioned diffeomorphism problem can be solved using a set of representatives from every connected component of $\phi^{-1}([0,x]) \bigcap Moduli(M_0^n)$ as the oracle information (see the previous section for the notations). Indeed, the diffeomorphism problem can be restated as the decision problem of recognizing whether or not a given element $\mu \in Moduli_n$ is in $Moduli(M_0^n)$; the oracle information enables one to solve the diffeomorphism problem for all $\mu \in Moduli_n$ such that $\phi(\mu) \leq x$. For this purpose one just needs to check whether or not an approximation to $\mu$ can be connected with one of the elements provided by the oracle by a finite sequence of sufficiently short "jumps" in $Moduli_n$. Now our lower bound for the time-bounded Kolmogorov complexity can be used to produce a lower bound for the number of the connected components of $\phi^{-1}([0,x])\bigcap Moduli(M^n)$. In many interesting cases this lower bound is at least exponential in $x$.

## 5. Disconnectedness of sublevel sets of Riemannian functionals.

In this section $M^n$ denotes a closed Riemannian manifold of dimension $n \geq 5$. Consider the space of Riemannian structures $Riem(M^n)$ (=isometry classes of Riemannain metrics) on $M^n$ endowed with the Gromov-Hausdorff metric. In this section we will consider geometry of sublevel sets of various Riemannian functionals on this space. Our goals are to prove that their sublevel sets are disconnected with a growing number of connected components, and, when possible, to prove the existence of infinitely many locally minimal values.

The first result of this kind was proven in [N2]: Let $I_{M^n}(\epsilon)$ denote the space of Riemannian structures on $M^n$ of volume equal to one and injectivity radius $\geq \epsilon$.

**Theorem 5.1.** *If $n \geq 5$, then for all sufficiently small $\epsilon$ $I_{M^n}(\epsilon)$ is not connected. Moreover, there exist two non-empty subsets of $I_{M^n}(\epsilon)$ such that the Gromov-Hausdroff distance between each point of one of these sets and each point of the other is at least $\epsilon/9$.*

In fact, one can use the notion of time-bounded Kolmogorov complexity as described in the previous section to show that there exist $\sim \frac{1}{\epsilon^n}$ non-empty subsets of $I_{M^n}(\epsilon)$ such that the distance between each pair of points in different subsets is at least $\epsilon/9$. Moreover, assume that one would like to connect a point in one of these subsets to a point in the other by a path in $I_{M^n}(\delta)$ for some positive $\delta < \epsilon$. It is not difficult to prove using a precompactness argument that some such $\delta = \delta_n(\epsilon)$ must exist. Yet $\frac{1}{\delta_n(\epsilon)}$ grows faster than any computable function of $\lfloor \frac{1}{\epsilon} \rfloor$.

Studies of variational problems for Riemannian functionals are motivated by the following problem posed by R. Thom (cf. [Be] , p. 499): "What is the best Riemannian structure on a given compact manifold?". (This question also appears in a well-known list of unsolved problems in Differential Geometry composed by S.T. Yau ([Y]).) A possible idea here is to choose a natural Riemannian functional and to look for its minima (or local minima) on the set of Riemannian structures on a given closed manifold $M^n$. However, for $n \geq 5$ (and probably $n = 4$) there is no really good notion of the "best" Riemannian structures on all $n$-dimensional manifolds ([N4]): Assume that for every $M^n$ there exists a non-empty subset $Best(M^n) \subset Riem(M^n)$. Also assume that there exists an algorithm recognizing when a given Riemannian metric is very close to one of the best Riemannian metrics. (This assumption is required to eliminate the following "solution" of the problem: Use the axiom of choice to choose one Riemannian structure on every $M^n$.) Then for every $M^n$ $Best(M^n)$ is an infinite set.

