APMA E3102 2007 Lectures Summary

Lecture 1.

Recalls on ODEs. An ordinary differential equation is an equation of the form

$$\dot{\mathbf{X}} = f(t, \mathbf{X}), \qquad t > 0, \qquad \mathbf{X}(0) = \mathbf{X}_0.$$

Here $\mathbf{X}(t)$ may be a vector depending on the one dimensional variable t (say time); $\mathbf{X}_0$ is the initial condition; $f(t, \mathbf{X})$ is a precribed well-behaved vector-valued function.

The solution of $\dot{x} + ax = 0$ with initial condition $x(0) = x_0$ is given by

$$x(t) = x_0 e^{-at}.$$

The solution of the harmonic oscillator equation $\ddot{x} + \omega^2 x = 0$ with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ is given by

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

Note that this equation may be written in the general framework written above with $\mathbf{X} = (x, \dot{x})$.

Partial differential equation. Definition: constraint on a function of several variables involving its partial derivatives. Example of partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} = 0.$$

For the above PDE, we verify that

$$f(t, x) = g(t - x),$$

for an arbitrary differentiable function $g(y)$, solves the equation. Most PDEs do not admit such a simple solution.

Derivation of PDEs in one space dimension. Let $\rho(t, x)$ be a density of particles and $u(t, x)$ the particle speed. Let

$$N(t, a, b) = \int_a^b \rho(t, x) A dx,$$

the number of particles inside a rod (a, b) of cross section A. We also assume that $Q(x, t)$ particles are *created* at position x and time t per unit time per unit volume.

The **conservation of particles** states that

$$\frac{d}{dt} N(t, a, b) = A(\rho u)(t, a) - A(\rho u)(t, b) + \int_a^b Q(t, x) A dx.$$

Conservation of number of particles on arbitrary domains allows us to differentiate the above expression with respect to b and obtain that

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} = Q(t, x).$$

Brief mention of boundary conditions and initial conditions for the above equation.

Lecture 2. Extension of the above result to the heat equation. Let $e(t,x)$ be the heat energy and $\phi(t,x)$ the energy flux. Then we find that

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} = 0.$$

Let $e(t,x) = c\rho u(t,x)$, where u is temperature. On physical grounds (experience), we posit that (Fourier's law)

$$\phi(t,x) = -K_0 \frac{\partial u}{\partial x}.$$

This models that heat energy flows from high to low temperatures. Note that this relation does not come from any conservation law. We then find that temperature solves the following heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad k = \frac{K_0}{c\rho} > 0.$$

The heat equation requires **initial conditions**

$$u(0,x) = u_0(x),$$

and **boundary conditions**. For instance:

$$u(a,t) \text{ and } u(b,t) \text{ prescribed : } \quad \text{Dirichlet conditions}$$

or

$$\frac{\partial u}{\partial x}(a,t) \text{ and } \frac{\partial u}{\partial x}(b,t) \text{ prescribed : } \quad \text{Neumann conditions.}$$

The above results are then extended to two and three dimensional geometries. Let A be a domain with boundary ∂A. Let $\boldsymbol{\phi}(\mathbf{x})$ defined for $\mathbf{x} \in A$. The divergence theorem states that

$$\int_A \text{div}\boldsymbol{\phi}(\mathbf{x})\text{d}\mathbf{x} = \int_{\partial \text{A}} \boldsymbol{\phi}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\text{d}\Sigma(\mathbf{x}),$$

where div is the divergence operator, $d\mathbf{x}$ is the volume measure in A, $d\Sigma$ is the surface measure on ∂A, and $\mathbf{n}(\mathbf{x})$ is the outward unit normal to A at $\mathbf{x} \in \partial A$.

Conservation of the heat energy on A is written as

$$\frac{d}{dt}\int_A e(t,\mathbf{x})d\mathbf{x} = -\int_{\partial A} \boldsymbol{\phi}(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})d\Sigma(\mathbf{x}).$$

Here $e(t,\mathbf{x})$ is the energy density and $\boldsymbol{\phi}(t,\mathbf{x})$ is the heat flux (per unit surface per unit time). Using the divergence theorem and the arbitrariness of the volume A, we find the local expression of the conservation law:

$$\frac{\partial e}{\partial t}(t,\mathbf{x}) + \text{div}\boldsymbol{\phi}(t,\mathbf{x}) = 0.$$

Now using Fourier's law

$$\boldsymbol{\phi}(t,\mathbf{x}) = -K_0 \nabla u,$$

where $e = c\rho u$ and ∇ is the gradient operator, we find that u solves the following heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u \equiv k\Delta u, \qquad k = \frac{K_0}{c\rho} > 0.$$

Here $\nabla^2 \equiv \Delta$ is the Laplace operator.

The heat equation needs to be augmented by initial conditions $u(0,\mathbf{x})$ and boundary conditions. For instance $u(t,\mathbf{x})$ is prescribed at $\mathbf{x} \in \partial A$ (Dirichlet conditions) or $\frac{\partial u}{\partial n}(t,\mathbf{x}) \equiv \mathbf{n} \cdot \nabla u(t,\mathbf{x})$ is prescribed at $\mathbf{x} \in \partial A$ (Neumann conditions).

Lecture 3. Linearity and superposition principle. Consider an equation of the form

$$Lu = f \quad \text{with boundary conditions} \quad Bu = g,$$

and define the operator

$$\mathcal{L} = \begin{pmatrix} L \\ B \end{pmatrix}.$$

The operator $\mathcal{L}$ is *linear* if

$$\mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L} u + \beta \mathcal{L} v,$$

where u and v are functions and α and β are real (or more generally complex) numbers.

Linearity allows us to derive the *principle of superposition*: the solution of a problem with two sources is the sum of the two solutions obtained with one source each. More specifically, if $\mathcal{L}u_1 = \mathcal{F}_1$ and $\mathcal{L}u_2 = \mathcal{F}_2$, then $u_1 + u_2$ is a solution of the equation $\mathcal{L}u = \mathcal{F}_1 + \mathcal{F}_2$.

Method of separation of variables. Consider the following heat equation with Dirichlet boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, && x \in (0, L),\ t > 0 \\ &u(0, x) = f(x) && x \in (0, L) \\ &u(t, 0) = u(t, L) = 0 && t > 0, \end{aligned} \tag{1}$$

defined on the rod $(0, L)$ for times $t > 0$.

The general methodology for the method of separation of variables is the following.

1. We neglect non-vanishing source terms at first (here the initial conditions) and look for non-trivial (i.e., non uniformly vanishing) solutions that separate variables, i.e., here, functions of the form $G(t)\phi(x)$, that solve all of the constraints in (1) but the initial condition constraint (here the first and third constraints). We find equations for $G(t)$ and $\phi(x)$.

2. We solve these equations. Typically, we'll find an infinite number of such *elementary* solutions, which we'll call $u_n(t, x) = G_n(t)\phi_n(x)$ for $n \geq 1$ (or $n \geq 0$ if the notation is more convenient).

3. We use the linearity of the equation and the principle of superposition to realize that any *linear combination* of the form

$$u(t, x) = \sum_{n=1}^{\infty} B_n u_n(t, x) \tag{2}$$

solves the PDE provided that the sum makes sense (i.e., converges). Note that the constants B_n are arbitrary (i.e., undetermined yet).

4. Finally (and only now) we come back to the source term, here $f(x)$ and see whether among the solutions of the form (2), one verifies the initial conditions. Namely, can we choose the coefficients B_n such that the following holds?

$$f(x) = u(0, x) = \sum_{n=1}^{\infty} B_n u_n(0, x). \tag{3}$$

Typically, we'll involve a general theory that answers yes to the above question and gives us a means to calculate the coefficients B_n from knowledge of $f(x)$. With these specific values of the coefficients B_n, the expression in (2) will then be the required solution to (1).

Let us apply the methodology to (1). Step 1 says that $u(t,x) = G(t)\phi(x)$ solves constraints 1 and 3 in (1), i.e.,

$$G'(t)\phi(x) = kG(t)\phi''(x), \quad G(t)u(0) = G(t)u(L) = 0.$$

Since $G(t)$ is non-trivial, $u(0) = u(L) = 0$. The first constraint states that

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda,$$

where $-\lambda$ is a constant since it is independent of both t and x. We thus have transformed the PDE into two ODEs

$$G'(t) + \lambda k G(t) = 0$$
$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = \phi(L) = 0.$$

Note that the equations are *eigenvalue* problems, in the sense that the solutions are defined up to a multiplicative constant. Now Step 2. We verify that

$$G(t) = G(0)e^{-\lambda k t}.$$

For the boundary value problem on ϕ, we verify that λ needs to be positive in order for ϕ to be non-trivial (since non-trivial sinh, cosh, and affine functions (of the form $ax+b$) cannot satisfy the boundary conditions). Thus ϕ satisfies:

$$\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x, \quad \phi(0) = \phi(L) = 0.$$

We obtain that $A = 0$ and that

$$\sqrt{\lambda}L = 0. \tag{4}$$

The latter constraint on the eigenvalues of the eigenvalue problem for ϕ impose that λ takes one of the following values

$$\lambda_n = \Big(\frac{n\pi}{L}\Big)^2. \tag{5}$$

Associated to λ_n, we thus find a solution of the PDE (1) (except for the source terms) given by

$$u_n(t,x) = B_n G_n(0)e^{-\lambda_n k t}\sin\sqrt{\lambda_n}x.$$

Since the solutions are defined up to a multiplicative constant, we might as well choose the constant simple and define the elementary solutions as:

$$u_n(t,x) = e^{-\lambda_n k t}\sin\sqrt{\lambda_n}x = e^{-(\frac{n\pi}{L})^2 k t}\sin\frac{n\pi x}{L}. \tag{6}$$

Step 3 is one line: we use superposition to obtain that

$$u(t,x) = \sum_{n=1}^{\infty} B_n e^{-(\frac{n\pi}{L})^2 k t}\sin\frac{n\pi x}{L}, \tag{7}$$

solves constraints 1 and 3 in (1) provided the series converges. Step 4 consists in identifying the coefficients B_n from the constraint on the source term:

$$f(x) = u(0,x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}. \tag{8}$$

We're almost there. The question now is: can an arbitrary (sufficiently nice) function $f(x)$ be decomposed into a superposition of sine functions, and how does one get the coefficients B_n from f?

The theory of Fourier series provides the answer.

Lecture 4. Statement of Fourier series theory (we'll come back to a more comprehensive study of Fourier series later in the class).

Consider an interval $[-L, L]$ and a piecewise smooth function $f(x)$ defined on that interval and extended to the whole real line $\mathbb{R}$ by periodicity; i.e., $f(x+2L) = f(x)$ for all $x \in \mathbb{R}$. Let us define the *Fourier coefficients* of f:

$$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \qquad n \geq 1.} \tag{9}$$

Define the truncated series

$$f_N(x) = a_0 + \sum_{n=1}^{N}\Big(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\Big). \tag{10}$$

Then we have the following Theorem:

(i) If $f(x)$ is continuous in the vicinity of a point x, then

$$\lim_{N\to\infty} f_N(x) = f(x).$$

(ii) If $f(x)$ is discontinuous at x, then

$$\lim_{N\to\infty} f_N(x) = \frac{1}{2}(f(x^+) - f(x^-)).$$

Here $f(x^+) = \lim_{0<\varepsilon\to 0} f(x+\varepsilon)$ and $f(x^-) = \lim_{0<\varepsilon\to 0} f(x-\varepsilon)$.

We then write the Fourier decomposition as

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\Big).} \tag{11}$$

The symbol $\sim$ accounts for the fact that we do not have equality at points of discontinuity of f unless the function f is redefined in such a way that when it jumps we have $f(x) = \frac{1}{2}(f(x^+) - f(x^-))$.

Note that we then recover the definition of the coefficients (9) from (11). For instance, for a_n, $n \geq 1$, we use the orthogonality conditions

$$\begin{aligned} &\frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \delta_{mn} \text{ for } n \geq 1 \text{ and } m \geq 0, \\ &\frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = 0, \text{ for } n, m \geq 0. \end{aligned}$$

where $\delta_{mn} = 1$ when $m = n$ and $\delta_{mn} = 0$ when $m \neq n$ (Kronecker symbol).

Back to (1)**.** Let us now apply the above theory to (8). We have a function $f(x)$ defined on $(0, L)$, which according to (8), should be a superposition of only sine functions.

To use the theory of Fourier series, we use the following trick: we first extend the function $f(x)$ by oddness on $(-L, L)$. This is done by positing that $f(x) = -f(-x)$ on $(0, L)$. Then we extend the function by periodicity on $\mathbb{R}$ by ensuring that $f(x+2L) = f(x)$ for all $x \in \mathbb{R}$.

Now we can use the decomposition (11). Moreover, we verify that because $f(x)$ is an odd function, $a_n = 0$ for $n \geq 0$ in (9). We have thus represented the function $f(x)$ as a superposition of sine functions as requested in (8). Moreover the coefficients B_n are given by b_n in (11). Because the function is odd, we actually verify that b_n are equivalently defined by

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Finally, the full solution to (1), satisfying all constraints including the initial conditions, is given by

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} \Big(\frac{2}{L}\int_0^L f(y)\sin\frac{n\pi y}{L}dy\Big)\sin\frac{n\pi x}{L}e^{-(\frac{n\pi}{L})^2 kt}.} \tag{12}$$

Note that in the above integration, the dummy variable y is used: we can no longer use x, which now means something else on the left hand side of the above equation (the position at which temperature is evaluated).

Lecture 5. Uniqueness of the solution to (1). Assume that u and v solve (1) and define $w = u - v$. By linearity, it satisfies the equation

$$\begin{aligned} &\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}, && x \in (0,L),\ t > 0 \\ &w(0,x) = 0(x) && x \in (0,L) \\ &w(t,0) = w(t,L) = 0 && t > 0. \end{aligned} \tag{13}$$

Multiply the first constraint by w and integrate over $(0,L)$. This yields

$$0 = \int_0^L (\frac{\partial w}{\partial t}w - k\frac{\partial^2 w}{\partial x^2}w)dx = \frac{d}{dt}\frac{1}{2}\int_0^L w^2 dx + k\int_0^L \Big(\frac{\partial w}{\partial x}\Big)^2 dx.$$

To obtain the latter term we have performed an integration by parts and have use the boundary conditions $w(t,0) = w(t,L) = 0$. Since both k and $(\frac{\partial w}{\partial x})^2$ are non-negative we deduce that

$$\frac{d}{dt}\int_0^L w^2 dx \le 0 \qquad \text{whence} \qquad \int_0^L w^2(t,x)dx \le \int_0^L w^2(0,x)dx.$$

This shows that energy is non-increasing. Since at time $t = 0$, $w^2(0,x)$ by hypothesis (these are the initial conditions), we deduce that $w(t,x) = 0$ at all times since $\int_0^L w^2(t,x)dx = 0$. But this implies that $u = v$ so that the solution to (1) is unique.

In order words, the three constraints imposed on u in (1) define it uniquely. We have seen by using the method of separation of variables that it admitted a solution given by (12). The problem (1) is then now completely solved.

Separation of variables for the heat equation with Neumann boundary conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, && x \in (0,L),\ t > 0 \\ &u(0,x) = f(x) && x \in (0,L) \\ &-\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) = 0 && t > 0, \end{aligned} \tag{14}$$

defined on the rod $(0,L)$ for times $t > 0$.

Lecture 6. Separation of variables for the Laplace equation with mixed boundary conditions

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 && 0 < x < L,\ 0 < y < H \\ &u(0,y) = 0, \quad u(L,y) = f(y) && 0 < y < H \\ &\frac{\partial u}{\partial n}(x,0) = 0, \quad \frac{\partial u}{\partial n}(x,H) = g(x) && 0 < x < L. \end{aligned} \tag{15}$$

(i) First realize that the solution can be written as the superposition of two solutions, one with $f(y)$ as the only source term, the other one with $g(x)$ as the only source term.

(ii) Consider the solution with $g(x) = 0$ and $f(y)$ as the only source term. Use the method of separation of variables.

The final solution turns out to have the form

$$u(x,y) = \frac{1}{LH}\int_0^H f(y')dy' + \sum_{n=1}^{\infty} \frac{1}{2H\sinh\frac{n\pi L}{H}}\Big(\int_0^H f(y')\cos\frac{n\pi y'}{H}dy'\Big)\sinh\frac{n\pi x}{L}\cos\frac{n\pi y}{L}. \tag{16}$$

Exercise: (i) solve the problem with $f(y) = 0$ and $g(x)$ as the unique source term. (ii) show that the solution to (15) is unique using the method presented in Lecture 5.

Lecture 7. Fourier series Theorem. Let us consider periodic functions of period $2L$ that are square integrable on $(-L, L)$:

$$\int_{-L}^{L} f^2(x)dx < +\infty, \tag{17}$$

which in addition we assume are piecewise continuous (with a finite number of pieces) to simplify. Then the Fourier theory says that such functions can be decomposed over an *orthogonal basis* of sine and cosine functions. The basis is composed of the functions:

$$\Big\{1, \dots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}, \dots\Big\}_{n\geq 1}. \tag{18}$$

Note that "1" above is the *function* $1(x)$ equal to 1 uniformly. The above basis elements are orthogonal with respect to the following *scalar product*:

$$(f,g) = \int_{-L}^{L} f(x)g(x)dx. \tag{19}$$

As usual, a scalar product (a.k.a. inner product) maps two functions to a real number, here an integral. The proof that the basis elements are orthogonal for the above scalar product can be found in the book. Note that the basis is not orthonormal for:

$$(1,1) = 2L, \qquad (\cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}) = L, \qquad (\sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}) = L. \tag{20}$$

So in order to normalize the basis (and make it orthonormal) the basis function 1 should be divided by $\sqrt{2L}$ and the other basis elements by $\sqrt{L}$.

Now consider an arbitrary function satisfying the hypotheses described above, and assume further that at points of discontinuity of $f(x)$, we have

$$f(x) = \frac{1}{2}(f(x^+) + f(x^-)). \tag{21}$$

Then the Fourier theory says that such a function can be decomposed over the basis we have introduced:

$$f(x) = a_0[f] + \sum_{n=1}^{\infty} a_n[f]\cos\frac{n\pi x}{L} + b_n[f]\sin\frac{n\pi x}{L}. \tag{22}$$

Moreover the above coefficients are uniquely defined by $f(x)$ and are given by:

$$a_0[f] = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad a_n[f] = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n[f] = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{23}$$

These relations stem directly from the orthogonality relations. For instance, assuming that (22) holds, we obtain that

$$(f, \cos\frac{m\pi x}{L}) = a_m[f](\cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}) = La_m[f],$$

from which we deduce the expression for a_n in (145).

The Fourier theorem states the following more accurate result. Define

$$f_N(x) = a_0[f] + \sum_{n=1}^{N} a_n[f] \cos \frac{n\pi x}{L} + b_n[f] \sin \frac{n\pi x}{L}, \tag{24}$$

where the coefficients are given by (145). Then $f_N(x)$ converges pointwise (i.e., for all x) to $f(x)$:

$$\lim_{N\to\infty} f_N(x) = f(x). \tag{25}$$

When (21) does not hold, the above series still converges to $\frac{1}{2}(f(x^+) + f(x^-))$. It is good to consider this issue as follows. As far as integrals are concerned (and specifically the integrals in (145) defining the coefficients), changing the function $f(x)$ at a finite number of points does not change anything. So for a given function, we can change it at the finite number of discontinuity points so that (21) holds: this changes nothing physically. Then (25) holds and there is no point to introduce the notation $\sim$ in the book. I have nothing against the notation. But you should really think that $f(x) \sim g(x)$ means that the two functions f and g are equal for all practical purposes.

Even and Oddness. We say that a function is odd when $f(-x) = -f(x)$ and that it is even when $f(x) = f(-x)$ for all $x \in \mathbb{R}$. An odd $2L$-periodic function $f(x)$ satisfies that $a_n[f] = 0$, $n \geq 0$. An even $2L$-periodic function $f(x)$ satisfies that $b_n[f] = 0$, $n \geq 1$. This comes from the fact that cos is an even function, sin is an odd function, and that the integral over $(-L, L)$ of product of an even with an odd function vanishes.

Lecture 8. Once the Fourier decomposition of a function $f(x)$ is known, the decomposition of the derivative $f'(x)$ is easily obtained: we differentiate (22) term by term. There is however one main restriction: the function $f(x)$ *needs* to be differentiable before one does so.

So let us assume that $f(x)$ is piecewise differentiable and continuous (as a function defined on $\mathbb{R}$, not as a function defined on $(-L, L)$) and that $f'(x)$ is piecewise continuous. Then we have that

$$a_0[f'] = 0, \qquad a_n[f'] = \frac{n\pi}{L} b_n[f], \qquad b_n[f'] = -\frac{n\pi}{L} a_n[f]. \tag{26}$$

Note that the function $f(x) = x$ on $(-L, L)$ and $2L-$periodic does not satisfy the above hypotheses. So (26) does not hold for that function.

Iterating (26) one more time yields

$$a_n[f''] = -\left(\frac{n\pi}{L}\right)^2 a_n[f], \qquad b_n[f''] = -\left(\frac{n\pi}{L}\right)^2 b_n[f]. \tag{27}$$

This obviously assumes that $f(x)$ is twice differentiable and that $f"(x)$ is piecewise continuous.

Application to the solution of PDEs. Let us consider again problem (1). Let us look for a solution $u(t, x)$ whose second-order partial derivative in x is piecewise continuous on $(0, L)$. Let us extend $u(t, x)$ by oddness on $(-L, L)$ and then by periodicity. Because $u(t, 0) = u(t, L) = 0$, we observe that this extension creates a function $u(t, x)$ whose second-order partial derivative in x is piecewise continuous on $\mathbb{R}$ (and not only on $(0, L)$). The function $f(x)$ is similarly extended by oddness and periodicity.

Since $u(t, x)$ is now an odd $2L$-periodic function in the x variable, it may be decomposed as

$$u(t, x) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}. \tag{28}$$

The unknown coefficients are now the functions $b_n(t)$. It remains to find an equation for the $b_n(t)$ and solve that equation to fully characterize $u(t, x)$. The equation for the $b_n(t)$ is obtained by transforming the first constraint in (1) as a constraint on the Fourier coefficients. Note that the boundary conditions in (1) are automatically satisfied by the decomposition (28).

Because $u(t,x)$ is twice differentiable in x, we obtain that

$$\frac{\partial^2 u}{\partial x^2}(t,x) = \sum_{n=1}^{\infty} -\Big(\frac{n\pi}{L}\Big)^2 b_n(t) \sin\frac{n\pi x}{L}.$$

The time derivative may be decomposed as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n'(t)\sin\frac{n\pi x}{L}.$$

Recall that $f(x) = g(x)$ if and only if the Fourier coefficients of these functions agree. This is the definition of (18) being a basis. So the first line in (1) is equivalent to the following equation *in the Fourier domain*:

$$b_n'(t) + k\Big(\frac{n\pi}{L}\Big)^2 b_n(t) = 0. \tag{29}$$

Similarly the initial conditions in (1) are equivalent to the following relation *in the Fourier domain*:

$$b_n[f] = b_n[u(0,\mathbf{x})] = b_n(0). \tag{30}$$

So for each $n \geq 1$, we have a first order ODE (29) and an initial condition (30). This can be solved:

$$b_n(t) = b_n(0)\exp\Big(-k\Big(\frac{n\pi}{L}\Big)^2 t\Big). \tag{31}$$

Now $b_n(t)$ is fully characterized and so is $u(t,x)$ via (28). We recover the same expression as in (12).

Let us consider the second application: solving

$$-u''(x) = f(x) \qquad x \in (0,L), \tag{32}$$

with $u(0) = u(L) = 0$. By extending u and f by oddness and then by periodicity, we verify that $u''(x)$ exists and is a piecewise continuous function on $\mathbb{R}$ (and not only on $(0,L)$). Using the Fourier decomposition and (27) again yields that

$$\Big(\frac{n\pi}{L}\Big)^2 b_n[u] = b_n[f]. \tag{33}$$

This thus gives an explicit expression for $b_n[u]$ and $u(x)$ is given by

$$u(x) = \sum_{n=1}^{\infty}\Big(\frac{L}{n\pi}\Big)^2 b_n[f]\sin\frac{n\pi x}{L}. \tag{34}$$

Lecture 9. **Integration term by term.** Let $f(x)$ be decomposed as in (22). Then the Fourier coefficients of an anti-derivative of $f(x)$, provided they exist, are easy to calculate. Let us define

$$F(x) = \int_0^x f(y)dy. \tag{35}$$

Since $f(x)$ is piecewise continuous, then $F(x)$ is continuous. Any other anti-derivative of $f(x)$, equal to $F(x)$ up to an additive constant, is continuous as well. Moreover we can integrate (22) term by term to get

$$F(x) = a_0 x + \sum_{n=1}^{\infty}\frac{La_n[f]}{n\pi}\sin\frac{n\pi x}{L} + \frac{Lb_n[f]}{n\pi}\Big(1-\cos\frac{n\pi x}{L}\Big).$$

This may be recast as

$$G(x) = F(x) - a_0 x = a_0[G] + \sum_{n=1}^{\infty} a_n[G]\cos\frac{n\pi x}{L} + b_n[G]\sin\frac{n\pi x}{L}, \tag{36}$$

where

$$a_0[G] = \sum_{n=1}^{\infty} \frac{Lb_n[f]}{n\pi}, \qquad a_n[G] = -\frac{Lb_n[f]}{n\pi}, \qquad b_n[G] = \frac{La_n[f]}{n\pi}. \tag{37}$$

Note that the function $F(x)$ is *not* $2L$-periodic unless $a_0[f] = 0$. As a consequence, it does not admit an expansion in Fourier coefficients on $\mathbb{R}$. Once $a_0[f]x$ is subtracted, then $G(x)$ is indeed a $2L$-periodic function, and the above formula shows how its coefficients are related to those of $f(x)$.

Complex form of Fourier series. Recall that

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \tag{38}$$

where $i = \sqrt{-1}$. Define

$$c_0 = a_0, \quad c_n = \frac{a_n + ib_n}{2}, \quad c_{-n} = \frac{a_n - ib_n}{2} = \overline{c_n}.$$

Here $\bar{z} = x - iy$ for $z = x + iy$ means complex conjugate. We verify that (22) may be recast as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} e^{i\frac{n\pi x}{L}} f(x)dx. \tag{39}$$

This is a more compact expression than the one involving the sines and cosines. Note that the above expression is a decomposition over an orthogonal basis, where the basis is

$$\{e^{-i\frac{n\pi x}{L}}\}_{-\infty<n<\infty}, \tag{40}$$

and the orthogonality is with respect to the (Hermitian) inner product

$$(f,g) = \int_0^L f(x)\bar{g}(x)dx. \tag{41}$$

Note the following. If $f(x)$ is piecewise differentiable and continuous, then we can differentiate term by term and obtain that

$$c_n[f'] = \frac{-in\pi}{L} c_n[f]. \tag{42}$$

This is again more compact than the equivalent expression involving sines and cosines. When the formula is applicable, we also have that

$$c_n[f''] = -\Big(\frac{n\pi}{L}\Big)^2 c_n[f]. \tag{43}$$

Changes of coordinates: Laplacian in curvilinear coordinates. We have seen that the separation of variables $\phi(x)g(y)$ allowed us to solve the Laplace equation on a domain which is a rectangle $\Omega = (0,L)\times(0,H)$. When the domain does not satisfy this decomposition the method of separation of Cartesian variables $\phi(x)g(y)$ will not work. In certain situations, other separation of variables will work. For instance if Ω is the disc of radius R, it may be decomposed as $\Omega = (0,R)\times(0,2\pi)$ in polar coordinates (r,θ). So a separation of variables of the form $\phi(\theta)G(r)$ will succeed.

The problem is that the Laplacian ∇^2 first needs to be expressed in the proper system of coordinates. Since $\nabla^2 = \nabla\cdot\nabla$, where $\nabla\cdot$ is the divergence operator and ∇ is the gradient operator, we first need to express these operators in curvilinear coordinates. This is what we now do.

We do this in two space dimensions. The generalization to 3D is easy. Let us assume that we want to change coordinates from Cartesian coordinates (x,y) to curvilinear coordinates (u,v). Assume that $x(u,v)$ and $y(u,v)$ is given. Let us denote a point $\mathbf{r}$ in the plane as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y.$$

Then

$$\frac{\partial\mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{e}_x + \frac{\partial y}{\partial u}\mathbf{e}_y \equiv |\frac{\partial\mathbf{r}}{\partial u}|\mathbf{e}_u \equiv h_u\mathbf{e}_u, \qquad \frac{\partial\mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{e}_x + \frac{\partial y}{\partial v}\mathbf{e}_y \equiv |\frac{\partial\mathbf{r}}{\partial v}|\mathbf{e}_v \equiv h_v\mathbf{e}_v.$$

The first equalities are just calculus. The $\equiv$ signs denote definitions. The above vectors have a norm, the scaling factors h_u and h_v by definition, and an orientation $\mathbf{e}_u$ and $\mathbf{e}_v$ also by definition. We *assume* that $(\mathbf{e}_u, \mathbf{e}_v)$ is an orthonormal basis, i.e., that $\mathbf{e}_u \cdot \mathbf{e}_v = 0$. This will be the cases in the examples we are interested in. The above expressions tell us how to change position in the new system of coordinates:

$$d\mathbf{r} = h_u du \mathbf{e}_u + h_v dv \mathbf{e}_v \qquad \Big(\text{ because } d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv\Big). \tag{44}$$

Now the definition of the gradient is as follows:

$$dg = \nabla g \cdot d\mathbf{r}. \tag{45}$$

The gradient, as any other vector, may be decomposed as

$$\nabla g = (\nabla g)_u \mathbf{e}_u + (\nabla g)_v \mathbf{e}_v, \qquad \text{or equivalently} \qquad \nabla g = \begin{pmatrix} (\nabla g)_u \\ \\ (\nabla g)_v \end{pmatrix}. \tag{46}$$

Since the chain rule gives us that

$$dg = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv = \nabla g \cdot d\mathbf{r} = (\nabla g)_u h_u du + (\nabla g)_v h_v dv,$$

we deduce that

$$\boxed{\nabla g = \begin{pmatrix} \dfrac{1}{h_u} \dfrac{\partial g}{\partial u} \\ \dfrac{1}{h_v} \dfrac{\partial g}{\partial v} \end{pmatrix}.} \tag{47}$$

Lecture 10. Now for the $(\nabla \cdot)$ operator, we use the divergence theorem

$$\int_{\Omega} \nabla \cdot \mathbf{p} d\mathbf{x} = \int_{\partial\Omega} \mathbf{p} \cdot \mathbf{n} d\Sigma. \tag{48}$$

Let us denote in curvilinear coordinates

$$\mathbf{p} = p_u \mathbf{e}_u + p_v \mathbf{e}_v.$$

Choosing the volume $\Omega = (u, u + du) \times (v, v + dv)$, we find that

$$\int_{\Omega} \nabla \cdot \mathbf{p} d\mathbf{x} \approx (\nabla \cdot \mathbf{p})(u, v) h_u h_v du dv.$$

Note that the area of the volume Ω is "distance in direction $\mathbf{e}_u$", equal to $h_u du$ thanks to (44), multiplied by "distance in direction $\mathbf{e}_v$", equal to $h_v dv$ thanks to (44). It remains to estimate the right hand side in (48), and we find that

$$\int_{\partial\Omega} \mathbf{p} \cdot \mathbf{n} d\Sigma \approx \frac{\partial}{\partial u}(p_u h_v) du dv + \frac{\partial}{\partial v}(p_v h_u) du dv.$$

This shows that

$$\boxed{(\nabla \cdot \mathbf{p})(u, v) = \frac{1}{h_u h_v}\Big(\frac{\partial}{\partial u}(p_u h_v) + \frac{\partial}{\partial v}(p_v h_u)\Big).} \tag{49}$$

Note that h_u and h_v depend on u and v in general. So the above expression does not simplify.

Remember that we were interested in the Laplacian in curvilinear coordinates. Using the above expressions for the divergence and gradient operators, it is given by

$$\boxed{\nabla^2 = \nabla \cdot \nabla = \frac{1}{h_u h_v}\Big(\frac{\partial}{\partial u}\Big(\frac{h_v}{h_u}\frac{\partial}{\partial u}\Big) + \frac{\partial}{\partial v}\Big(\frac{h_u}{h_v}\frac{\partial}{\partial v}\Big)\Big).} \tag{50}$$

Polar Coordinates. Let us apply the above to polar coordinates. They are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{51}$$

So

$$d\mathbf{r} = (\cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y)dr + (-r\sin\theta\mathbf{e}_x + r\cos\theta\mathbf{e}_y)d\theta = \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta.$$

Thus $h_r = 1$ and $h_\theta = r$, and $\mathbf{e}_r = (\cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y)$ and $\mathbf{e}_\theta = (-\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y)$. We indeed verify that $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$ so that we can apply the formulas obtained above. In polar coordinates, we thus obtain that

$$\nabla g = \begin{pmatrix} \dfrac{\partial g}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial g}{\partial \theta} \end{pmatrix}, \qquad (\nabla \cdot \mathbf{p})(r,\theta) = \frac{1}{r}\Big(\frac{\partial (rp_r)}{\partial r} + \frac{\partial p_\theta}{\partial \theta}\Big). \tag{52}$$

The Laplacian thus takes the form

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{53}$$

Let us now use the above formula and consider the problem

$$\begin{aligned} \nabla^2 u &= 0, \qquad \Omega, \\ u &= f, \qquad \partial\Omega \end{aligned} \tag{54}$$

where Ω is the disc of radius R. Note that Ω can be written as $\Omega = (0,R) \times (0,2\pi)$ in polar coordinates and that the boundary can be written as $\partial\Omega = \{R\} \times (0,2\pi)$.

Let us look for elementary solutions of the form $u(r,\theta) = \phi(r)g(\theta)$. Then, using (54) and (53), we find that

$$\frac{1}{r}(r\phi')'(r)g(\theta) + \frac{1}{r^2}\phi(r)g''(\theta) = 0.$$

This yields

$$\frac{r(r\phi')'}{\phi}(r) = -\frac{g''}{g}(\theta) = \lambda, \tag{55}$$

where λ is a constant. We now need additional boundary conditions for g and ϕ. They are obtained as follows. There is nothing special about the points $\theta = 0$ and $\theta = 2\pi$ in the definition of polar coordinates. Nothing special happens there physically. So physically, we really want $u(r,\theta)$ to be a 2π-*periodic* function in the variable θ. This implies that $g(\theta)$ is a 2π-periodic function as well. So the equation for $g(\theta)$ ought to be

$$\begin{aligned} &g''(\theta) + \lambda g(\theta) = 0 \\ &g(0) = g(2\pi), \quad g'(0) = g'(2\pi). \end{aligned} \tag{56}$$

Let us solve this problem. For $\lambda < 0$ the solutions of the above ODE are sinh and cosh functions that are not periodic. So $\lambda < 0$ leads to trivial solutions. When $\lambda = 0$, we obtain that $g(\theta) = a + b\theta$. Since $\theta \mapsto \theta$ is not periodic, b has to vanish and we find that

$$\lambda_0 = 0, \quad g_0(\theta) = 1, \tag{57}$$

is a non-trivial solution to (56). For $\lambda > 0$, the solutions are of the form $\cos\sqrt{\lambda}\theta$ and $\sin\sqrt{\lambda}\theta$. We verify that they are periodic (or verify the boundary conditions in (56)) if and only if $\lambda = n^2$ for $n \geq 1$. So we have the new solutions to (56):

$$\lambda_n = n^2, \quad g_{n,1}(\theta) = \cos n\theta, \quad g_{n,2}(\theta) = \sin n\theta; \qquad n \geq 1. \tag{58}$$

Note that the eigenvalue λ_n is associated with *two* (linearly independent) eigenvectors $\cos n\theta$ and $\sin n\theta$.

Let us now focus on $\phi(r)$, which becomes $\phi_n(r)$ with $\lambda = n^2$. It turns out that polar coordinates introduce a singularity at $r = 0$, since the point $(x,y) = (0,0)$ does not have a unique expression in polar coordinates. Physically however, there is no reason for the solution to (56) to behave differently at 0, and it should certainly be *bounded*. This translates into the following equation for ϕ_n:

$$
\begin{aligned}
&r(r\phi_n')' - n^2\phi_n = 0 \\
&|\phi_n(0)| \text{ is bounded.}
\end{aligned}
\tag{59}
$$

For $n \geq 1$, let us try the solution r^α. We verify that $\alpha = \pm n$ provides two linearly independent solutions to the first line in (59). Since r^{-n} is not bounded at $r = 0$, this is not an admissible solution for the above equation, so we deduce that

$$\phi_n(r) = r^n, \qquad n \geq 1.$$

It remains to solve the equation for $n = 0$ and find that $\ln r$ and 1 provide two linearly independent solutions. Since $\ln r$ is not bounded at 0 however, only $\phi_0(r) = 1$ is admissible.