Indeed, assume the opposite. Then there exists the following algorithm deciding whether or not a given $n$-dimensional manifold is diffeomorphic to $M^n$ yielding a contradiction with S.P. Novikov theorem: Start from any Riemannian metric on a given manifold. Do a trial and error search until we find a Riemannian metric close to one of the best Riemannian metrics on the considered manifold. As we assumed that the set of the best Riemannian metrics on $M^n$ is finite, we can assume that the algorithm "knows" them all (or, more precisely, it knows a sufficiently close approximation to each of them). Now we can check if the found approximation to a best metric is sufficiently close to one of the known best Riemannian metrics on $M^n$. The given manifold is diffeomorphic to $M^n$ if and only if the answer is positive.

This argument strongly suggests that if a Riemannian functional $\phi$ has local minima on $Riem(M^n)$ for every $M^n$, and the set of its local minima is locally compact, then for every $M^n$ the set of local minima of $\phi$ must have an infinite set of connected components. Thus, the following result obtained by the author and Shmuel Weinberger seems to provide a reasonably good solution of the problem posed by R. Thom. The naive idea is that one can try to define the best Riemannian metrics by fixing a scale (i.e. the diameter or volume) and looking for (local) minima of a curvature functional, for example, $\sup |K|$, where $K$ denotes the sectional curvatuture. Equivalently, one can consider Riemannian metrics with $\sup |K| \leq 1$ and to look for local minima of the diameter. More formally, let $Al(M^n)$ denote the Gromov-Hausdorff closure of the subset of $Riem(M^n)$ formed by all Riemannian structures satisfying $\sup |K| \leq 1$ in the space of all metric spaces homeomorphic to $M^n$. The elements of this space are Alexandrov structures on $M^n$ with curvature bounded above and below. They have virtually the same nice analytic and geometric properties as smooth Riemannian manifolds with sectional curvature between $-1$ and 1 (see [BN]). In particular, they are $C^{1,\alpha}$-smooth Riemannian manifolds for each $\alpha < 1$. For each element of $Al(M^n)$ its sectional curvature is defined at almost all points, and the absolute value of the sectional curvature does not exceed 1. It is well-known that sublevel sets of the diameter $d$ regarded as a functional on $\bigcup_{M^n} Al(M^n)$ are precompact. However, there exist manifolds $M^n$ such that $Al(M^n)$ is complete, and, therefore, sublevel sets of $d$ on $Al(M^n)$ are compact, as well as manifolds $M^n$ such that sublevel sets of $d$ on $Al(M^n)$ are not compact. For example, tori $T^n$ admit flat metrics with arbitrarily small diameter, so that $\inf_{Al(T^n)} d = 0$ for every $n$. Therefore, even the existence part in the following theorem proven by the author and Shmuel Weinberger is non-obvious:

**Theorem 5.2.** *([NW1]) For every closed manifold $M^n$ of dimension $n > 4$ the set of locally minimal values of $d$ on $Al(M^n)$ is an unbounded set.*

In particular, this theorem implies that the set of locally minimal values of $d$ on $Al(M^n)$ is infinite. However, it is not difficult to see that the set of its locally minimal values is countable. We also proved many additional results about distribution of local minima of $d$ on $Al(M^n)$ and geometry of connected components of sublevel sets $d^{-1}((0,x])$ of $d : Al(M^n) \longrightarrow (0,\infty)$. For example, we proved that the assertion of Theorem 5.2 will remain true for the values of $d$ at its "very deep" local minima. Here one can define "very deep" local minima by first choosing a (preferably rapidly growing) strictly increasing *computable* function $\phi : N \longrightarrow N$ and postulating that a local minimum $\mu$ of $d$ is "very deep" if there is no path