To summarize, we have found that

$$\phi_0(r) = 1, \qquad \phi_n(r) = r^n, \quad n \geq 1. \tag{60}$$

In other words, this means that the elementary solutions of (54) are of the form

$$u_0(r,\theta) = 1, \qquad u_n(r,\theta) = (A_n \cos n\theta + B_n \sin n\theta)r^n, \quad n \geq 1. \tag{61}$$

By the principle of superposition, we deduce that

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta)r^n, \tag{62}$$

is a solution of the first line in (54). It remains to see how the coefficients A_n and B_n should be chosen so that $u(R,\theta) = f(\theta)$, i.e., so that the boundary conditions holds. This is done by using the Fourier theory once more.

By the method of superposition, we obtain the following general solution to (54):

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta). \tag{63}$$

We now choose the coefficients A_n and B_n such that the boundary condition is satisfied. The latter reads

$$g(\theta) = A_0 + \sum_{n=1}^{\infty} R^n(A_n \cos n\theta + B_n \sin n\theta). \tag{64}$$

Now $g(\theta)$ is a 2π-periodic function, and thus can be decomposed as a Fourier series. This yields

$$A_0 = a_0[g], \quad R^n A_n = a_n[g], \quad R^n B_n = b_n[g],$$

whence

$$A_0 = a_0[g], \quad A_n = \frac{a_n[g]}{R^n}, \quad B_n = \frac{b_n[g]}{R^n}. \tag{65}$$

We have thus solved (54) and found that

$$u(r,\theta) = a_0[g] + \sum_{n=1}^{\infty} \frac{r^n}{R^n}\Big(a_n[g] \cos n\theta + b_n[g] \sin n\theta\Big). \tag{66}$$

Lecture 11. Application: the Mean Value Theorem. Consider the Laplace equation $\nabla^2 u = 0$ on an arbitrary two dimensional open domain Ω. Take a point $\mathbf{x}$ inside Ω and use polar coordinates so that $\mathbf{x}$ becomes the origin. Now the above solution (66) evaluated at $u(\mathbf{x}) \equiv u(0)$ in polar coordinates gives that

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R,\theta)d\theta, \qquad \text{for all} \qquad R > 0 \quad \text{sufficiently small.} \tag{67}$$

Here, sufficiently small means that the disc of center $\mathbf{x}$ and radius R is inside the domain Ω. Indeed, apply (66) on the disc of radius R, with $g(\theta) = u(R,\theta)$ and observe that all the terms in (66) converge to 0 at $r = 0$ except for the first one. Changing back to Cartesian coordinates, and using the notation $\mathbf{x} = (x, y)$, the above formula is equivalent to

$$u(\mathbf{x}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x + R\cos\theta, y + R\sin\theta)d\theta, \qquad \text{for all} \qquad R > 0 \quad \text{sufficiently small.} \tag{68}$$

Please verify the above in detail. What this means is that *harmonic* functions, i.e., solutions of the Laplace equation, have the *mean value* property: they are equal at a point $\mathbf{x}$ to their average over any circle within the domain Ω centered at $\mathbf{x}$ and of radius R.

Application: Maximum Principle. The maximum principle theorem says that a harmonic function (solution of $\nabla^2 u = 0$) cannot attain its maximum (nor its minimum) inside the domain Ω. As a consequence it has to attain its maximum at the domain boundary. The proof uses the mean value theorem. Assume a strict maximum is attained at $\mathbf{x}$. Then $u(\mathbf{x}) > u(x + R\cos\theta, y + R\sin\theta)$ for all R small enough and all θ. But this contradicts (68) as can easily be verified.

Application: Uniqueness of solution to the Laplace equation. We have seen uniqueness results already using energy methods (integrations by parts). Here is another method based on the maximum principle. Assume that u and v are solutions of the following Dirichlet problem:

$$\nabla^2 u = 0 \text{ on } \Omega, \qquad u = g \text{ on } \partial\Omega.$$

Then $u - v$ solves the above equation as well, by linearity, with g replaced by 0. But then, $u - v$ attains its maximum at the domain boundary thanks to the maximum principle. Since $u - v = 0$ there, we deduce that $u \le v$. Now revert the roles of u and v and obtain that $v \le u$ for the same reason. This shows that $u = v$, whence that the solution is unique. Note that for Ω the disc of radius R, we now know existence and uniqueness of a solutions since we have constructed on in (66).

Solvability condition. We have now covered all the material in Haberman's chapters 1 to 3. Let me stress one last point. Consider u a solution of the following Neumann problem:

$$\nabla^2 u = 0 \text{ on } \Omega, \qquad \frac{\partial u}{\partial n} = f \text{ on } \partial\Omega.$$

Use of the divergence theorem, independently of any equation, shows that

$$\int_\Omega \nabla^2 u d\mathbf{x} = \int_\Omega \nabla\cdot\Big(\nabla u\Big)d\mathbf{x} = \int_{\partial\Omega} \mathbf{n}\cdot\nabla u d\Sigma = \int_{\partial\Omega}\frac{\partial u}{\partial n}d\Sigma.$$

For the above Neumann problem, this implies that

$$0 = \int_\Omega \nabla^2 u d\mathbf{x} = \int_{\partial\Omega}\frac{\partial u}{\partial n}d\Sigma = \int_{\partial\Omega} f d\Sigma.$$

The source term $f(\mathbf{x})$ thus needs to satisfy the compatibility condition

$$\int_{\partial\Omega} f d\Sigma = 0. \tag{69}$$

Otherwise the above Neumann problem cannot admit any solution. The reason is that we are looking for a steady state solution of a problem where heat energy is constantly sent into or extracted from the domain

Ω. There is no such solution. Note that once (69) is satisfied, the solution to the above Neumann problem is given up to the addition of an arbitrary constant. There is therefore no uniqueness of the Laplace equation with Neumann boundary conditions.

Integrations by parts and non-homogeneous equations. Let us come back to the one-dimensional heat equation with non-zero boundary conditions

$$\begin{aligned}
&\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in (0,L), \quad t > 0 \\
&u(0,t) = A(t), \qquad u(L,t) = B(t), \qquad t > 0 \\
&u(x,0) = 0, \qquad x \in (0,L).
\end{aligned} \tag{70}$$

The method of separation of variables does not work directly because of the non-homogeneous boundary conditions. We have also seen (in homeworks) that differentiation term-by-term does not apply directly either because $u(x,t)$ is not differentiable at $x = 0$ and $x = L$.

Here is the (best) way to solve this problem: use integration term by term, which is always allowed. Before doing so, we remark that $u(t,x)$ will be as usual a superposition of sine functions (thus an odd function) because of the Dirichlet boundary conditions. Consequently, $\frac{\partial u}{\partial x}$ will be a superposition of cosines (because it is an even function) as $\frac{\partial^2 u}{\partial x^2}$ will be again a superposition of sines (because it is odd).

Since $\frac{\partial^2 u}{\partial x^2}(x,t)$ is a superposition of sine functions, it takes the form

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n\geq 1} \gamma_n(t) \sin\frac{n\pi x}{L}, \qquad \gamma_n(t) = \frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\sin\frac{n\pi x}{L}dx.$$

We can now integrate $\gamma_n(t)$ by parts to replace $\frac{\partial^2 u}{\partial x^2}(t,x)$ by $u(t,x)$. This will allow us to obtain an ODE for $b_n[u]$ as usual. More precisely, we have:

$$\frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\sin\frac{n\pi x}{L}dx = -\frac{2}{L}\int_0^L \frac{\partial u}{\partial x}\frac{n\pi}{L}\cos\frac{n\pi x}{L}dx + \frac{2}{L}\frac{\partial u}{\partial x}\sin\frac{n\pi x}{L}\Big|_0^L.$$

Since the boundary terms vanish, we get that

$$\gamma_n(t) = -\frac{2}{L}\int_0^L \frac{\partial u}{\partial x}\frac{n\pi}{L}\cos\frac{n\pi x}{L}dx.$$

Another integration by parts shows that

$$\gamma_n(t) = \frac{2}{L}\int_0^L \Big[-\Big(\frac{n\pi}{L}\Big)^2\Big]u(t,x)\sin\frac{n\pi x}{L}dx - \frac{2}{L}\frac{n\pi}{L}u(t,x)\cos\frac{n\pi x}{L}\Big|_0^L.$$

Using the boundary conditions for u, this is

$$\gamma_n(t) = -\Big(\frac{n\pi}{L}\Big)^2 b_n[u](t) + \frac{2n\pi}{L^2}\big[A(t) - B(t)(-1)^n\big].$$

However, the equation stipulates that

$$b_n'[u](t) = k\gamma_n(t).$$

Therefore,

$$b_n'[u](t) + k\Big(\frac{n\pi}{L}\Big)^2 b_n[u](t) = \frac{2n\pi}{L^2}\big[B(t)(-1)^n - A(t)\big] := S_n(t).$$

It remains to integrate this ODE to get that

$$b_n[u](t) = \int_0^t e^{-k(\frac{n\pi}{L})^2(t-s)}S_n(s)ds, \tag{71}$$

where we have used that $u(0,x) = 0$.

Lecture 12. The Wave Equation. I shall not derive the wave equation in class. Please refer to Haberman's textbook. The wave equation in one dimension of space reads

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad x \in 0, L. \tag{72}$$

where c is speed of propagation (sound speed for instance). Since it is second-order in time, it needs two initial conditions, for instance $u(0,x)$ and $\frac{\partial u}{\partial t}(0,x)$. It still needs two boundary conditions since it is second-order in space. For instance $u(t,0)$ and $u(t,L)$ given.

Consider the problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, && t > 0, \quad x \in (0,L), \\ &u(t,0) = u(t,L) = 0, && t > 0 \\ &u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x), && x \in (0,L). \end{aligned} \tag{73}$$

Here, f and g are given initial conditions and our objective now is to solve the above problem by the method of separation of variables.

Assuming that $u(t,x) = \phi(x)h(t)$, we find as usual the following equations

$$h''(t) + c^2 \lambda h(t) = 0, \qquad \phi''(x) + \lambda \phi(x) = 0, \quad \phi(0) = \phi(L) = 0.$$

We have solved the problem for ϕ already: $\phi_n(x) = \sin \frac{n\pi x}{L}$, associated to the eigenvalue $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

Since we know that $\lambda_n > 0$, the equation for $h_n(t)$ gives us the solutions:

$$h_n(t) = A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}.$$

Note that the equation for h_n is a second-order ordinary differential equation, whence has two linearly independent solutions. By superposition, the most general solution to the wave equation may be written as

$$u(t,x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \Big(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \Big).$$

The first initial condition written in the Fourier domain provides

$$A_n = b_n[f].$$

Note that

$$\frac{\partial u}{\partial t}(t,x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \Big(- A_n \frac{n\pi c}{L} \sin \frac{n\pi ct}{L} + B_n \frac{n\pi c}{L} \cos \frac{n\pi ct}{L} \Big).$$

The second initial condition written in the Fourier domain thus yields

$$B_n = \frac{L}{n\pi c} b_n[g].$$

This concludes the derivation.

Note that we can introduce the following notation

$$\omega_n = \frac{n\pi c}{L} = 2\pi f_n, \qquad \theta_n = \arctan \frac{A_n}{B_n}.$$

Here ω_n is frequency (physicists would call f_n frequency) and θ_n is a phase. Using these variables, the solution to the wave equation may be written (in polar-type coordinates if you wish, with $A_n = \rho_n \sin \theta_n$ and $B_n = \rho_n \cos \theta_n$) as

$$u(t,x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sqrt{A_n^2 + B_n^2} \sin(\omega_n t + \theta_n).$$

Sturm Liouville Theory. The motivation for the theory is to look at equations of the form

$$\begin{aligned} &c\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\big(K_0(x)\frac{\partial u}{\partial x}\big) + \alpha(x)u(x) && t>0,\ 0<x<L, \\ &u(t,0) = u(t,L) = 0 && t>0, \\ &u(0,x) = f(x) && 0<x<L. \end{aligned} \tag{74}$$

The method of separation of variables $u(t,x) = G(t)\phi(x)$ provides the usual equation

$$G'(t) + \lambda G(t) = 0,$$

and the less-usual equation

$$\frac{d}{dx}\Big(K_0\frac{d\phi}{dx}(x)\Big) + \alpha\phi(x) + \lambda c\rho(x)\phi(x) = 0, \qquad \phi(0) = \phi(L) = 0.$$

The Sturm Liouville theory is the following. Consider the problem

$$\begin{aligned} &\frac{d}{dx}\Big(p(x)\frac{d\phi}{dx}\Big) + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a<x<b \\ &\beta_1\phi(a) + \beta_2\phi'(a) = 0 \\ &\beta_3\phi(b) + \beta_4\phi'(b) = 0, \end{aligned} \tag{75}$$

with the conditions that p, q, σ are real functions with $p(x) > 0$ and $\sigma(x) > 0$ for all $x \in [a,b]$. Then we have the following properties:

1. All the eigenvalues $\lambda \in \mathbb{R}$.
2. $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$.
3. λ_n is simple. $\phi_n(x)$ has exactly $n-1$ zeros on $[a,b]$.
4. The ϕ_n form a complete set (a basis). In practice, this means that any function $f(x)$ (say piecewise continuous and equal to the half sum of its limiting points where it jumps), satisfies

$$f(x) = \sum_{n=1}^{\infty} a_n[f]\phi_n(x), \tag{76}$$

   for some set of coefficients A_n.
5. Eigenfunctions are orthogonal in the following sense

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0, \qquad \lambda_m \neq \lambda_n \tag{77}$$

6. We have

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b \Big(p\Big(\frac{\partial\phi}{\partial x}\Big)^2 - q\phi^2\Big)dx}{\int_a^b \phi^2\sigma dx}. \tag{78}$$

We'll come back to the derivation of some of these properties, but let us conclude here by the main application of the orthogonality condition stated above, namely the calculation of the coefficients $a_n[f]$. Multiplying both sides in (76) by $\phi_m(x)\sigma(x)dx$ and integrating over (a,b) yields

$$\int_a^b f(x)\phi_m(x)\sigma(x)dx = \sum_{n=1}^{\infty} a_n[f]\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx.$$

However using property (5) above, we conclude that

$$a_n[f] = \frac{\displaystyle\int_a^b f(x)\phi_m(x)\sigma(x)dx}{\displaystyle\int_a^b \phi_m^2(x)\sigma(x)dx}. \tag{79}$$

Similarly to the theory of Fourier series, we can expand any (sufficiently regular) function $f(x)$ over the basis of the $\phi_n(x)$, and thanks to the above orthogonality property, we have the rule (79) to calculate the coefficients of the expansion. Note that the above theory also states that $f(x) = g(x)$ for all $a < x < b$ is equivalent to the fact that $a_n[f] = a_n[g]$ for all $n \geq 1$. So a constraint in the physical domain can be replaced by a constraint in the domain of coefficient, which will be also referred to as the domain of Fourier coefficients.

Lecture 13. Sturm Liouville Theory; some details on the derivation. Let us define the operator

$$L(\phi) = \frac{d}{dx}\Big[p(x)\frac{d\phi}{dx}\Big] + q(x)\phi(x), \tag{80}$$

so that the Sturm Liouville problem takes the form $L\phi = -\lambda\sigma(x)\phi$ on (a, b) with proper boundary conditions at a and b. It has the form of a *generalized eigenvalue problem.* It is a "classical" eigenvalue problem when $\sigma(x) \equiv 1$ and is "generalized" otherwise.

We first obtain the **Lagrange identity**:

$$L(u)v - L(v)u = \frac{d}{dx}\Big[p(x)\Big(v\frac{du}{dx} - u\frac{dv}{dx}\Big)\Big]. \tag{81}$$

This is easily verified. The advantage of the formula is that it is in *divergence* form, i.e., the right hand side is the divergence (derivative in 1D) of something. It is then advantageous to integrate such a relationship, and this is called **Green's formula**:

$$\begin{aligned}\int_a^b \Big(L(u)v - L(v)u\Big)dx &= -p(x)(uv' - vu')(x)\Big|_a^b \\ &= -p(b)\big(u(b)v'(b) - u'(b)v(b)\big) + p(a)\big(u(a)v'(a) - u'(a)v(a)\big).\end{aligned} \tag{82}$$

Let us assume moreover that u and v, which are arbitrary functions in the above expression, satisfy the boundary conditions in (75). Let us also assume that $\beta_2 \neq 0$ and $\beta_4 \neq 0$. Then the Green's formula takes the form

$$\begin{aligned}\int_a^b \Big(L(u)v - L(v)u\Big)dx &= -p(b)\Big(u(b)\frac{-\beta_3}{\beta_4}v(b) - \frac{-\beta_3}{\beta_4}u(b)v(b)\Big) + p(a)\Big(u(a)\frac{-\beta_1}{\beta_2}v(a) - \frac{-\beta_1}{\beta_2}u(a)v(a)\Big) \\ &= 0.\end{aligned}$$

We can verify that the above still holds when $\beta_2 = 0$ or $\beta_4 = 0$ since then u and v satisfy Dirichlet conditions at such boundary points. So for any arbitrary functions u and v satisfying the boundary conditions in (75), we find that

$$\int_a^b \Big(L(u)v - L(v)u\Big)dx = 0. \tag{83}$$

Application to orthogonality. Let us assume that (λ_m, ϕ_m) and (λ_n, ϕ_n) are solutions of the Sturm-Liouville problem (75). Then from $L(\phi_m) + \lambda_m\sigma(x)\phi_m = 0$ and $L(\phi_n) + \lambda_n\sigma(x)\phi_n = 0$, we deduce that

$$\int_a^b \Big(L(\phi_m)\phi_n - L(\phi_n)\phi_m\Big) = (\lambda_n - \lambda_m)\int_a^b \sigma(x)\phi_n(x)\phi_m(x).$$

However, with $u = \phi_m$ and $b = \phi_n$ in (83), which holds since both ϕ_n and ϕ_m satisfy the boundary conditions in (75), we deduce that

$$(\lambda_n - \lambda_m) \int_a^b \sigma(x)\phi_n(x)\phi_m(x) = 0.$$

This implies that

$$\lambda_n \neq \lambda_m \quad \Rightarrow \quad \int_a^b \sigma(x)\phi_n(x)\phi_m(x) = 0. \tag{84}$$

This is the *orthogonality property* (5), with the proper inner product (integral) involving the weight $\sigma(x)$.

Application to realness of λ_n. Let us now choose $u = \phi_n$ and $v = \phi_n^*$, its complex conjugate. Then the above Green formula (83) shows that

$$0 = (\lambda_n - \lambda_n^*) \int_a^b \sigma(x)|\phi_n|^2(x)dx.$$

Since ϕ_n is not uniformly 0, the latter integral is positive. This implies that $\lambda_n - \lambda_n^* = 0$, whence λ_n is real.