$\gamma : [0,1] \longrightarrow Al(M^n)$ starting at $\mu$ such that $d(\gamma(1)) < d(\gamma(0)) = d(\mu)$, and $d(\gamma(t)) \leq \phi(\lfloor d(\mu) \rfloor)$ for each $t \in [0,1]$. Moreover, the result remains true if one considers only those "very deep" local minima of $d$, where the value of the volume is not less than 1 (or any other fixed value). Furthermore, we proved that the number of these "very deep" local minima of $d$ on $Al(M^n)$, such that the value of $d$ does exceed $x$, grows at least exponentially with $x^n$. Later Shmuel Weinberger observed that this distribution function for the number of very deep local minima has even a doubly exponential lower bound ([We]). (To explain the last observation note that the volume of manifolds with $|K| \leq 1$ and $diam \leq x$ can be as large as $\exp(c(n)x)$. Thus, one can "fit" an exponential number of nonintersecting metric balls of radius $\sim 1$ and volume $\sim 1$ inside such a manifold. Therefore, one can reduce the halting problem for a universal Turing machine with inputs of lengths up to $\exp(const(n)x)$ to a certain version of the diffeomorphism problem relevant here and explaned in the next section. This version of diffeomorphism problem involves only Riemannian manifolds with $|K| \leq 1$ and $diam \leq x$. The time-bounded Kolmogorov complexity of the halting problem grows exponentially with the length of the inputs, and the number of the local minima grows at least as the time-bounded Kolmogorov complexity, as it was explained in the previous section.)

We refer the reader to our paper [NW2] for further results about depths of the local minima of $d$ on $Al(M^n)$ and the distribution of local minima of different depths.

## 6. Methods III: Simplicial norm, homology surgery, arithmetic groups.

Our proof of Theorem 5.2 follows the scheme outlined in section 3 but contains several new ideas. We start from recalling a classical result of Gromov ([G1], [Gr]) that if a closed Riemannian manifold has a positive simplicial volume, then a lower bound for the Ricci curvature implies a positive lower bound for the volume. More precisely, if $Ric \geq -(n-1)$, then $vol(M^n) \geq c(n)\|M^n\|$, where $\|M^n\|$ denotes the simplicial volume, and $c(n)$ is an explicit constant depending only on the dimension. Simplicial volume is a homotopy invariant of manifolds (see [G1] for its definition and basic properties.) It depends only on the fundamental group of the manifold and the image of its fundamental class under the classifying map. As the isomorphism problem for groups is algorithmically unsolvable, the following theorem proven in [NW1] is not especially surprising:

**Theorem 6.1.** *Let $M^n$ be a closed manifold of dimension $n > 4$. There is no algorithm that decides whether or not a given manifold $N^n$ is diffeomorphic to $M^n$ even if it is a priori known that, if $N^n$ is not diffeomorphic to $M^n$, then it has a simplicial volume greater than* 1.

Here one can replace 1 by any constant, if desired. Thus, having a large simplicial volume is not helpful, when one tries to distinguish between manifolds by means of an algorithm. This theorem immediately follows from its particular case, when $M^n = S^n$. To prove this theorem we need a large stock of $n$-dimensional smooth homology spheres of non-zero simplicial volume. (Homology groups are computable, and there exists an easy algorithm that is able to distinguish between $S^n$ and a manifold which is not a homology $n$-sphere.) Further, it turned out that given *one* homology $n$-sphere with a non-zero simplicial volume, one is able to construct a collection of different homology spheres with simplicial volume $> 1$ which is sufficiently rich to prove Theorem 6.1. Thus, proving the following theorem turned out to be by far the most difficult part of the proof of Theorem 6.1:

**Theorem 6.2.** *([NW1]) For every $n \geq 5$ there exists an $n$-dimensional smooth homology sphere of a non-zero simplicial volume.*

Prior to our work such homology spheres were known only for $n = 3$. Very informally speaking, such manifolds enjoy simultaneously certain hyperbolicity properties (namely, non-zero simplicial volume) as well as ellipticity properties (homology of a sphere). Their construction starts from an application of work of J.P. Hausmann and P. Vogel ([H], [V]) based on the theory of homology surgery by S. Cappell and J. Shaneson ([CS]). This work enables us to reduce the topological problem to an algebraic problem of constructing finitely presented groups with certain homological properties. The resulting algebraic problem can be essentially resolved by using certain discrete cocompact subgroups of $SU(2n-1,1)$ investigated by L. Clozel ([Cl]), who proved that these groups have very few non-trivial real homology classes below dimension $n$. We obtain the desired groups from the groups investigated by Clozel by taking certain amalgamated free products and passing to the universal central extension to kill the remaining real homology classes below dimension $n$.