Application to the fact that λ_n is simple. Let us assume that (λ, ϕ_1) and (λ, ϕ_2) are solutions of (75) and let us apply (191) on the interval (a, y). Since $p(x) > 0$, we deduce that

$$\phi_1(y)\phi_2'(y) - \phi_2(y)\phi_1'(y) = 0 = \phi_1^2(y)\frac{d}{dx}\Big(\frac{\phi_2}{\phi_1}\Big)(y).$$

This is because the above equality holds at $y = a$ thanks to the boundary conditions. If we assume that the above equality implies that $\frac{\phi_2}{\phi_1}(y) = 0$ (which is not completely correct, but this gives the flavor of the derivation), then we deduce that $\phi_2 = c\phi_1$ for some constant c. This implies that the eigenvector associated to λ is unique up to multiplication by a constant, whence that λ is simple.

Self-Adjoint operator. An operator L that satisfies (83) is called a *self-adjoint* operator. Note that the boundary conditions are important to derive (83). So what is self adjoint is really the operator L as in (80) defined on functions that satisfy the boundary conditions in (75). (Defined on more general functions that do not satisfy these boundary conditions, the operator L may not be self-adjoint.)

Why is this notion important? Self-adjoint operators to a large extent extend the notion of symmetric matrices to the infinite dimensional case. And they share many of their properties, for instance that they can be diagonalized. The Sturm Liouville theory is nothing but a diagonalization (over an orthogonal basis) of the Sturm-Liouville operator L.

Rayleigh quotient. Let us come back to the problem (75). We multiply the PDE by $\phi(x)$ and integrate by parts. This yields

$$-\int_a^b p(\phi')^2 + p\phi'\phi\Big|_a^b + \int_a^b q\phi^2 + \int_a^b \lambda\sigma\phi^2 = 0.$$

This is clearly equivalent to (78). Now assuming that $\beta_2 \neq 0$ and $\beta_4 \neq 0$, we obtain that

$$-\phi'\phi\Big|_a^b = \frac{\beta_3}{\beta_4}\phi^2(b) - \frac{\beta_1}{\beta_2}\phi^2(a).$$

So if

$$\frac{\beta_3}{\beta_4} \geq 0, \qquad \text{and} \qquad \frac{\beta_1}{\beta_2} \leq 0, \tag{85}$$

then we find that $-\phi'\phi\Big|_a^b \geq 0$ and $-p\phi'\phi\Big|_a^b \geq 0$ since $p > 0$. The same result holds obviously if $\beta_2 = 0$ or $\beta_4 = 0$. Note that (85) is equivalent to saying that

$$\beta_3 \text{ and } \beta_4 \text{ have the same sign} \qquad \textit{and} \qquad \beta_1 \text{ and } \beta_2 \text{ have different signs.} \tag{86}$$

The latter expression includes the cases where $\beta_1 = 0$ or $\beta_3 = 0$. In such cases, we deduce from (78) that

$$\lambda \geq 0. \tag{87}$$

This will be the main application of the formula (78) in this course.

Note that the Rayleigh quotient is also used as the minimization principle

$$\lambda_1 = \min_{\phi} \frac{-p\phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \Big(p\Big(\frac{\partial \phi}{\partial x}\Big)^2 - q\phi^2\Big)dx}{\int_a^b \phi^2 \sigma dx}. \tag{88}$$

Here λ_1 is the smallest eigenvalue of (75). This is a very important property in many practical applications; see Haberman's textbook for more on this.

Application to the non-uniform wave equation. Consider the equation

$$\begin{aligned}
&\frac{\partial^2 p}{\partial t^2} = c^2(x)\frac{\partial^2 p}{\partial x^2}, && t>0, \quad 0<x<L,\\
&p(t,0) = p(t,L) = 0, && t>0,\\
&p(0,x) = f(x), \quad \frac{\partial p}{\partial t}(0,x) = g(x), && 0<x<L.
\end{aligned} \tag{89}$$

The method of separation of variables $p(t,x) = h(t)\phi(x)$ provides the equations $h'' + \lambda h = 0$ and

$$\phi''(x) + \lambda \frac{1}{c^2(x)}\phi(x) = 0, \quad \phi(0) = \phi(L) = 0. \tag{90}$$

This is a Sturm Liouville problem with $p = 1 > 0$, $q = 0$, $\sigma(x) = c^{-2}(x) > 0$. Moreover the boundary conditions are of the right form so that the Sturm Liouville theory applies. We thus know the existence of (λ_n, ϕ_n) for $n \geq 1$ solutions of the above eigenvalue problem. Moreover the Rayleigh quotient tells us that $\lambda_n \geq 0$ as in (87) since

$$\lambda_n = \frac{\int_0^L (\phi_n')^2(x)dx}{\int_0^L \phi_n^2(x)\frac{1}{c^2(x)}dx} \geq 0. \tag{91}$$

We can actually say more: $\lambda_n > 0$. Indeed let us assume that $\lambda_n = 0$. Then the Rayleigh quotient formula implies that $(\phi_n)' = 0$ since $\int_a^b u^2(x)dx = 0$ implies that $u = 0$ on (a,b). This in turn implies that $\phi_n = Cte$. The boundary condition finally yields $\phi(0) = Cte = 0$ so that $\phi \equiv 0$ is a trivial solution. As a conclusion $\lambda_n = 0$ is impossible. Since it is non-negative, it has to be positive.

Let us come back to the temporal behavior of the modes:

$$h_n''(t) + \lambda_n h_n = 0.$$

Since $\lambda_n > 0$, we deduce that the most general solution to the above ODE is

$$h_n(t) = A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t.$$

By superposition, we thus have

$$p(t,x) = \sum_{n=1}^{\infty}\Big(A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t\Big)\phi_n(x). \tag{92}$$

Evaluating the above at $t = 0$ yields

$$f(x) = \sum_{n=1}^{\infty} A_n \phi_n(x).$$

From the completeness of the set of functions $\phi_n(x)$, we deduce that such an expansion is possible, and from the orthogonality of the eigenvectors, we obtain that

$$A_n = a_n[g] = \frac{\displaystyle\int_0^L f(x)\phi_n(x)\frac{1}{c^2(x)dx}}{\displaystyle\int_0^L \phi_n^2(x)\frac{1}{c^2(x)dx}}. \tag{93}$$

It remains to consider the second initial condition. A differentiation of (92) in time yields

$$\frac{\partial}{\partial t}p(t,x) = \sum_{n=1}^{\infty}\Big(-A_n\sqrt{\lambda_n}\sin\sqrt{\lambda_n}t + B_n\sqrt{\lambda_n}\cos\sqrt{\lambda_n}t\Big)\phi_n(x).$$

Evaluated at $t = 0$, this is

$$B_n\sqrt{\lambda_n} = a_n[g] = \frac{\displaystyle\int_0^L g(x)\phi_n(x)\frac{1}{c^2(x)dx}}{\displaystyle\int_0^L \phi_n^2(x)\frac{1}{c^2(x)dx}}.$$

Using the method of separation of variables, the principle of superposition, and the Sturm Liouville theory, we have thus obtained that

$$p(t,x) = \sum_{n=1}^{\infty}\Big(a_n[f]\cos\sqrt{\lambda_n}t + \frac{a_n[g]}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}t\Big)\phi_n(x). \tag{94}$$

Lecture 14. A little bit of approximation theory. Let us come back to the Sturm Liouville theory. We know that arbitrary (square integrable) functions $f(x)$ admit the decomposition

$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x). \tag{95}$$

We now want to address the question of the accuracy of the following finite summation:

$$f_N(x) = \sum_{n=1}^{N} a_n\phi_n(x). \tag{96}$$

Let us define

$$\varepsilon_N(x) = f(x) - f_N(x) = \sum_{n=N+1}^{\infty} a_n\phi_n(x). \tag{97}$$

We will estimate ε_N in the following *norm*

$$\|\varepsilon_N\| = \Big(\int_a^b \varepsilon_N^2(x)\sigma(x)dx\Big)^{1/2}. \tag{98}$$

Note that $\|\varepsilon_N\| = 0$ implies that $\varepsilon_N = 0$. The orthogonality properties of the ϕ_m's allows us to deduce that

$$\int_a^b \varepsilon_N^2(x)\sigma(x)dx = \sum_{n=N+1}^{\infty} a_n^2\int_a^b \phi_n^2(x)\sigma(x)dx.$$

So if we normalize the eigenvectors ϕ_n such that

$$\int_a^b \phi_n^2(x)\sigma(x)dx = 1,$$

then we find that

$$\|\varepsilon_N\| = \Big(\sum_{n=N+1}^{\infty} a_n^2 \Big)^{1/2}. \qquad (99)$$

So the error can be obtained by summing the coefficients a_n that are not accounted for in $f_N(x)$. Note that the coefficients a_n decay faster when the solution $f(x)$ is smoother. So, for smooth functions, the error term (99) can be very small very quickly as $N \to \infty$. For $f(x)$ less smooth, more terms need to be incorporated into (96) to obtain the same accuracy. Note that for $N = 0$, we deduce that

$$\int_a^b f^2(x)\sigma(x)dx = \sum_{n=1}^{\infty} a_n^2. \qquad (100)$$

This is the celebrated *Parseval* relation. The left hand side may be interpreted as an energy. The Parseval relation shows that the energy can be estimated in the physical domain (left hand side) as well as in the Fourier domain (right hand side).

PDEs on arbitrary multi-dimensional bounded domains. Let us consider the heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nabla^2 u = 0, && t > 0, \mathbf{x} \in \Omega \\ &a(\mathbf{x})u(t,\mathbf{x}) + b(\mathbf{x})\frac{\partial u}{\partial n}(t,\mathbf{x}) = 0 && t > 0, \mathbf{x} \in \partial\Omega \\ &u(0,\mathbf{x}) = f(\mathbf{x}) && x \in \Omega. \end{aligned} \qquad (101)$$

Here $a(\mathbf{x})$ and $b(\mathbf{x})$ are both non-negative and do not vanish at the same time.

If we try separation of variables on such an equation, $u(t,\mathbf{x}) = G(t)\phi(\mathbf{x})$, we obtain that $G' + \lambda G = 0$ as usual and that

$$\begin{aligned} &\nabla^2\phi + \lambda\phi = 0, && \mathbf{x} \in \Omega \\ &a\phi + b\frac{\partial \phi}{\partial n} = 0, && \mathbf{x} \in \partial\Omega. \end{aligned} \qquad (102)$$

This equation is sometimes referred to as the Helmholtz equation. It is an eigenvalue problem and the question is: what are its solutions? The answer is very similar to what we obtained in the Sturm Liouville theory. Here is an abstract result.

1. All the eigenvalues $\lambda \in \mathbb{R}$.

2. $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$.

3. λ_n is not necessarily simple for $n \geq 2$ but the number of linearly independent eigenvectors associated to λ_n is finite.

4. The ϕ_n form a complete set (a basis). In practice, this means that any function $f(\mathbf{x})$ (say continuous to simplify), satisfies

   $$f(\mathbf{x}) = \sum_{n=1}^{\infty} a_n[f]\phi_n(\mathbf{x}), \qquad (103)$$

   for some set of coefficients A_n.

5. Eigenfunctions are orthogonal in the following sense

   $$\int_\Omega \phi_n(\mathbf{x})\phi_m(\mathbf{x})d\mathbf{x} = 0, \qquad \lambda_m \neq \lambda_n \qquad (104)$$

   The linearly independent eigenvectors associated to a same eigenvalue can also be chosen orthogonal to each other by the Gram-Schmidt orthogonalization procedure. In the sequel we assume the eigenvectors $\phi_m(\mathbf{x})$ form an orthogonal basis of functions.

6. We have the Rayleigh quotient:

$$\lambda = \frac{-\displaystyle\int_{\partial\Omega} \phi \frac{\partial \phi}{\partial n} d\sigma(\mathbf{x}) + \int_\Omega |\nabla \phi|^2 d\mathbf{x}}{\displaystyle\int_\Omega \phi^2(\mathbf{x}) d\mathbf{x}}. \tag{105}$$

By orthogonality of the ϕ_n's, the coefficients $a_n[f]$ are given by

$$a_n[f] = \frac{\displaystyle\int_\Omega \phi_n(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}}{\displaystyle\int_\Omega \phi^2(\mathbf{x}) d\mathbf{x}}. \tag{106}$$

This is verified as in the Sturm Liouville theory.

Rayleigh quotient. The Rayleigh quotient formula is obtained as follows. We multiply the equation $\nabla^2\phi + \lambda\phi = 0$ by ϕ and integrate over Ω. Using that

$$\nabla \cdot (\phi \nabla \phi) = \phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi,$$

we obtain that

$$\lambda \int_\Omega \phi^2 d\mathbf{x} = \int_\Omega |\nabla \phi|^2 d\mathbf{x} + \int_\Omega \nabla \cdot (\phi \nabla \phi) d\mathbf{x}.$$

It remains to use the divergence theorem:

$$\int_\Omega \nabla \cdot (\phi \nabla \phi) d\mathbf{x} = \int_{\partial\Omega} \mathbf{n} \cdot \phi \nabla \phi d\sigma = \int_{\partial\Omega} \phi \frac{\partial \phi}{\partial n} d\sigma,$$

to obtain (105). Note that when either $a = 0$ or $b = 0$, we have

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla \phi|^2 d\mathbf{x}}{\displaystyle\int_\Omega \phi^2(\mathbf{x}) d\mathbf{x}} \geq 0.$$

Moreover $\lambda = 0$ implies that $\nabla\phi = 0$ on Ω. This implies that $\phi = Cte$ is a constant. When $a = 0$, then $\phi = 1$ is indeed a non-trivial eigenvector associated to $\lambda = 0$. However, when $b = 0$ so that $\phi = 0$ on $\partial\Omega$, we deduce that $\phi = Cte = 0$ is a trivial solution so that $\lambda = 0$ cannot be an eigenvalue.

More generally assuming that $b \neq 0$, we deduce that

$$\lambda = \frac{\displaystyle\int_{\partial\Omega} \frac{a}{b} \phi^2 d\sigma(\mathbf{x}) + \int_\Omega |\nabla \phi|^2 d\mathbf{x}}{\displaystyle\int_\Omega \phi^2(\mathbf{x}) d\mathbf{x}} \geq 0,$$

since we have assumed that a and b had the same sign. Here however, $\lambda = 0$ implies that $\nabla\phi = 0$ and $\phi = 0$ on $\partial\Omega$ when $a > 0$. Thus $\phi = 0$ is a trivial solution and $\lambda = 0$ cannot be a solution when $a > 0$.

Lagrange identity and Green's formula. The Lagrange identity states that

$$v \nabla^2 u - u \nabla^2 v = \nabla \cdot (v \nabla u - u \nabla v). \tag{107}$$

It is based on the calculation

$$\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v,$$

which you are encouraged to verify carefully using the representation of ∇ and $\nabla\cdot$ in Cartesian coordinates.

Upon integrating the Lagrange identity on Ω and using the divergence theorem, we obtain **Green's formula**:

$$\int_\Omega \Big(v\nabla^2 u - u\nabla^2 v\Big) d\mathbf{x} = \int_{\partial\Omega} \Big(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\Big) d\sigma(\mathbf{x}). \tag{108}$$

When u and v satisfy the boundary conditions in (102), we verify that

$$\int_{\partial\Omega} \Big(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\Big) d\sigma(\mathbf{x}) = 0.$$

As before, separate the cases $b = 0$ and $b \neq 0$. Thus for functions u and v satisfying the boundary conditions in (102), we deduce that

$$\int_\Omega \Big(v\nabla^2 u - u\nabla^2 v\Big) d\mathbf{x} = 0. \tag{109}$$

Application to orthogonality. Assume that (λ_m, ϕ_m) and (λ_n, ϕ_n) are solutions to (102). Then with $u = \phi_m$ and $v = \phi_n$ in (109) we deduce that

$$0 = \int_\Omega \Big(\phi_n\nabla^2\phi_m - \phi_m\nabla^2\phi_n\Big) d\mathbf{x} = (\lambda_n - \lambda_m)\int_\Omega \phi_m\phi_n d\mathbf{x}.$$

When $\lambda_m \neq \lambda_n$ we thus deduce that ϕ_m and ϕ_n are orthogonal to each-other. When $\lambda_n = \lambda_m$ for $n \neq m$ (which can happen here unlike in the Sturm Liouville theory), the eigenvectors ϕ_n and ϕ_m can be *chosen* to be orthogonal by the Gram-Schmidt procedure of orthogonalization.

Lecture 15. Let us briefly come back to the multidimensional heat equation (101). By separation of variables, we find that the elementary solutions $u_n(t, \mathbf{x})$ are given by

$$u_n(t, \mathbf{x}) = e^{-\lambda_n t}\phi_n(\mathbf{x}), \qquad n \geq 1. \tag{110}$$

It remains to verify the initial condition $u(0, \mathbf{x}) = f(\mathbf{x})$. This is done by superposition as usual and the constraint we need to satisfy is

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} a_n\phi_n(\mathbf{x}).$$

The above abstract theory says that $f(\mathbf{x})$ can indeed be decomposed as was just written since the family $\phi_n(\mathbf{x})$ for $n \geq 1$ is complete. Moreover by orthogonality, we deduce that

$$a_n = a_n[f] = \frac{\displaystyle\int_\Omega \phi_n(\mathbf{x})f(\mathbf{x})d\mathbf{x}}{\displaystyle\int_\Omega \phi^2(\mathbf{x})d\mathbf{x}},$$

as we have seen in (106). The solution to (101) is thus given by

$$u(t, \mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{\displaystyle\int_\Omega \phi_n(\mathbf{y})f(\mathbf{y})d\mathbf{y}}{\displaystyle\int_\Omega \phi^2(\mathbf{y})d\mathbf{y}}\right) e^{-\lambda_n t}\phi_n(\mathbf{x}). \tag{111}$$

Note that the dummy variable $\mathbf{y}$ in the above integrals is not $\mathbf{x}$, which means position on the left hand side.

Heat equation in Cartesian geometry. Let us now consider the equation (101) for $\Omega = (0, L)\times(0, H)$ and with Dirichlet boundary conditions. Then the geometry is sufficiently simple so that more can be said about the modes $\phi_n(\mathbf{x})$; namely, they can be written explicitly as products of sine functions. In Cartesian geometry, the Laplace operator ∇^2 takes a familiar form and we recast (101) as

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) = 0, && t > 0, (x,y) \in (0,L) \times (0,H) \\
&u(t,0,y) = u(t,L,y) = 0 && t > 0, 0 < y < H \\
&u(t,x,0) = u(t,x,H) = 0 && t > 0, 0 < x < L \\
&u(0,x,y) = f(x,y) && (x,y) \in (0,L) \times (0,H).
\end{aligned} \tag{112}$$

Writing $u(t,x,y) = G(t)\phi(x,y)$, we still obtain that $G' + \lambda G = 0$ as before and that ϕ solves the equation

$$\begin{aligned}
&\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \lambda\phi = 0, && (x,y) \in (0,L) \times (0,H) \\
&\phi(0,y) = \phi(L,y) = 0 && 0 < y < H \\
&\phi(x,0) = \phi(x,H) = 0 && 0 < x < L.
\end{aligned} \tag{113}$$

Not surprisingly, this partial differential equation (eigenvalue problem) can be solved by the method of separation of variables $\phi(x,y) = f(x)g(y)$. We find that

$$f''g + fg'' + \lambda fg = 0.$$

This may be recast as

$$\frac{f''}{f}(x) = -\lambda - \frac{g''}{g}(y) = -\mu.$$

Here a *new* constant μ is rendered necessary to separate the variables x and y. The first constant λ was here to separate the time variable t from the spatial variables (x,y).

We thus have the two equations

$$\begin{aligned}
&f'' + \lambda f = 0, && g'' + (\lambda - \mu)g = 0, \\
&f(0) = f(L) = 0, && g(0) = g(H) = 0.
\end{aligned}$$

The equation for f is solved as usual and has for solutions

$$\mu_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad f_n(x) = \sin\frac{n\pi x}{L}, \qquad n \geq 1.$$

The equation for g now becomes

$$\begin{aligned}
&g_n'' + (\lambda - \mu_n)g_n = 0, \\
&g_n(0) = g_n(H) = 0.
\end{aligned}$$

This is because μ is no longer arbitrary. Note here that the above equation also admits an infinite number of solutions, which we will need to label, and thus denote by $g_{mn}(y)$ and λ_{mn}. They are given by

$$(\lambda_{mn} - \mu_n) = \Big(\frac{m\pi}{H}\Big)^2, \qquad g_{mn}(y) = \sin\frac{m\pi y}{H}$$

To summarize, we have thus obtained that the solutions to (113) were given by

$$\phi_{mn}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \qquad \lambda_{mn} = \Big(\frac{m\pi}{H}\Big)^2 + \Big(\frac{n\pi}{L}\Big)^2, \qquad m,n \geq 1. \tag{114}$$

By superposition, the solution to (112) is thus of the form

$$u(t,x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-\lambda_{mn}t} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}. \tag{115}$$

It remains to find the coefficients A_{mn} using the initial conditions:

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}. \tag{116}$$

Here however, we use the theory of Fourier series in each variable to deduce that the above expansion is indeed possible since the sine functions form a complete family, and moreover that

$$A_{mn} = b_{mn}[f] = \frac{2}{L}\frac{2}{H}\int_0^L \int_0^H f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} dxdy. \tag{117}$$

Note that the solutions $(\lambda_{mn}, \phi_{mn}(x,y))$ for $m,n \geq 1$ should be the same as the $(\lambda_p, \phi_p(\mathbf{x}))$ for $p \geq 1$ obtained in the abstract theory. The abstract theory clearly applies to Cartesian geometries. The reason why the single index p in the abstract theory is replaced by a double index (m,n) in Cartesian geometry is only a matter of convenience. It is convenient to label the solutions $\phi_{mn}(x,y)$ with two indices, one for each Cartesian variable. However, they could be *relabeled* as $\phi_p(\mathbf{x})$, and the same thing for λ_p. Even though that may look counter-intuitive at first, there is the "same" number of elements in the set $\{m,n \geq 1\}$ as in the set $\{p \geq 1\}$. This can be verified by introducing

$$p(m,n) = \frac{(m+n-1)(m+n-2)}{2} + n, \tag{118}$$

which, as one can verify, maps $\{m,n \geq 1\}$ onto $\{p \geq 1\}$ and is one-to-one. This means that for each $p \geq 1$, there is a unique $m \geq 1$ and $n \geq 1$ such that (118) holds.

The sets $\{m,n \geq 1\}$ and $\{p \geq 1\}$ are then the same (*isomorphic* is the mathematical term). With the above map, we can *identify* $\phi_{mn}(x,y)$ with $\phi_p(\mathbf{x})$ with $\mathbf{x} = (x,y)$. Note that (116) and (103) can also be identified, as well as (117) and (106). The former are always the application of the latter to the specific choice of a Cartesian geometry.

Lecture 16. Let us now consider the problem (101) with Dirichlet conditions in the case where Ω is a disc of radius R, and thus may be written in polar coordinates as $\Omega = (0,R) \times (0, 2\pi)$. Recalling that

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

the heat equation (101) with Dirichlet conditions may be recast as

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{1}{r}\frac{\partial u}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, && t>0, 0<r<R, 0<\theta<2\pi,\\
&u(t,R,\theta) = 0 && t>0, 0<\theta<2\pi,\\
&|u(t,0,\theta)| < \infty && t>0, 0<\theta<2\pi,\\
&u(0,r,\theta) = f(r,\theta), && 0<r<R, 0<\theta<2\pi.
\end{aligned} \tag{119}$$

Writing $u(t,x,y) = G(t)\phi(x,y)$, we still obtain that $G' + \lambda G = 0$ as before and that ϕ solves the equation

$$\begin{aligned}
&\frac{1}{r}\frac{\partial \phi}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \lambda\phi = 0, && 0<r<R, 0<\theta<2\pi,\\
&\phi(R,\theta) = 0 && 0<\theta<2\pi,\\
&|\phi(0,\theta)| < \infty && 0<\theta<2\pi.
\end{aligned} \tag{120}$$

This is an eigenvalue problem that we solve by the method of separation of variables $\phi(r,\theta) = f(r)g(\theta)$. This yields first

$$\frac{1}{r}(rf')'g + \frac{1}{r^2}fg'' + \lambda fg = 0,$$

so that after multiplication by $r^2/(fg)$,

$$\frac{r(rf')'}{f} + \frac{r^2\lambda}{f} = -\frac{g''}{g} = \mu.$$

Along with the boundary conditions we thus have to solve the following ordinary differential equations

$$\begin{array}{ll} g'' + \mu g = 0, & r(rf')' + (r^2\lambda - \mu)f = 0 \\ g(\theta)\, 2\pi - \text{periodic} & f(R) = 0, \quad |f(0)| < \infty. \end{array} \tag{121}$$

We have solved the equation for g already. The solutions are

$$\mu_n = n^2, \qquad g_n(\theta) = A\cos n\theta + B\sin n\theta, \qquad n \geq 0. \tag{122}$$

The equation for (λ, f) thus becomes an equation for (λ_n, f_n) for $n \geq 0$ of the form

$$\begin{array}{l} r(rf_n')' + (r^2\lambda_n - n^2)f_n = 0 \\ f_n(R) = 0, \quad |f_n(0)| < \infty. \end{array} \tag{123}$$

This is a second-order ordinary differential equation, for which no explicit solutions can be obtained. Moreover, these solutions have different behaviors depending on the sign of λ_n. However, we know from the abstract theory (using the Rayleigh quotient) that the problem (102) with $a = 1$ and $b = 0$ admits positive eigenvalues $\lambda_n > 0$. We are thus allowed to introduce the following change of variables

$$r \to z = \sqrt{\lambda_n} r \quad \text{from} \quad (0, R) \text{ to } (0, \sqrt{\lambda_n} R).$$

Define $\tilde{f}_n(z) = f_n(r) = \tilde{f}_n(\sqrt{\lambda_n} r)$. Then

$$\frac{d\tilde{f}_n}{dz}(z) = \tilde{f}_n'(z) = \frac{dz}{dr} f_n'(r) = \sqrt{\lambda_n} f_n'(r),$$

so that

$$z\tilde{f}_n'(z) = rf_n'(r).$$

One more time shows that

$$z(z\tilde{f}_n')'(z) = r(rf_n')'(r).$$

The equation (123) may thus be recast equivalently as

$$\begin{array}{l} z(z\tilde{f}_n')' + (z^2 - n^2)\tilde{f}_n = 0 \\ \tilde{f}_n(\sqrt{\lambda_n} R) = 0, \quad |\tilde{f}_n(0)| < \infty. \end{array} \tag{124}$$

What have we gained with this trick? The eigenvalue λ_n no longer appears in the ODE in (124). It only appears in the boundary conditions. Now let us consider the first line in (124). This is a second-order ordinary differential equation with non-constant coefficients. It turns out that it cannot be written in terms of "explicit" solutions such as sines, cosines, logs, exps, and the likes. Yet these two linearly independent solutions exist and they have names:
$J_n(z)$ is the *Bessel function of the first kind of order* n; and
$Y_n(z)$ is the *Bessel function of the second kind of order* n.
Moreover these two solutions have the following asymptotic behavior at $z = 0$:

$$J_n(z) \sim \begin{cases} 1 & n = 0 \\ \dfrac{1}{2^n n!} z^n & n > 0 \end{cases}, \qquad Y_n(z) \sim \begin{cases} \dfrac{2}{\pi} \ln z & n = 0 \\ -\dfrac{2^n(n-1)!}{\pi} z^{-n} & n > 0. \end{cases}$$

Note that $Y_n(z)$ is unbounded as $z \to 0$. So it does not satisfy the boundary conditions in (124). We have thus obtained that

$$f_n(r) = \tilde{f}_n(z) = J_n(\sqrt{\lambda_n} r).$$

However, it remains one boundary condition to satisfy: $f_n(R) = 0$, or equivalently $J_n(\sqrt{\lambda_n} R) = 0$. This is a constraint on λ_n, very similar to the constraint $\sin\sqrt{\lambda} L = 0$ in Cartesian geometry.

It turns out that the Bessel function J_n is an oscillatory function, like the sine function, and that it admits an infinite number of zeros. We denote by z_{mn} for $m \geq 1$ the zeros of the Bessel function J_n. Note again that we need two indices to represent them. n is the index of the Bessel function, and m is for the mth zero of that specific Bessel function. The constraint $J_n(\sqrt{\lambda_n} R) = 0$ implies that $\sqrt{\lambda_n} R$ has to be one of these zeros z_{mn}. So for each n, there is an infinite number of solutions λ_{mn} of $J_n(\sqrt{\lambda_n} R) = 0$, and they are given by

$$\lambda_{mn} = \Big(\frac{z_{mn}}{R}\Big)^2, \qquad m \geq 1. \tag{125}$$

Associate to these eigenvalues are eigenvectors $f_{mn}(r)$ (again for each fixed n, there is an infinite number of eigenvectors solution so they need to be labeled by (m, n)), given by

$$f_{mn}(r) = J_n(\frac{z_{mn} r}{R}). \tag{126}$$

To come back to the problem (120), we have thus found the following solutions

$$\lambda_{mn} = \Big(\frac{z_{mn}}{R}\Big)^2, \qquad \phi_{mn1} = J_n(\frac{z_{mn} r}{R}) \cos n\theta, \qquad \phi_{mn2} = J_n(\frac{z_{mn} r}{R}) \sin n\theta, \quad n \geq 0, m \geq 1. \tag{127}$$

The physical modes $\phi_{mnk}(r, \theta)$ are thus parameterized by $n \geq 0$, $m \geq 1$, and $k = 1, 2$ (with the small abuse of notation that $k = 1$ only when $n = 0$). The elementary solutions of (119) are thus given by

$$u_{mn}(t, r, \theta) = e^{-\lambda_{mn} t} J_n(\frac{z_{mn} r}{R})\Big(A_{mn} \cos n\theta + B_{mn} \sin \theta\Big), \quad n \geq 0, m \geq 1, \tag{128}$$

so that by superposition we have

$$u(t, r, \theta) = \sum_{m=1,n=0}^{\infty} e^{-\lambda_{mn} t} J_n(\frac{z_{mn} r}{R})\Big(A_{mn} \cos n\theta + B_{mn} \sin \theta\Big). \tag{129}$$

The initial conditions are thus satisfied if

$$f(r, \theta) = \sum_{m=1,n=0}^{\infty} J_n(\frac{z_{mn} r}{R})\Big(A_{mn} \cos n\theta + B_{mn} \sin \theta\Big) = \sum_{m,n=1}^{\infty} A_{mn}\phi_{mn1}(r, \theta) + B_{mn}\phi_{mn2}(r, \theta). \tag{130}$$

Note that the above sum is nothing but a sum over the $\phi_{mnk}(r, \theta)$, and we know from the abstract theory that the latter form a complete set. It remains to figure out how the coefficients A_{mn} and B_{mn} can be obtained. Here we have to be a bit careful. The abstract theory also tells us that the eigenvectors ϕ_{mnk} are orthogonal in the sense that (104) holds. Note that orthogonality holds for an integral over Ω. This integral, in polar coordinates, is given by

$$\int_0^R \int_0^{2\pi} \phi_{mnk}(r, \theta)\phi_{m'n'k'}(r, \theta) r dr d\theta = 0, \qquad (m, n, k) \neq (m', n', k'). \tag{131}$$

This is the orthogonality condition verified by the spatial modes on a disc. The coefficients A_{mn} and B_{mn} are thus given, using the above orthogonality conditions, by

$$A_{mn} = \frac{\displaystyle\int_0^R \int_0^{2\pi} \phi_{mn1}(r, \theta) f(r, \theta) r dr d\theta}{\displaystyle\int_0^R \int_0^{2\pi} \phi_{mn1}^2(r, \theta) r dr d\theta}, \quad m \geq 1, n \geq 0, \tag{132}$$

and a similar expression for B_{mn} with ϕ_{mn1} above replaced by ϕ_{mn2}. The denominator can be simplified somewhat since the $\phi_{mnk}(r,\theta)$ are explicit in the θ variable so that the integration can be performed.

Let us conclude by a remark on the Bessel functions. Let $n \geq 0$ fixed. For a function $f(r,\theta) = g(r)\cos n\theta$, where $g(r)$ is independent of θ, we find that (130) simplifies to

$$g(r) = \sum_{m=1}^{\infty} A_m J_n(\frac{z_{mn}r}{R}). \tag{133}$$

Since $g(r)$ is arbitrary, this shows that the set of functions $\Big\{J_n(\frac{z_{mn}r}{R})\Big\}_{m\geq 1}$ is *complete*. Each function of r defined on $(0,R)$ can be decomposed over the functions $J_n(\frac{z_{mn}r}{R})$, as for sine functions for instance. Moreover, we observe that (131) for $n = n'$ and $k = k'$ provides that

$$\int_0^R J_n(\frac{z_{mn}r}{R})J_n(\frac{z_{m'n}r}{R})rdr = 0, \quad m \neq m'. \tag{134}$$

Consequently, the coefficients A_m in (133) are given by

$$A_m = \frac{\displaystyle\int_0^r J_n(\frac{z_{mn}r}{R})g(r)rdr}{\displaystyle\int_0^r J_n^2(\frac{z_{mn}r}{R})rdr}. \tag{135}$$

The orthogonality conditions (134) can directly be obtained from the analysis of the Bessel functions. However, as we have seen, they can easily be derived from the abstract theory for the Helmholtz equation (102) on Ω the disc of radius R.

A final word on the weight r in the above integrals (do not forget that the integrals involve rdr and not dr). The equation (123) can be recast as

$$(rf_n')' - \frac{n^2}{r}f + r\lambda f = 0.$$

It is in Sturm Liouville form with $p = \sigma = r$ and $q = n^2/r$. The Sturm Liouville theory for (75) does not quite apply because the boundary condition, $f(0)$ bounded, is not of the type given in (75), and because $p(r) = \sigma(r) = r$ vanishes at r (and also $q(r)$ is not bounded at $r = 0$). But nonetheless, the above equation can be seen as a limiting case of the Sturm Liouville theory, and in any event, this helps explain why the weight $\sigma(r) = r$ is used in the scalar product as it is in e.g. (77).

Let us conclude this lecture by an asymptotic expression for the Bessel functions. As $z \to \infty$, we find that

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}}\cos(z - \frac{\pi}{4} - m\frac{\pi}{2}), \qquad Y_m(z) \sim \sqrt{\frac{2}{\pi z}}\sin(z - \frac{\pi}{4} - m\frac{\pi}{2}).$$

This gives a fairly good idea of the location of the zeros of the Bessel functions as z becomes large.

Lecture 17. This lecture is on modified Bessel functions and cylindrical geometry. Let us consider the Laplace equation $\nabla^2 u = 0$ on a cylinder Ω. In cylindrical coordinates (r,θ,z), this is $\Omega = (0,R)\times(0,2\pi)\times(0,H)$. We consider the Laplace equation with Dirichlet boundary conditions: $u(R,\theta,z) = \gamma(\theta,z)$ known on $(0,2\pi)\times(0,H)$ and $u(r,\theta,0) = u(r,\theta,H) = 0$ for $(r,\theta)\in(0,R)\times(0,2\pi)$ on top and at the bottom of the cylinder.

We can now use separation of variables: $u(r,\theta,z) = f(r)g(\theta)h(z) = \phi(r,\theta)h(z)$. The equation for h is

$$h'' + \lambda h = 0, \qquad h(0) = h(H) = 0.$$

We know the solutions:

$$h_n(z) = \sin\frac{n\pi z}{H}, \qquad \lambda_n = \Big(\frac{n\pi}{H}\Big)^2, \qquad n \geq 1.$$

The equation for $\phi = \phi_n$ is

$$\nabla^2 \phi_n = \lambda_n \phi_n.$$

And we do not have any additional boundary conditions. Note that this is *NOT* the equation $(\nabla^2 + \lambda)\phi = 0$ with $\lambda > 0$. The sign of λ is different, and thus so will be the solutions.

Upon separating the variables r and θ, we find

$$g_n'' + \mu g_n = 0, \quad g_n : 2\pi - \text{periodic}.$$

The constant μ is here to separate r from θ. We thus find as usual that

$$g_{mn}(\theta) = A_{mn} \cos m\theta + B_{mn} \sin m\theta, \qquad \mu_m = m^2, \qquad m \geq 0.$$

The equation for $f(r) = f_{mn}(r)$ is now

$$r^2 f_{mn}'' + r f_{mn}' - \big(r^2 \lambda_n + m^2\big) f_{mn} = 0, \qquad |f_{mn}(0)| < \infty$$

with no additional boundary condition. We introduce the change of variables $\omega = \sqrt{\lambda_n} r$ (because $\lambda_n > 0$) and find that $\tilde{f}_{mn}(\omega) = f_{mn}(r)$ solves

$$\omega^2 \tilde{f}_{mn}'' + \omega \tilde{f}_{mn}' - (\omega^2 + m^2)\tilde{f}_{mn} = 0, \qquad |\tilde{f}_{mn}(0)| < \infty.$$

The solutions to the above ODE are called modified Bessel functions:

$$c_1 K_m(\omega) + c_2 I_m(\omega).$$

It turns out that I_m is bounded at 0, blows up at ∞, and is positive in between, while K_m is bounded at ∞ but blows up at 0. So $c_1 = 0$ above. By superposition, we thus obtain that

$$u(r,\theta,z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} I_m\Big(\frac{n\pi r}{H}\Big) \sin \frac{n\pi z}{H} \cos m\theta + B_{mn} I_m\Big(\frac{n\pi r}{H}\Big) \sin \frac{n\pi z}{H} \sin m\theta.$$

It remains to estimate the coefficients A_{mn} and B_{mn} from $u(R,\theta,z) = \gamma(\theta,z)$. This yields equations

$$A_{mn} I_m\Big(\frac{n\pi R}{H}\Big) = a_{mn}[\gamma], \qquad B_{mn} I_m\Big(\frac{n\pi R}{H}\Big) = b_{mn}[\gamma].$$

Here for instance,

$$a_{mn}[\gamma] = \frac{\displaystyle\int_0^H \int_0^{2\pi} \gamma(\theta,z) \sin \frac{n\pi z}{H} \cos m\theta dz d\theta}{\displaystyle\int_0^H \int_0^{2\pi} \sin^2 \frac{n\pi z}{H} \cos^2 m\theta dz d\theta.}$$

This concludes the derivation.