Once Theorem 6.1 was established, we followed a line of reasoning similar to the outline described in section 3. In particular, we needed to design an algorithm that constructed sufficiently dense nets in the spaces of Riemannian structures on all closed $n$-dimensional manifolds satisfying $|K| \leq 1$, $vol \geq const > 0$ and $diam \leq x$ for a variable $x$. For this purpose we used the Ricci flow to smooth out the Riemannian metric and to obtain a control over derivatives of the curvature tensor, and a subsequent algebraic approximation to reduce the infinite-dimensional situation to a finite dimensional one.

# 7. Disconnectedness of sublevel sets of Riemannian functionals: current work and some open questions.

The smoothing out of Riemannian metrics by means of the Ricci flow is not available if one replaces the two-sided bound for the sectional curvature by the lower bound (or by the two-sided bound for the Ricci curvature). Therefore, we do not know how to prove the existence of an algorithm constructing a sufficiently dense net in the space of Riemannian structures on all $n$-dimensional manifolds satisfying $K \geq -1$ (or $|Ric| \leq 1$) and $diam \leq x$ despite the fact that these spaces are well-known to be precompact. The difficulty can be captured in the following problem:

**Problem:** Does there exist an algorithm that given a positive $\epsilon$ and a finite metric space $X$ decides whether or not $X$ is $\epsilon$-close (in the Gromov-Hausdorff metric) to an $n$-dimensional Riemannian manifold with $K \geq -1$? We allow here a certain room for an error: a positive answer must imply only the $1.01\epsilon$-closeness, whenever a negative answer needs to imply only that $X$ is not $0.99\epsilon$-close to any such manifold. (Here we assume that $\epsilon$ and all distances between points of $X$ are algebraic numbers.)

The problem remains open if one would consider the class of $n$-dimensional Alexandrov spaces with $K \geq -1$ instead of Riemannian manifolds with $K \geq -1$. (It is also open, if one would replace the condition $K \geq -1$ by $Ric \geq -(n-1)$ or $|Ric| \leq n-1$.)

The main purpose of our paper [NW3] is to bypass this difficulty, and to prove the analogues of Theorem 5.2 and all results about geometry of sublevel sets of diameter on $Al(M^n)$ mentioned in section 5 in the situation, when the two-sided bound for the sectional curvature is replaced by the lower bound. In other words, we replace $Al(M^n)$ by a (larger) space $al(M^n)$ of Alexandrov structures with curvature $\geq -1$ on $M^n$. More formally, $al(M^n)$ is the Gromov-Hausdorff closure of the set of all Riemannian structures on $M^n$ satisfying $K \geq -1$ in the space of isometry classes of metric spaces homeomorphic to $M^n$.

Our basic idea is to "approximate" the space of Riemannian structures of sectional curvature bounded below on closed $n$-manifolds by a space of isometry classes of simplicial length spaces that share some important metric and topological properties with manifolds with curvature bounded from below. One chooses these properties so that they can be verified by means of an algorithm (in order to be able to construct the desired nets). For example, one needs to have a lower bound for the volume in terms of the simplicial volume for these length spaces. To ensure this property one can use Theorem 5.38 in [Gr]. This theorem yelds a desired generalization of the mentioned result from [G 1] providing a lower bound for the volume in terms of simplicial volume in the case, when the Ricci curvature is bounded from below. According to Theorem 5.38 in [Gr] an analogous lower bound will be valid for a length space if a packing function for its universal covering admits an upper bound which is similar to Bishop-Gromov upper bounds for manifolds with Ricci curvature bounded below. However, note that universal coverings cannot be constructed by means of an algorithm (as, for example, there is no algorithm deciding whether or not the fundamental group is trivial). Nevertheless, one can modify this constraint so that it becomes verifiable by means of an algorithm: It is sifficient to require the desired upper bound for the packing function for the *effective universal covering* (that will be explained below in section 9) instead of the usual universal covering.

We believe that this approach can also be used to generalize Theorem 5.2 to the situation, when the bound $|K| \leq 1$ is replaced by $|Ric| \leq n-1$ (or by

$Ric \geq -(n-1)$).