Lecture 18. We now consider non-homogeneous problems (Chapter 8 in Haberman's book). The first problem, which was already treated in a preceding lecture, is:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(t,x), && 0 < x < \pi, \qquad t > 0 \\ &u(0,t) = A(t), \qquad u(\pi,t) = 0 && t > 0 \\ &u(x,0) = 0 && 0 < x < \pi. \end{aligned} \tag{136}$$

The function $u(x,t)$ is extended by oddness on $(-\pi,\pi)$. We recall that $\dfrac{\partial u}{\partial x}(x,t)$ is an even function in x and that $\dfrac{\partial^2 u}{\partial x^2}(x,t)$ is again odd in x, so that $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \displaystyle\sum_{n=1}^{\infty} \gamma_n(t) \sin nx$, with $\gamma_n(t) = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} \frac{\partial^2 u}{\partial x^2}(y,t) \sin ny dy$.

We integrate by parts with $f'(y) = \dfrac{\partial^2 u}{\partial x^2}(y,t)$ and $g(y) = \sin ny$ so that $f(y) = \dfrac{\partial u}{\partial x}(y,t)$ and $g'(y) = n\cos ny$ to get

$$\gamma_n(t) = \frac{2}{\pi}\int_0^{\pi} -\frac{\partial u}{\partial x}(y,t) n\cos ny dy + \frac{2}{\pi}\frac{\partial u}{\partial x}(y,t)\sin ny\Big|_0^{\pi} = -n\beta_n(t) + 0 - 0.$$

Integrating by parts in the expression for $\beta_n(x)$ with $f'(y) = \dfrac{\partial u}{\partial x}(y,t)$ and $g(y) = \cos ny$ so that $f(y) = u(y,t)$ and $g'(y) = -n\sin ny$, we get

$$\beta_n(t) = \frac{2}{\pi}\int_0^{\pi} u(y,t) n\sin ny dy + \frac{2}{\pi}u(y,t)\cos ny\Big|_0^{\pi} = nb_n(t) + \frac{2}{\pi}[(-1)^n u(\pi,t) - u(0,t)].$$

Note that $b_n(t)$ are the sine Fourier coefficients of $u(x,t)$. Upon calculating the Fourier coefficients of both sides in the first equation of (136), we obtain that

$$b_n'(t) = \gamma_n(t) + \alpha_n(t),$$

since $b_n'(t)$ are the Fourier coefficients of the odd function $\dfrac{\partial u}{\partial t}(x,t)$ and $\gamma_n(t)$ are the Fourier coefficients of the odd function $\dfrac{\partial^2 u}{\partial x^2}(x,t)$. From the results of questions 1 and 2 we get

$$\begin{aligned} & b_n'(t) - \alpha_n(t) = -n\beta_n(t) = -n^2 b_n(t) - n\frac{2}{\pi}[(-1)^n u(\pi,t) - u(0,t)] \\ & = -n^2 b_n(t) + \frac{2n}{\pi}A(t). \end{aligned}$$

Note that $b_n(0)$ are the Fourier coefficients of $u(x,0) = 0$. So $b_n(0) = 0$. This implies that

$$b_n(t) = \int_0^t e^{-n^2(t-s)}\Big(\frac{2n}{\pi}A(s) + \alpha_n(s)\Big)ds.$$

The solution to the PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty}\Big(\int_0^t e^{-n^2(t-s)}\Big(\frac{2n}{\pi}A(s) + \alpha_n(s)\Big)ds\Big)\sin nx.$$

Let us consider the second problem:

$$\begin{aligned} & \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + Q(t,\mathbf{x}), && t > 0, \quad \mathbf{x} \in \Omega \\ & u(t,\mathbf{x}) = g(t,\mathbf{x}) && t > 0, \quad \mathbf{x} \in \partial\Omega \\ & u(0,\mathbf{x}) = \alpha(\mathbf{x}), \quad \frac{\partial u}{\partial t}(0,\mathbf{x}) = \beta(\mathbf{x}) && \mathbf{x} \in \Omega. \end{aligned} \tag{137}$$

This is a wave equation with source terms; here c is a constant. The method of separation of variables does not work. However, in the absence of source terms, we know that the following eigenvalue problem is useful:

$$\begin{aligned} & \nabla^2\phi_n + \lambda_n\phi_n = 0 && \Omega \\ & \phi_n = 0 && \partial\Omega. \end{aligned} \tag{138}$$

We have seen this problem and know that the solutions $\phi_n(\mathbf{x})$, $n \geq 1$, form an orthogonal basis, and that the associated $\lambda_n > 0$ because of the Dirichlet conditions and the use of the Rayleigh quotient. We thus adopt a slightly different from yet very similar strategy to the previous example: we decompose

$$u(t,\mathbf{x}) = \sum_{n=1}^{\infty} A_n(t)\phi_n(\mathbf{x}), \qquad A_n(t) = \frac{\displaystyle\int_{\Omega} u(t,\mathbf{x})\phi_n(\mathbf{x})d\mathbf{x}}{\displaystyle\int_{\Omega}\phi_n^2(\mathbf{x})d\mathbf{x}} \equiv a_n[u(t,\mathbf{x})]. \tag{139}$$

There is no problem in differentiating in time, so we get

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} A_n''(t)\phi_n(\mathbf{x}) = c^2\nabla^2 u + Q. \tag{140}$$

Because of the non-homogeneous conditions $u = g$, we are *not allowed* to differentiate (139) term by term. Rather, we use the only secure technique available to us: integrations by parts. From the orthogonality of the ϕ_n's, we obtain that

$$A_n''(t) = \frac{\displaystyle\int_\Omega (c^2\nabla^2 u + Q)\phi_n d\mathbf{x}}{\displaystyle\int_\Omega \phi_n^2 d\mathbf{x}} = q_n(t) + \frac{\displaystyle\int_\Omega c^2(\nabla^2 u)\phi_n d\mathbf{x}}{\displaystyle\int_\Omega \phi_n^2 d\mathbf{x}}; \qquad q_n(t) = \frac{\displaystyle\int_\Omega Q\phi_n d\mathbf{x}}{\displaystyle\int_\Omega \phi_n^2 d\mathbf{x}}. \tag{141}$$

By integrations by parts (use of the divergence theorem in dimensions higher than one), we can now hope to get an equation for A_n.

As we have seen several times (and this is based on $\nabla\cdot(u\nabla v) = \nabla u\cdot\nabla v + u\nabla^2 v$ and the use of the divergence theorem), we obtain that

$$\int_\Omega \nabla^2 u\phi_n d\mathbf{x} = -\int_\Omega \nabla u\cdot\nabla\phi_n d\mathbf{x} + \int_{\partial\Omega}\phi_n\frac{\partial u}{\partial n}d\sigma(\mathbf{x}) = -\int_\Omega \nabla u\cdot\nabla\phi_n d\mathbf{x},$$

since $\phi_n(\mathbf{x}) = 0$ on $\partial\Omega$. Turning the crank once more, we deduce that

$$-\int_\Omega \nabla u\cdot\nabla\phi_n d\mathbf{x} = \int_\Omega \nabla^2\phi_n u d\mathbf{x} - \int_{\partial\Omega}\frac{\partial\phi_n}{\partial n}u d\sigma(\mathbf{x}) = -\lambda_n\int_\Omega \phi_n u d\mathbf{x} - r_n(t)\int_\Omega \phi_n^2 d\mathbf{x},$$

where we have defined

$$r_n(t) = \frac{\displaystyle\int_{\partial\Omega}\frac{\partial\phi_n}{\partial n}(\mathbf{x})g(t,\mathbf{x})d\sigma(\mathbf{x})}{\displaystyle\int_\Omega \phi_n^2 d\mathbf{x}}. \tag{142}$$

Note that since g and the ϕ_n's are supposed to be known, then so is $r_n(t)$. Wrapping up what we have and using (139), we deduce that

$$A_n''(t) = q_n(t) - c^2\lambda_n A_n(t) - c^2 r_n(t) = -\omega_n^2 A_n(t) + s_n(t), \quad \omega_n = c\sqrt{\lambda_n}, \quad s_n(t) = q_n(t) - c^2 r_n(t).$$

We thus have to solve the ordinary differential equation

$$A_n''(t) + \omega_n^2 A_n(t) = s_n(t). \tag{143}$$

Some calculations involving the method of variation of parameter, tells us that

$$A_n(t) = C_n\cos(\omega_n t) + D_n\sin(\omega_n t) + \int_0^t \frac{s_n(\tau)}{\omega_n}\sin(\omega_n(t-\tau))d\tau. \tag{144}$$

Finally we use the initial conditions in (137), which we had left out so far. We verify that $A_n(0) = C_n$ and $A_n'(0) = \omega_n D_n$. At the same time $A_n(0) = a_n[\alpha]$ and $A_n'(0) = a_n[\beta]$ so that, details spelled out,

$$C_n = \frac{\displaystyle\int_\Omega \alpha(\mathbf{x})\phi_n(\mathbf{x})d\mathbf{x}}{\displaystyle\int_\Omega \phi_n^2(\mathbf{x})d\mathbf{x}}, \qquad D_n = \frac{1}{\omega_n}\frac{\displaystyle\int_\Omega \beta(\mathbf{x})\phi_n(\mathbf{x})d\mathbf{x}}{\displaystyle\int_\Omega \phi_n^2(\mathbf{x})d\mathbf{x}}. \tag{145}$$

Thus, $u(t,\mathbf{x})$ is given by (139), and the $A_n(t)$ are given by (144), where C_n and D_n are given by (145), and $s_n(t) = q_n(t) - c^2 r_n(t)$ with $q_n(t)$ given in (141) and $r_n(t)$ in (142).

As we see, the method of eigenfunction expansion is quite powerful. It allows us to handle arbitrary source terms: volume source Q, boundary source q and initial source (α,β). Of course, it works because the expansion over the ϕ_n's is adapted to the geometry. We have seen in the course of the development that $\nabla^2\phi_n$ is replaced by $-\lambda_n\phi_n$. This step is obviously crucial in ensuring that a *simple* ODE is available for $A_n(t)$. No other expansion than that based on the ϕ_n's will work!

Lecture 19. MIDTERM.

Lecture 20. Spherical harmonics.

Let us consider the wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0, && t>0, \quad \mathbf{x}\in\Omega \\
&u(0,\mathbf{x}) = f(\mathbf{x}), \quad \frac{\partial u}{\partial t}(0,\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x}\in\Omega && \\
&u(t\mathbf{x}) = 0 && \mathbf{x}\in\partial\Omega, \quad t>0,
\end{aligned} \tag{146}$$

when Ω is a ball of radius R in three space dimensions. The usual theory of separation of variables tells us that

$$u(t,\mathbf{x}) = \sum_{p=1}^{\infty} \Big(A_p\cos\omega_p t + B_p \sin\omega_p t\Big)\varphi_p(\mathbf{x}), \tag{147}$$

where the frequencies are given by

$$\omega_p = c\sqrt{\lambda_p}, \tag{148}$$

and the coefficients A_p and B_p are given by

$$A_p = \frac{\displaystyle\int_\Omega f(\mathbf{x})\varphi_p(\mathbf{x})d\mathbf{x}}{\displaystyle\int_\Omega \varphi_p^2(\mathbf{x})d\mathbf{x}}, \qquad B_p = \frac{1}{\omega_p}\frac{\displaystyle\int_\Omega g(\mathbf{x})\varphi_p(\mathbf{x})d\mathbf{x}}{\displaystyle\int_\Omega \varphi_p^2(\mathbf{x})d\mathbf{x}}. \tag{149}$$

The couples (λ_p,φ_p) solve the usual eigenvalue problem

$$\nabla^2\varphi_p(\mathbf{x}) + \lambda_p\varphi_p(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega, \qquad \varphi_p(\mathbf{x}) = 0, \quad \mathbf{x}\in\partial\Omega. \tag{150}$$

The general theory for such problems gives us existence of an infinite number of solutions and tells us that the φ_p form a complete family of functions. When Ω is a ball, we can however be a little more explicit about the structure of these eigenfunctions. This is what we do now.

Since the domain Ω is a ball, it can be written as $\Omega = (0,R)\times(0,2\pi)\times(0,\pi)$ in spherical coordinates (r,θ,ϕ), which are given by

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi. \tag{151}$$

We verify that $h_r = 1$, $h_\theta = r\sin\phi$, and $h_\phi = r$ so that the Laplacian ∇^2 can be represented as

$$\nabla^2 u(r,\phi,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial u}{\partial\phi}\Big) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}. \tag{152}$$

(see also exercise 1.5.21 in Haberman). As a consequence, the couples $(\lambda,\varphi(r,\theta,\phi))$ solve the equation (after multiplication through by $r^2\sin^2\phi$)

$$\sin^2\phi\frac{\partial}{\partial r}\Big(r^2\frac{\partial\varphi}{\partial r}\Big) + \sin\phi\frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial\varphi}{\partial\phi}\Big) + \frac{\partial^2\varphi}{\partial\theta^2} + \lambda r^2\sin^2\phi\,\varphi = 0.$$

Here φ can be decomposed over modes of the form $f(r)q(\theta)g(\phi)$. Separation of variables thus yields that

$$q'' + \nu q = 0, \qquad q(\theta):\ 2\pi - \text{periodic},$$

whose solutions are $q_m = A_m\cos m\theta + B_m\sin m\theta$ with $\nu_m = m^2$ for $m\geq 0$.

We thus find that

$$\sin^2\phi\frac{\partial}{\partial r}\Big(r^2\frac{\partial(fg)}{\partial r}\Big) + \sin\phi\frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial(fg)}{\partial\phi}\Big) - m^2(fg) + \lambda r^2\sin^2\phi\,(fg) = 0.$$

Thus, separating ϕ and r, we get

$$\frac{1}{f}\frac{d}{dr}\Big(r^2\frac{df}{dr}\Big) + \lambda r^2 = -\frac{1}{g\sin\phi}\frac{d}{d\phi}\Big(\sin\phi\frac{dg}{d\phi}\Big) - \frac{m^2}{\sin^2\phi} = \mu.$$

Both ordinary differential equations are "difficult" to solve in the sense that no explicit solutions are available. We concentrate on the equation for $g(\theta)$, given by

$$\frac{d}{d\phi}\Big(\sin\phi\frac{dg}{d\phi}\Big) + \Big(\mu\sin\phi - \frac{m^2}{\sin\phi}\Big)g = 0.$$

The only "boundary" conditions we can impose here is that g be bounded on the whole interval $[0,\pi]$, since there is no reason for singularities to arise at the poles (the north pole $\phi = 0$ and the south pole $\phi = \pi$). The first step is actually a (one-to-one and onto) change of variables

$$x : [0,\pi] \to [-1,1], \qquad x \mapsto x(\phi) = \cos\phi.$$

With an abuse of notation (done in Haberman) we still call $g(x) = g(\theta)$ the function after the change of variables. The chain rule shows that the new $g(x)$ solves the ODE

$$\frac{d}{dx}\Big((1-x^2)\frac{dg}{dx}\Big) + \Big(\mu - \frac{m^2}{1-x^2}\Big)g = 0.$$

It goes beyond the scope of this course to show how one can solve the above equation. Note that it is almost in Sturm Liouville form except that the weights p and σ either vanish or blow up at $x = \pm 1$. In any event, the above ODE admits a bounded solution on the whole interval $[-1,1]$ if and only if

$$\mu = \mu_{mn} = n(n+1), \qquad n \geq m. \tag{153}$$

The associated eigenvectors are called the **spherical harmonics** (or associated Legendre functions) and are denoted by

$$g(x) = g_{mn}(x) = P_n^m(x).$$

Because of their importance in practice, there is an infinite number of formulas for the spherical harmonics in the literature. Let us just mention the Rodrigues formula

$$P_n(x) \equiv P_n^0(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

We behavior for $g(\theta)$ is thus, in the original variables

$$g_{mn}(\theta) = P_n^m(\cos\theta), \qquad \mu_{mn} = n(n+1), \quad n \geq m \geq 0. \tag{154}$$

It now remains to address the equation for $f(r)$. So far, we have two indices, m and n, and one series eigenvalues left to be found, λ. Taking into account boundary conditions, the equation for $f(r)$ is

$$\begin{aligned}&\frac{d}{dr}\Big(r^2\frac{df}{dr}\Big) + (\lambda r^2 - n(n+1))f = 0,\\ &f(R) = 0, \qquad |f(0)| < \infty.\end{aligned} \tag{155}$$

Now that ν and μ are known explicitly, $f(r)$ is in fact a function $f_{mn}(r)$. Since m does not appear explicitly, we're somewhat "lucky" that $f(r)$ is in fact $f_n(r)$ (ie depends on n). We drop the reference to the index n to simplify notation. Yet another change of variables shows that it is useful to introduce

$$f(r) = r^{-1/2}Z_{n+1/2}(\sqrt{\lambda}r).$$

Indeed, we know that $\lambda > 0$ from the general theory and the use of the Rayleigh quotient (since we have Dirichlet boundary conditions). Moreover the function $Z_{n+1/2}$ then satisfies the equation

$$\begin{aligned} &\frac{d}{dz}\Big(z^2\frac{dZ_{n+1/2}}{dz}\Big) + (z^2 - (n+\frac{1}{2})^2)Z_{n+1/2} = 0 \\ &|Z_{n+1/2}(0)| < \infty. \end{aligned} \tag{156}$$

The only solution bounded at the origin is $J_{n+1/2}$ the Bessel function of the first kind of order $n + 1/2$ (a new kind of Bessel functions we had not encountered before). The eigenvalues λ are thus defined such that

$$J_{n+1/2}(\sqrt{\lambda}R) = 0,$$

or in other words

$$\lambda_{nl} = \Big(\frac{z_{nl}}{R}\Big)^2, \tag{157}$$

where z_{nl} is the lth zero of the Bessel function $J_{n+1/2}$.