Furthermore, we conjecture that an analogue of Theorem 5.2 will hold in the situation, when one replaces *diam* by *vol*. In particular, we would like to establish disconnectedness of sublevel sets of *vol* on $Al(M^n)$ (and $al(M^n)$). This problem is interesting, because sets $vol^{-1}([v,V]) \subset Al(M^n)$ are not precompact, yet the failure of the precompactness is not too "severe".

Finally note, that it is possible that the technique used to prove Theorem 5.2 is applicable to the Einstein-Hilbert action, and can even lead to a proof of the existence of infinitely many isometry classes of singular Einstein metrics of scalar curvature equal to $-1$ on every compact manifold of dimension $> 4$.

# 8. Higher-dimensional cycles in sublevel sets.

In the previous sections we discussed deep local minima (or, more generally, deep basins on graphs) of some functionals. In principle, one can regard a non-trivial deep local minimum of a functional $F$ on a simply-connected space $X$ as a homologically non-trivial 0-dimensional cycle in a sublevel set of $F$ that becomes trivial in an ambient sublevel set that corresponds to a much higher value of $F$. (This 0-cycle is the linear combination of the deep non-trivial local minimum and the global minimum taken with opposite signs.)

One can provide the following intuitive explanation of the appearance of the non-trivial deep basins in situations that we have considered: As the manifold of interest is algorithmically indistinguishable from other manifolds of the same dimension but with different fundamental groups, there will be deep basins where the manifold "looks" like it has a certain fundamental group which is different from what it actually is.

Similar phenomena for higher-dimensional cycles were explored in [NW2]. We found various sources of higher-dimensional cycles in sublevel sets of Riemannian functionals, say *diam* on $Al(M^n)$, that become null-homologous only in a much larger sublevel set corresponding to a much higher value of the functional.

To explain this phenomenon note that $Al(M^n)$ is weakly homotopy equivalent to the space $Riem(M^n)$ of Riemannian structures on $M^n$. This last space is the quotient of the contractible space of Riemannian metrics on $M^n$ by the pullback action of the diffeomorphism group. Therefore, the topology of $Riem(M^n)$ (and $Al(M^n)$) is closely related to the topology of $BDiff(M^n)$. (For example, if all compact groups non-trivially acting on $M^n$ are finite, then $Riem(M^n)$ is rationally homotopy equivalent to $BDiff(M^n)$.). On the other hand, $BDiff(M^n)$ has a very rich topology - especially in the case, when $M^n$ has a non-trivial fundamental group. For example, in many interesting cases one can identify a subgroup of a homology group of $BDiff(M^n)$ isomorphic to a lattice in a homology group of the fundamental group of $M^n$ with real coefficients. (For this purpose one can use $H\rho$-invariant introduced by Shmuel Weinberger in [We0].) Now we can use the logical method, and to argue that as $M^n$ is algorithmically undistinguishable from manifolds $N^n$ with arbitrary large fundamental groups, the homology classes of

$\pi_1(N^n)$ corresponding to non-trivial homology classes of $BDiff(N^n)$ and $Al(N^n)$ will correspond to "virtual" homological classes of $M^n$, that is, to cycles in sublevel sets of *diam* on $Al(M^n)$ that will become null-homologous only in much large sublevel sets. This approach works under some restrictions on topology of $M^n$, and produces "virtual" $k$-cycles for $k << n$. Another approach to constructing "virtual" $k$-cycles in $Al(M^n)$ is based on connections between $Diff(N^n)$ and $Out(\pi_1(N^n))$ and works for all closed manifolds $M^n$ of dimension $> 4$ and all $k$. For example, one can always choose $N^n$ (algorithmically undistinguishable from $M^n$), so that $Diff(N^n)$ admits a split surjection on $Z^m$ for arbitrarily large values of $m$. As the result, for every $k$ one obtains "virtual" $k$-cycles in $Al(M^n)$.