When we look carefully at what we have done, we have constructed the eigenvectors $\varphi_p(\mathbf{x}) \equiv \varphi_{mnl,k}(r,\theta,\phi)$ which are defined explicitly as

$$\begin{aligned} &\varphi_{mnl,1}(r,\theta,\phi) = r^{-1/2}J_{n+1/2}\Big(\frac{z_{nl}\,r}{R}\Big)\cos m\theta P_m^n(\cos\phi), \\ &\varphi_{mnl,2}(r,\theta,\phi) = r^{-1/2}J_{n+1/2}\Big(\frac{z_{nl}\,r}{R}\Big)\sin m\theta P_m^n(\cos\phi), \\ &l \geq 1, \quad m \geq 0, \quad n \geq m. \end{aligned} \tag{158}$$

These eigenvectors are associated to the eigenvalues $\lambda_p \equiv \lambda_{mnl}$ given by

$$\lambda_{mnl} = \Big(\frac{z_{nl}}{R}\Big)^2, \qquad l \geq 1, \quad m \geq 0, \qquad n \geq m. \tag{159}$$

Note that these eigenvalues do not depend on m and on the index k introduced to define the $\varphi_{mnl,k}$. Since $m \leq n$, we deduce that the redundancy of each eigenvalue is at least of order $2m$ (ie there are at least $2m$ eigenvectors associated to the eigenvalue λ_{mnl}). This comes from the fact that the unit ball has quite a lot of symmetries (by rotation of course).

Let us come back to the decomposition

$$f(\mathbf{x}) = \sum_{p=1}^{\infty} A_p\varphi_p(\mathbf{x}), \qquad A_p = \frac{\displaystyle\int_\Omega f(\mathbf{x})\varphi_p(\mathbf{x})d\mathbf{x}}{\displaystyle\int_\Omega \varphi_p^2(\mathbf{x})d\mathbf{x}}.$$

This comes from the orthogonality condition

$$\int_\Omega \varphi_p(\mathbf{x})\varphi_{p'}(\mathbf{x})d\mathbf{x} = 0, \qquad p \neq p'.$$

In spherical coordinates, $d\mathbf{x} = r^2\sin\phi dr d\theta d\phi$ since this is $h_r h_\theta h_\phi dr d\theta d\phi$. We thus have that

$$\int_0^R\int_0^{2\pi}\int_0^\pi \varphi_{mnl,k}\varphi_{m'n'l',k'}r^2\sin\phi dr d\theta d\phi = 0, \quad (m,n,l,k) \neq (m',n',l',k').$$

Let $\mathcal{S} = \{m \geq 0,\, l \geq 1, n \geq m\, k = 1,2\}$ be the bag of indices useful in spherical harmonics decompositions. Note that there are as many points in $\mathcal{S}$ as in the set $\{p \geq 1\}$ as $\mathbb{N}$ is isomorphic to $\mathbb{N}^2$, hence to $\mathbb{N}^3$, hence to $\mathcal{S}$ with a little bit of imagination. Then the above decomposition may be recast as

$$f(r,\theta,\phi) = \sum_{(mnlk)\in\mathcal{S}} A_{mnl,k}\varphi_{mnl,k}(r,\theta,\phi), \tag{160}$$

where the coefficients are given explicitly by

$$A_{mnl,k} = \frac{\displaystyle\int_0^R \int_0^{2\pi} \int_0^{\pi} f(r,\theta,\phi)\varphi_{mnl,k}(r,\theta,\phi) r^2 \sin\phi dr d\theta d\phi}{\displaystyle\int_0^R \int_0^{2\pi} \int_0^{\pi} \varphi^2_{mnl,k}(r,\theta,\phi) r^2 \sin\phi dr d\theta d\phi}, \ (mnlk) \in \mathcal{S}. \tag{161}$$

As ugly as they are, these decompositions are very useful in practice to solve partial differential equations in spherical geometries.

Lecture 21. One-dimensional Green's functions Consider the problem

$$\begin{aligned} -Lu(x) &= f(x), \qquad x \in (a,b), \quad Lu = \frac{d}{dx}\Big(p(x)\frac{du}{dx}\Big) - q(x)u \\ u(a) &= u(b) = 0. \end{aligned} \tag{162}$$

By the Sturm Liouville theory, assuming that $p(x) > 0$, we know the existence of the complete family ϕ_n, $n \geq 1$, solution of

$$L\phi_n + \lambda_n \phi_n = 0, \qquad \phi_n(a) = \phi_n(b) = 0.$$

Using the decomposition

$$u(x) = \sum_{n=1}^{\infty} a_n[u]\phi_n(x), \qquad a_n[u] = \frac{\displaystyle\int_a^b u(x)\phi_n(x)dx}{\displaystyle\int_a^b \phi_n^2(x)dx},$$

we deduce from the Green formula that

$$0 = \int_a^b [(Lu)\phi_n - (L\phi_n)u]dx = \Big(\int_a^b \phi_n^2 dx\Big)\Big(\lambda_n a_n[u] - a_n[f]\Big).$$

This implies that $a_n[u] = \lambda_n^{-1} a_n[f]$, in other words

$$u(x) = \sum_{n=1}^{\infty} \frac{a_n[f]}{\lambda_n}\phi_n(x),$$

which may be recast as

$$u(x) = \int_a^b f(y)\Big(\sum_{n=1}^{\infty} \frac{\phi_n(y)\phi_n(x)}{\lambda_n \int_a^b \phi_n^2 dx}\Big)dy = \int_a^b G(x,y)f(y)dy. \tag{163}$$

We have defined the Green's function

$$G(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(y)\phi_n(x)}{\lambda_n \int_a^b \phi_n^2 dx}. \tag{164}$$

Note that we easily verify the reciprocity principle

$$G(x,y) = G(y,x). \tag{165}$$

We have thus replaced the differential equation (162) by an integral equation (163). This is the advantage of using Green's functions. However, calculating the Green's function is quite difficult, except in very simple cases. The formula (164) is by no means easy to evaluate.

Delta (δ) function. Before we can get an equation for G, we need to introduce the notion of a delta function. The delta function is not a function but rather a *generalized* function. Its definition is as follows. Take an arbitrary interval $(-a, b)$ with $a, b > 0$ so that 0 is inside the interval. Then *for each* continuous function $f(x)$ on (a, b) we define the delta function as the object that satisfies:

$$\int_{-a}^{b} \delta(x) f(x) dx = f(0). \tag{166}$$

It is useful to see the delta function as the limit of bona fide functions $\delta_\eta(x)$ defined as

$$\delta_\eta(x) = \begin{cases} 0 & |x| > \frac{\eta}{2} \\ \frac{1}{\eta} & |x| < \frac{\eta}{2}. \end{cases} \tag{167}$$

It does not matter how the function is defined at $|x| = \eta/2$. Note that the function is increasingly pinched in the vicinity of 0 as η goes to 0 and that the area of the function $\int_{-a}^{b} \delta_\eta(x) dx = 1$ for all η sufficiently small. For f sufficiently smooth, we verify that

$$\int_{-a}^{b} \delta_\eta(x) f(x) dx = \int_{-a}^{b} \delta_\eta(x) f(0) dx + O(\eta) = f(0) + O(\eta),$$

where $O(\eta)$ means a term of order η, ie small as $\eta \to 0$. In the limit $\eta = 0$, we get (166). The δ function should thus be seen as a mass one concentrated at the point 0.

Let now $H_\eta(x)$ be the antiderivative of $\delta_\eta(x)$ such that $H_\eta(-a) = 0$. We easily verify that

$$H_\eta(x) = \begin{cases} 0 & x < -\frac{\eta}{2} \\ \frac{1}{\eta}(x + \frac{\eta}{2}) & -\frac{\eta}{2} < x < \frac{\eta}{2} \\ 1 & x > \frac{\eta}{2}. \end{cases} \tag{168}$$

In the limit $\eta = 0$, we find that the above function converges to the *Heaviside* function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{169}$$

This shows that $H(x)$ is the "anti derivative" of the δ function, which I repeat is not a function. We thus have the important formula for us

$$H'(x) = \delta(x). \tag{170}$$

This is an equality of generalized functions. The way we use it is as follows. If we have an equation of the form $f' = \delta$, then we know that $f(x) = H(x) + \alpha$, where α is a constant.

Finally, let us define the shifted δ function

$$\int_{z-\alpha}^{z+\beta} f(y) \delta(y - z) dy = f(z). \tag{171}$$

This may be obtained easily from (166) by the change of variables $y \mapsto x = y - z$. We now verify that the antiderivative of the shifted delta function is the shifted Heaviside function:

$$H'(x - z) = \delta(x - z), \tag{172}$$

where, spelling out the details, the shifted delta function is given by

$$H(x - z) = \begin{cases} 0 & x < z \\ 1 & x > z. \end{cases} \tag{173}$$

The latter two equalities are important to us, so I've spelled them out.

Green's function: an equation. Let us come back to (163). If we choose $f(y) = \delta(y-z)$ for some $a < z < b$, then we deduce from (171) that $u(x) = G(x,z)$. This implies that the Green's function satisfies the equation

$$\begin{aligned} -LG(x,z) &= \delta(x-z), \qquad x \in (a,b), \\ G(a,z) &= G(b,z) = 0. \end{aligned} \tag{174}$$

Note that the differential operator L acts on the x variable here. This means that the Green's function $G(x,y)$ gives the solution as a function of x for a source term located at y. Then (163) may be seen as a superposition principle. Once we know the solution for all possible local sources, we sum over sources that are present in the system, and obtain that the solution is the appropriate superposition of Green's functions, solutions corresponding to elementary (localized) sources.

We now can verify the *reciprocity principle*

$$G(x,y) = G(y,x). \tag{175}$$

Indeed consider $G(x,z)$ and $G(x,y)$ and the Green's formula

$$0 = -\int_a^b \Big(LG(x,y)G(x,z) - LG(x,z)G(x,y) \Big) dx = \int_a^b \Big(\delta(x-y)G(x,z) - \delta(x-z)G(x,y) \Big) dx = G(y,z) - G(z,y).$$

The influence at y of a source at x is thus the same as the influence at x of a source at y. This is a non-trivial result.

1D Green's function. Let us consider the simplest of equations for the Green's function

$$\begin{aligned} -G''(x,z) &= \delta(x-z), \qquad x \in (a,b), \\ G(0,z) &= G(L,z) = 0. \end{aligned} \tag{176}$$

We want to solve the above equation in two ways. The first method realizes that the derivative of $-G'$ is a shifted delta function. As a consequence, we have that

$$-G'(x,z) = H(x-z) + \alpha. \tag{177}$$

Note that G' now is a *function*, whereas G'' was a generalized function. The antiderivative of the function G' is therefore a *continuous* function. The Green's function is therefore *continuous*. Integrating once more, we get

$$-G(x,z) = \int_0^x H(y-z)dy + \alpha x + \beta. \tag{178}$$

We find that

$$-G(0,z) = \beta = 0.$$

It remains to use $G(L,z) = 0$ to find α. Note that

$$\int_a^x H(y-z)dy = \begin{cases} 0 & x < z \\ x - z & x > z. \end{cases}$$

Make sure you see where the above comes from. Also draw a graph of the Heaviside function and look at the above integral graphically. As a consequence we have that

$$0 = L - z + \alpha L = 0, \qquad \text{whence } \alpha = \frac{z-L}{L}.$$

This provides the formula

$$G(x,z) = \begin{cases} \dfrac{L-z}{L} x & x < z \\ \dfrac{L-x}{L} z & x > z. \end{cases} \tag{179}$$

The function is thus piecewise linear, continuous, and looks like a tent. Note that the tent is negative in Haberman's book because his source is $-\delta$ with the above notation, while I prefer to stick to a source of $+\delta$. All of Haberman's Green's functions are negative valued. Mine are positive valued. Needless to say, I prefer mine, but watch out for the signs.

Let us now look at the second method, which can be applied to more general settings that the preceding method. We realize that the delta function is concentrated at $x = z$. So there are no sources in (220) for $x < z$ and for $x > z$. As a consequence, the equation $G'' = 0$ holds on these intervals. This means that

$$G(x, z) = ax + b, \quad x < z, \qquad \text{and} \qquad G(x, z) = cx + d, \quad x > z,$$

for some constants a, b, x, d to be determined. Since $G = 0$ at 0 and L, we find that $G(x, z) = ax$ for $0 < x < z$ and $G(x, z) = c(x - L)$ for $z < x < L$.

We now need two equations for a and c. They have to come from the source term. Let us integrate (220) on a small interval $(z - \varepsilon, z + \varepsilon)$. We get

$$-\big(G'(z + \varepsilon) - G'(z - \varepsilon)\big) = \int_{z-\varepsilon}^{z+\varepsilon} \delta(x, z)dx = 1.$$

Passing to the limit $\varepsilon \to 0$, we get that

$$-\big(G'(z^+) - G'(z^-)\big) = 1. \tag{180}$$

Here $G'(z^+)$ is the derivative of $G(z)$ on the right of z and $G'(z^-)$ is the derivative of $G(z)$ on the left of z. Note that the function G' is discontinuous, as the Heaviside function is. The above translates, for our problem, into

$$-(c - a) = 1.$$

Note that (180) generalizes to more complicated problems than (220) and is very useful in many situations. It remains to find another equation for a and c. This is done by remembering that $x \mapsto G(x, z)$ is a continuous function, since it is differentiable (as G' is a function, not a generalized function).

Continuity implies thus that

$$az = c(z - L).$$

Solving for a and c yields $c = -z/L$ and $a = (L - z)/L$ so that (179) holds.

Lecture 22. Multidimensional Green's functions.

Let us come back to Green's functions and consider the problem

$$\begin{aligned} -\nabla^2 u &= f(\mathbf{x}) \qquad && \mathbf{x} \in \Omega \\ u &= 0 && \mathbf{x} \in \partial\Omega. \end{aligned} \tag{181}$$

Here Ω is an arbitrary domain. We want to find an integral representation of the solution to the above equation. We know the existence of solutions to

$$\nabla^2 \phi_n + \lambda_n \phi_n = 0, \quad \text{in} \quad \Omega, \qquad \phi_n = 0 \quad \text{on} \quad \partial\Omega. \tag{182}$$

Using the Green's formula as in 1D, we obtain using the boundary conditions for u and ϕ_n that

$$0 = \int_\Omega (\nabla^2 u \phi_n - \nabla^2 \phi_n u) d\mathbf{x} = \int_\Omega (-f \phi_n + \lambda_n \phi_n u) d\mathbf{x}.$$

Therefore,

$$u(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{\displaystyle\int_\Omega f(\mathbf{y})\phi_n(\mathbf{y})d\mathbf{y}}{\lambda_n \displaystyle\int_\Omega \phi_n^2(\mathbf{y})d\mathbf{y}} \phi_n(\mathbf{x}) = \int_\Omega \Bigg(\sum_{n=1}^{\infty} \frac{\phi_n(\mathbf{y})\phi_n(\mathbf{x})}{\lambda_n \displaystyle\int_\Omega \phi_n^2(\mathbf{z})d\mathbf{z}} \Bigg) f(\mathbf{y})d\mathbf{y}. \tag{183}$$

We may recast this as

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \qquad G(\mathbf{x},\mathbf{y}) = \left(\sum_{n=1}^\infty \frac{\phi_n(\mathbf{y})\phi_n(\mathbf{x})}{\lambda_n \displaystyle\int_\Omega \phi_n^2(\mathbf{z}) d\mathbf{z}} \right) = G(\mathbf{y},\mathbf{x}). \tag{184}$$

To find an equation for the Green's function $G(\mathbf{x},\mathbf{y})$, we need to generalize the notion of delta functions to dimensions higher than one. For instance for $\mathbf{x} = (x, y)$, we defined

$$\delta(\mathbf{x}) = \delta(x)\delta(y),$$

with a straightforward generalization to higher dimensions. An equivalent (more useful) generalization is that for each function $f(\mathbf{x})$ continuous on a domain Ω such that $\mathbf{0} \in \Omega$, we have that

$$\int_\Omega \delta(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = f(\mathbf{0}). \tag{185}$$

More generally, for each domain Ω such that $\mathbf{y} \in \Omega$, we have that

$$\int_\Omega \delta(\mathbf{x}-\mathbf{y}) f(\mathbf{x}) d\mathbf{x} = f(\mathbf{y}). \tag{186}$$

If $f(\mathbf{x})$ is chosen as $\delta(\mathbf{x}-\mathbf{y})$ in (181) and (184), we find that $u(\mathbf{x}) = G(\mathbf{x},\mathbf{y})$ so that $G(\mathbf{x},\mathbf{y})$ has to satisfy the following equation

$$\begin{aligned} -\nabla^2 G(\mathbf{x},\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}) && \mathbf{x} \in \Omega \\ G(\mathbf{x},\mathbf{y}) &= 0 && \mathbf{x} \in \partial\Omega. \end{aligned} \tag{187}$$

We already know that the solution is given by (184).

We can now consider a more general problem

$$\begin{aligned} -\nabla^2 u &= f(\mathbf{x}) && \mathbf{x} \in \Omega \\ u &= g(\mathbf{x}) && \mathbf{x} \in \partial\Omega, \end{aligned} \tag{188}$$

with non-homogeneous boundary conditions. Using the Green's formula, we get

$$\int_\Omega (u\nabla^2 G - G\nabla^2 u) d\mathbf{x} = \int_{\partial\Omega} u \frac{\partial G}{\partial n} d\sigma(\mathbf{x}),$$

since $G(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{x} \in \partial\Omega$. Thus,

$$u(\mathbf{y}) = \int_\Omega f(\mathbf{y}) G(\mathbf{x},\mathbf{y}) d\mathbf{x} - \int_{\partial\Omega} g(\mathbf{x}) \frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) d\sigma(\mathbf{x}). \tag{189}$$

The *same* Green's function shows us how to handle volume as well as boundary source terms.

Green's functions in infinite space.

Let us consider the two dimensional problem first:

$$-\nabla^2 G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^2, \tag{190}$$

for some $\mathbf{y} \in \mathbb{R}^2$. We first observe that by invariance of $\mathbb{R}^2$ by translation, nothing changes if $\mathbf{x}$ is translated by $-\mathbf{y}$, so that $G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}-\mathbf{y},\mathbf{0})$, a function of $\mathbf{x}-\mathbf{y}$, which to simplify we will still denote by $G(\mathbf{x}-\mathbf{y})$. Note that this invariance by translation does not necessarily hold when the problem is defined on an domain $\Omega \subset \mathbb{R}^2$.

So we have a problem for $G(\mathbf{x})$ to solve. The domain $\mathbb{R}^2$ is also invariant by rotation. It is therefore useful to recast the problem using polar coordinates, and solve for $G(r,\theta)$, which solves the problem

$$-\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial G}{\partial r}\Big) - \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} = \delta(\mathbf{x}),$$

for $r > 0$ and $\theta \in (0, 2\pi)$. Note that we have not transformed the δ function into polar coordinates. Because δ is a generalized function, this needs to be done very carefully and we will actually avoid the problem altogether.