On the other hand, Shmuel Weinberger noticed that if $M^n$ admits a non-trivial smooth compact group action, then one can similarly exploit a part of topology of $Riem(M^n)$ based on singularities that does not come from the topology of $BDiff(M^n)$. In particular, in [NW2] we used the non-existence of an algorithm deciding whether or not the fixed point set of an $S^1$-action on $S^n$ is diffeomorphic to $S^{n-2}$ to prove the existence of 5-dimensional (or, more generally, $(4i+1)$-dimensional) "virtual" rational cycles in $Al(S^n)$ that are close to the round metric in the path metric on $Al(S^n)$ (see Theorem 17.1 in [NW2] and Theorem 5 in section 4.1 of [We] for precise statements).

## 9. Morse landscapes of the length functional.

Assume that $M^n$ is a simply-connected Riemannian manifold, $p \in M^n$. Consider the length functional $l$ on the space $\Omega_p M^n$ of loops on $M^n$ based at $p$. Note that, in principle, $l$ can have no local minima other than the trivial loop. If there exists another local minimum $\alpha$, we can define its depth as the minimal possible difference between the length of the longest loop in a path homotopy connecting $\alpha$ with a loop of a smaller length and the length of $\alpha$. One can generalize this definition for the situation, when the length is regarded as a functional on the space $\Omega_{p,q}M^n$ of paths connecting a pair of points $p, q \in M^n$. (Of course, $\Omega_{p,p}M^n = \Omega_p M^n$.)

It is clear that one can give a similar definition of depth in the case, when $\alpha$ is a critical point of the length functional of a higher index $i$. (One needs to look at the minimal $x \geq l(\alpha)$ such that an appropriately defined $i$-cycle in $l^{-1}([0, l(\alpha)])$ that "hangs" at $\alpha$ becomes a boundary in $l^{-1}([0, x])$; the depth is then defined as $x - l(\alpha)$. If no such $x$ exists, then we say that $\alpha$ has infinite depth.)

In [N5] we proved a theorem with the following informal meaning (see Theorem 2.1 in [N5] for an exact statement):

**Theorem 9.1.** *Let $M^n$ be a simply-connected Riemannian manifold. Assume that the length functional has a "very deep" non-trivial local minimum on $\Omega_p M^n$. Then it has "many" "deep" local minima.*

In other words, this theorem asserts that if there exists a loop $\gamma$ based at $p$ that cannot be contracted to a point via loops of length $\leq L+$ length($\gamma$), then there exist at least $k(L)$ geodesic loops providing "deep" local minima for the length functional on $\Omega_p M^n$, where $k(L) \longrightarrow \infty$, as $L \longrightarrow \infty$.

Note that a counterexample to Theorem 9.1 must "look" like a Riemannian manifold with a "small" finite fundamental group. Otherwise we will be able to construct "many" deep local minima of the length functional by taking powers and products of powers of the already constructed geodesic loops based at $p$ and shortening them to geodesic loops providing new local minima. Therefore, informally speaking, Theorem 9.1 implies that a closed simply-connected Riemannian manifold cannot "look" like it has a finite fundamental group. (Of course, we saw in the previous sections that a closed simply-connected Riemannain manifold can "look" like it has an infinite fundamental group, and this fact was one of the cornerstones of all applications of recursion theory to geometry discussed in this paper.)

The proof of this theorem given in [N5] is based on the idea of the "effective univeral covering". (Recall that this concept can also be used for proving analogues of Theorem 5.2 for weaker curvature constraints - see section 7 above.) This idea can be explained as follows: [4]

The universal covering space of a topological space $X$ is usually constructed as the quotient of the space of paths on $X$ starting at a base point $x \in X$ by the following equivalence relation: Two paths are equivalent if they end at the same point, and together form a contractible loop (based at $x$). Let $X = M^n$ be a closed Riemannian manifold. One can try to make the following natural modification of this construction: Assume that one takes into consideration the length of paths, and allows only a controlled increase of length during a homotopy contracting the loop formed by two paths. More specifically, one can choose parameters $U$ and $V > 2U$, consider the set $P(U)$ of all paths of length $\leq U$ based at $x$, and then try to introduce the equivalence relation $\sim_V$ on this set by identifying paths forming loops contractible via loops of length $\leq V$. However, in general, this relation will not be an equivalence relation. Nevertheless, we observed that there exists a "large" set of values of $V$ such that $\sim_V$ is an equivalence relation. In particular, one can choose a "controllably" large $V = V(U, M^n)$, and to obtain an effectively constructible connected space $P(U, V)$ of $\sim_V$-equivalence classes of elements of $P(U)$ so that the map $P(U, V) \longrightarrow M^n$ sending each equivalence class of paths into their common endpoint is a covering "away from the boundary" in the sense of Definition 1.1 in [N5]. One can regard sets $P(U, V)$ as constructive analogs of metric balls of radius $U$ in the universal covering of $M^n$.

Now one can demonstrate Theorem 9.1 by contradiction. Assume that there exists a counterexample. It must "look" like a manifold with a finite fundamental group formed by "few" "deep" local minima of the length functional on $\Omega_p M^n$. Observe that when one constructs the usual universal covering of a closed Riemannian manifold with a finite fundamental group, one does not need to consider arbitrarily long paths. Paths of length $\leq 2d|\pi_1 M^n|$ are sufficient. (Longer paths are equivalent to some of the shorter paths.) A similar phenomenon occurs, when we construct the "effective universal covering" $P(U, V)$ of $M^n$ for appropriately

[4]Note similarities between this idea and the notion of fundamental pseudogroups introduced by Gromov in [G2].

chosen $U$ and $V$ using $p$ as the base point: Longer paths become equivalent to shorter paths, $P(U,V)$ becomes a closed manifold and the covering "away from the boundary" becomes the covering of $M^n$ in the usual sense. Our assumption about the existence of at least one "very deep" non-trivial local minimum of the length functional implies that the cardinality of the fiber of this covering is at least two, and so it cannot be a homeomorphism. However, all coverings of $M^n$ are trivial, as $M^n$ was assumed to be simply-connected, and we obtain a desired contradiction.

Note that this proof implies that if the depth of a non-trivial local minimum is $\lambda d$, then there exist at least $k(\lambda) \sim \sqrt{\lambda}$ local minima. Moreover, according to Theorem 2.1 in [N5] the lengths of the geodesic loops $\gamma_i$ providing these local minima do not exceed $4id, i = 1, \ldots, k$.

Both these estimates were recently improved in [NR] using a different approach that was based on geometric constructions invented largely by Regina Rotman. In particular, we demonstrated that if the the length functional on $\Omega_p M^n$ has a non-trivial local minimum of depth $> \lambda d + S$, for some $S \geq 2d$ and $\lambda$, then there exist $k \geq [\frac{\lambda}{6} + \frac{1}{2}]$ non-trivial local minima of depth $> S$. In addition, one can ensure that the length of $\gamma_i$ is in the interval $(2(i-1)d, 2id]$. (This is a direct corollary of Theorem 7.3 in [NR] for $m = 1$.) The same technique also implies that the existence of a "very deep" critical point of any index $m \geq 0$ of a finite depth of the length functional on $\Omega_p M^n$ also implies the existence of "many" "deep" local minima of the length functional; explicit bounds for the number, lengths and depths of these minima are available. For example, if the depth of a critical point of index $m$ is finite but greater than $\lambda d + (2m-1)S$, for some $S \geq 2d$ and $\lambda$, then one is guaranteed $k \geq [\frac{\lambda}{4m+2} - \frac{2m-5}{4m+2}]$ local minima of depth $> S$ with lengths in the intervals $(2(i-1)d, 2id]$, $i = 1, \ldots, k$. Furthermore, Theorems 7.3, 7.4 in [NR] immediately imply similar results for the length functional on spaces $\Omega_{p,q}(M^n)$. Thus, in particular, the results of [NR] imply that:

**Theorem 9.2.** *(Imprecise version) If the length functional $l$ on $\Omega_{p,q}(M^n)$ has a critical point of an arbitrary index of a "large" finite depth, then $l$ has "many" "deep" local minima.*

For the lack of time I will not attempt to give a more detailed presentation of these and related results and methods, and most notably applications of these methods to quantitative geometric calculus of variations. Instead I refer the readers to [NR], [NR0], [R1], [R2], [R3] for some of the highlights of this emerging theory that has its origins in some of Gromov's ideas from [G3].

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