Now the $\delta(\mathbf{x})$ function is concentrated at the point $\mathbf{x} = 0$, ie at the point $r = 0$ in polar coordinates. This is a point, which is clearly invariant by rotation. As a consequence, since the domain $\mathbb{R}^2$ and the source $\delta(\mathbf{x})$ are invariant by rotation, we should look for a solution of the above problem that is also invariant by rotation; namely $G = G(r)$ only. We we're after a solution that solves

$$-\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial G}{\partial r}\Big) = \delta(\mathbf{x}).$$

Now one more piece of information. For $r > 0$ there is no source since the delta function is concentrated at $r = 0$, so we have

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{dG}{dr}\Big) = 0, \qquad r > 0.$$

This is an ODE, and we can solve it:

$$G(r) = \alpha \ln r + \beta. \tag{191}$$

β will be an unspecified constant. Note that solutions of (192) are clearly defined up to additive constants. However α is not arbitrary. And to obtain it, we finally remember that there is a source term at $\mathbf{x} = 0$ and that this source term is a delta function. The best way to obtain α is to use the divergence theorem (yes, once more). Let us go back to $G(\mathbf{x}) = G(|\mathbf{x}|)$ and remember that it solves (192) with $\mathbf{y} = 0$ on any ball $B = B(\mathbf{0}, R)$ centered at $\mathbf{0}$ and of radius $R > 0$. Integrating the equation over the ball and using the divergence theorem gives

$$-\int_{\partial B} \frac{\partial G}{\partial n} d\sigma(\mathbf{x}) = -\int_B \nabla^2 G d\mathbf{x} = \int_B \delta(\mathbf{x}) d\mathbf{x} = 1.$$

However $\mathbf{n}$ above is $\mathbf{e}_r$ so that $\dfrac{\partial G}{\partial n} = G'(R)$ at the boundary of B, of perimeter $2\pi R$. This yields

$$1 = -G'(R)2\pi R = -\frac{\alpha}{R}2\pi R = -2\pi\alpha, \qquad \text{whence} \qquad \alpha = \frac{-1}{2\pi}.$$

This provides the result

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}-\mathbf{y}) = \frac{-1}{2\pi}\ln|\mathbf{x}-\mathbf{y}| + \beta. \tag{192}$$

How about in 3D? The same symmetries by translation and rotation provide now the equation

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dG}{dr}\Big) = 0, \qquad r > 0, \tag{193}$$

using the proper expression for the Laplacian in spherical coordinates recalled in the preceding lecture. The solutions are

$$G(r) = \frac{\alpha}{r} + \beta.$$

Note that the first term converges to 0. Here β is still undetermined by $\beta = 0$ is the only choice that makes the Green function converge to 0 as $r \to \infty$. (Note that this was not possible in 2D.) Is remains to find α and we use the same divergence theorem as before:

$$-\int_{\partial B} \frac{\partial G}{\partial n} d\sigma(\mathbf{x}) = -\int_B \nabla^2 G d\mathbf{x} = \int_B \delta(\mathbf{x}) d\mathbf{x} = 1.$$

We still have that $\dfrac{\partial G}{\partial n} = G'(R)$ at the boundary of B, which now has a surface equal to $4\pi R^2$. As a consequence we have

$$1 = -G'(R)4\pi R^2 = \frac{\alpha}{R}^2 4\pi R^2 = 2\pi\alpha, \qquad \text{whence} \qquad \alpha = \frac{1}{4\pi}.$$

This provides the result

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{194}$$

Needless to say, these two explicit expressions for the Green's functions in two and three space dimensions are very useful in applications. I would like to stress one more time that there are very few geometries in which explicit expressions for the Green's function are feasible. Not even in a square (or a cube) can one get a nice expression for the Green's functions.

One such example of a nice geometry is the half space $\mathbb{R}^2_+$ where $y > 0$ with Dirichlet boundary conditions. Let $\mathbf{y} = (y_1, y_2)$ be a source term in $\mathbb{R}^2_+$ and let us denote its image $\mathbf{y}^* = (y_1, -y_2)$ in the lower half space. Let us then construct the function

$$G(\mathbf{x},\mathbf{y}) = \frac{-1}{2\pi}\Big(\ln|\mathbf{x}-\mathbf{y}| - \ln|\mathbf{x}-\mathbf{y}^*|\Big) = \frac{-1}{4\pi}\ln\frac{|\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}-\mathbf{y}^*|^2}. \tag{195}$$

We verify that

$$\begin{aligned} -\nabla^2 G(\mathbf{x},\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}), && \mathbb{R}^2_+ \\ G(\mathbf{x},\mathbf{y}) &= 0 && \partial\mathbb{R}^2_+. \end{aligned} \tag{196}$$

Here $\partial\mathbb{R}^2_+$ are the points $\mathbf{x} = (x_1, 0)$. So we have found the Green's function of the half space with Dirichlet boundary conditions by using the method of images.

We verify that

$$\frac{\partial G}{\partial x_2}\Big|_{x_2=0} = \frac{1}{\pi}\frac{y_2}{(x_1-y_1)^2 + y_2^2}$$

Using (189), we find that the solution to the problem

$$\begin{aligned} -\nabla^2 u(\mathbf{x}) &= 0, && \mathbb{R}^2_+ \\ u(x_1) &= h(x_1) && \partial\mathbb{R}^2_+, \end{aligned} \tag{197}$$

is given by

$$u(y_1,y_2) = \int_{\mathbb{R}} \frac{\partial G}{\partial x_2}(x_1, 0, \mathbf{y}) h(x_1) dx_1 = \int_{\mathbb{R}} \frac{h(x_1)y_2}{\pi((x_1-y_1)^2 + y_2^2)} dx_1. \tag{198}$$

This is a non-trivial result that can be obtained relatively painlessly using Green's functions.

Lecture 23. Time dependent Green's functions.

We will not dwell on time dependent Green's functions and refer to Chapter 11 in Haberman's textbook. Still, let us consider the one-dimensional problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = Q(x,t), \qquad x \in (0, L), \quad t > 0 \\ &u(0,t) = A(t), \quad u(L,t) = B(t), \qquad t > 0 \\ &u(x, 0) = g(x), \qquad x \in (0, L). \end{aligned} \tag{199}$$

Assume first that A, B, and Q vanish. Then the usual method of separation of variables yields

$$u(x,t) = \int_0^L g(y)\Big(\sum_{n=1}^\infty \frac{2}{L}\sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}\Big)dy = \int_0^L g(y)G(x,t;y,0)dy,$$

where we have defined the Green's function

$$G(x,t;y,0)=G(y,t;x,0)\equiv G(x,t;y)=G(y,t;x)=\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}. \tag{200}$$

The Green's function thus gives the influence of a delta source at y and at time 0 on the solution at position x and later time t. By invariance of the equation with respect to time shifts (because the coefficients in the equation are independent of time), we realize that $G(x,t;y,t_0)=G(x,t-t_0;y,0)$, where $G(x,t;y,t_0)$ is the influence of a delta source at y and at time t_0 on the solution at position x and later time t. So with a slight abuse of notation, we write the latter function as $G(x,t-t_0;y)$.

Using the usual properties of the δ function, we thus obtain that G solves the following equation:

$$\begin{aligned}&\frac{\partial G}{\partial t}-k\frac{\partial^2 G}{\partial x^2}=0, \qquad x\in(0,L), \quad t>0\\ &G(x,0;y)=\delta(x-y), \qquad x\in(0,L).\end{aligned} \tag{201}$$

Note that an equivalent way of writing the above equation is:

$$\begin{aligned}&\frac{\partial G}{\partial t}-k\frac{\partial^2 G}{\partial x^2}=\delta(x-y)\delta(t), \qquad x\in(0,L), \quad t>0\\ &G(x,t;y)=0 \quad \text{for } t<0.\end{aligned} \tag{202}$$

To convince yourself of this, find the corresponding differential equations for the Heaviside function $H(t)$.

A final useful property of the Green's function is that $G(x,T-t;y)$ solves the following equation

$$\begin{aligned}&-\frac{\partial}{\partial t}[G(x,T-t;y)]-k\frac{\partial^2}{\partial x^2}[G(x,T-t;y)]=0, \qquad x\in(0,L), \quad t>0\\ &G(x,T-T;y)=\delta(x-y), \qquad x\in(0,L).\end{aligned} \tag{203}$$

This is just because $[u(-t)]'=-u'(-t)$.

We are now ready to go back to (199). Get ready for a somewhat lengthy calculation. We want to show that the solution of (199) is completely determined by knowledge of G. The tools we use are appropriate integrations by parts. However, one has to be careful with the integration in the time domain.

Let $T>0$ and $y\in(0,L)$ be fixed and let us find $u(y,T)$. We use $G(T-t)$ instead of $G(x,T-t;y)$ to simplify notation. We first need to realize that the following Lagrange-type identity holds:

$$\Big(\frac{\partial u}{\partial t}-k\frac{\partial^2 u}{\partial x^2}\Big)G(T-t)-\Big(-\frac{\partial}{\partial t}-k\frac{\partial^2}{\partial x^2}\Big)[G(T-t)]u=Q(t,x)G(T-t). \tag{204}$$

This comes from the volume equations for u and $G(T-t)$. The differentiations in time provide

$$\frac{\partial u}{\partial t}G(T-t)+\frac{\partial}{\partial t}[G(T-t)]u=\frac{\partial}{\partial t}[u(t)G(T-t)], \tag{205}$$

so that

$$\int_0^T\Big(\frac{\partial u}{\partial t}G(T-t)+\frac{\partial}{\partial t}[G(T-t)]u\Big)dt=u(x,T)G(x,0;y)-u(x,0)G(x,T;y)=u(x,T)\delta(x-y)-g(x)G(x,T;y). \tag{206}$$

The Green's identity tells us that

$$\int_0^L\Big(\frac{\partial^2}{\partial x^2}[G(T-t)]u-\frac{\partial^2 u}{\partial x^2}G(T-t)\Big)dx=\frac{\partial G(L,T-t;y)}{\partial x}u(L,t)-\frac{\partial G(0,T-t;y)}{\partial x}u(0,t), \tag{207}$$

thanks to the boundary conditions for G.

We are now ready to integrate (204) in time and in space. The contributions from the differentiations in time yield, thanks to (206), that

$$\int_0^L \int_0^T \Big(\frac{\partial u}{\partial t} G(T-t) + \frac{\partial}{\partial t}[G(T-t)]u\Big) dt dx = u(y,T) - \int_0^L g(x)G(x,T;y)dx. \tag{208}$$

The contributions from the spatial differentiations are given by

$$\int_0^T \int_0^L \Big(\frac{\partial^2}{\partial x^2}[G(T-t)]u - \frac{\partial^2 u}{\partial x^2} G(T-t)\Big) dx dt = \int_0^T \Big(\frac{\partial G(L,T-t;y)}{\partial x} B(t) - \frac{\partial G(0,T-t;y)}{\partial x} A(t)\Big) dt, \tag{209}$$

thanks to the boundary conditions satisfied by $u(t,x)$.

Now, integrating (204) over $(0,T)\times(0,L)$ and using the above results, we get the formula

$$\begin{aligned} u(T,y) \quad &= \quad \int_0^L g(x)G(x,T;y)dx + \int_0^L \int_0^T Q(x,t)G(x,T-t;y)dxdt \\ &\quad + \int_0^T \Big(\frac{\partial G(0,T-t;y)}{\partial x} A(t) - \frac{\partial G(L,T-t;y)}{\partial x} B(t)\Big) dt. \end{aligned} \tag{210}$$

This justifies that claim that $u(t,x)$ is completely determined by knowledge of the Green's function $G(x,t;y)$, even for non-homogeneous PDEs and non-homogeneous boundary conditions.

Lecture 24. Fourier transform.

In many applications, boundaries may be far away from the location of the quantity we are interested in. In such situations, it may be preferable to replace the domain, say $(0,L)$ in one space dimension, by the real line. For instance let us consider the heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad && t>0, \quad x \in \mathbb{R} \\ &u(0,x) = f(x) && x \in \mathbb{R}, \end{aligned} \tag{211}$$

and assume that $f(x)$ goes to 0 as $|x| \to \infty$. We are interested in solutions such that $u(t,x) \to 0$ as well as $|x| \to \infty$.

Since the domain for t and x can be written as $\mathbb{R}^+ \times \mathbb{R}$, there is no reason for not using the method of separation of variables: $u(t,x) = \phi(x)h(t)$ and obtain

$$h' + \lambda k h = 0, \qquad \phi'' + \lambda\phi = 0.$$

However we no longer have boundary conditions for ϕ. We now only impose that $\phi(x)$ be bounded uniformly on $\mathbb{R}$. For λ fixed, we have

$$\phi(x) = ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x}, \quad \lambda > 0; \qquad \phi(x) = ce^{\sqrt{-\lambda}x} + de^{-\sqrt{-\lambda}x}, \quad \lambda < 0; \qquad \phi(x) = e + fx, \quad \lambda = 0.$$

Boundedness of the solution (for both positive and negative x) implies that $c = d = f = 0$. The admissible solutions are thus constant and oscillatory functions. Note however that $\lambda \geq 0$ is arbitrary, and no longer quantized as in the case of bounded domains. The spectrum of the eigenvalue problem $\phi'' + \lambda\phi = 0$ is thus continuous and no longer discrete.

The principle of superposition still holds. We can sum elementary solutions of the form $h_\lambda(t)\phi_\lambda(x)$ and obtain that

$$u(t,x) = \int_0^\infty \Big(A_1(\lambda)e^{i\sqrt{\lambda}x} + B_1(\lambda)e^{-i\sqrt{\lambda}x}\Big)e^{-k\lambda t} d\lambda,$$

at least provided that the above integral makes sense. We recast the above integral as follows. For the first term involving A_1, we change variables $\lambda \to \omega(\lambda) = -\sqrt{\lambda}$ and define $c(\omega) = 2|\omega|A_1(\lambda)$, and for the second term involving B_1 we change variables $\lambda \to \omega(\lambda) = \sqrt{\lambda}$ and introduce $c(\omega) = 2|\omega|B_1(\lambda)$. In both cases, we have that $\omega^2 = \lambda$ so that $2|\omega|d\omega = d\lambda$. We then find that

$$u(t,x) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}e^{-k\omega^2 t}d\omega. \tag{212}$$

It thus remains to figure out what the function $c(\omega)$ is. We now use the initial condition and realize that the following relation has to hold:

$$u(0,x) = f(x) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}d\omega.$$

So there are two questions as usual. Can $f(x)$ be decomposed as a superposition of oscillatory functions? And if yes, how does one calculate $c(\omega)$?

The answer is the theory of Fourier transforms. When $f(x)$ is $2L$-periodic, we have seen that

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} f(y)e^{i\frac{n\pi y}{L}}dy \quad e^{-i\frac{n\pi x}{n}} = \int_{-L}^{L} f(y)\sum_{n=-\infty}^{\infty}\frac{1}{2L}e^{i\frac{n\pi(y-x)}{L}}\ dy.$$

Now using $\Delta x = \frac{1}{2L}$ and sending L to infinity, we obtain, using the approximation

$$\int_{-\infty}^{\infty} g(z)dz = \sum_{n=-\infty}^{\infty} \Delta x g(n\Delta x),$$

and the change of variables $2\pi z = \omega$ that

$$f(x) = \lim_{L\to\infty}\int_{-L}^{L} f(y)\int_{\mathbb{R}} e^{i2\pi(y-x)z}dz\ dy = \int_{\mathbb{R}} e^{-i\omega x}\Big(\frac{1}{2\pi}\int_{\mathbb{R}} f(y)e^{i\omega y}dy\Big)d\omega.$$

This shows the following result. Let $f(y)$ be a sufficiently smooth function decaying sufficiently fast as $|x| \to \infty$. Then we define $F(\omega)$ the **Fourier transform** of $f(x)$ as:

$$F(\omega) = \mathcal{F}[f(x)](\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega x}f(x)dx. \tag{213}$$

We define $f(x)$ the **inverse Fourier transform** of $F(\omega)$ as:

$$f(x) = \mathcal{F}^{-1}[F(\omega)](x) = \int_{\mathbb{R}} e^{-i\omega x}F(\omega)d\omega. \tag{214}$$

Note that the factor $\frac{1}{2\pi}$ appears in the definition of the Fourier transform as well as term $e^{i\omega x}$. We could have chosen to put $\frac{1}{2\pi}$ and $e^{i\omega x}$ in the definition of the inverse Fourier transform. The Fourier transform would then have no factor $\frac{1}{2\pi}$ and a phase $e^{-i\omega x}$ instead. The reason I mention this is that all possible conventions appear in the literature. I have followed Haberman's convention, which is far from being the most natural in fact.

To come back to the solution of the 1D heat equation, we obtain that

$$u(t,x) = \int_{\mathbb{R}}\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega y}f(y)dy\Big)e^{-i\omega x}e^{-k\omega^2 t}d\omega. \tag{215}$$

This may be recast as

$$u(t,x) = \int_{\mathbb{R}} G(t,x-y)f(y)dy, \tag{216}$$

where we have defined the Green's function

$$G(t,x;0,y) = G(t,x-y) = \int_{\mathbb{R}} \frac{1}{2\pi}e^{-i\omega(x-y)}e^{-k\omega^2 t}d\omega. \tag{217}$$

We thus observe that the Green's function is the inverse Fourier transform of the function $F(\omega) = e^{-k\omega^2 t}$, which is a *Gaussian* function. It turns out, and we will admit this important result (see appendix to 10.3 p.453 in Haberman), that the inverse Fourier transform of a Gaussian is a Gaussian (so that the Fourier transform of a Gaussian is also a Gaussian), and more precisely that:

$$\mathcal{F}[e^{-\beta x^2}](\omega) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{\omega^2}{4\beta}}, \qquad \mathcal{F}^{-1}[e^{-\alpha\omega^2}](x) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}}. \tag{218}$$

As a consequence, we find the very important result:

$$\boxed{G(t,x) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.} \tag{219}$$

Thus,

$$u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} f(y) dy.$$

We can show that

$$\lim_{t\to 0^+} G(t,x) = \delta(x),$$

so that the Green's function $G(t, x-y) = G(t,x;y)$ satisfies the following equation

$$\begin{aligned} &\frac{\partial G}{\partial t} = k \frac{\partial^2 G}{\partial x^2} && t > 0, \quad x \in \mathbb{R} \\ &G(0,x;y) = \delta(x-y) && x \in \mathbb{R}. \end{aligned} \tag{220}$$

The same result can be found by choosing $f(x) = \delta(x-y)$ in the equation for $u(t,x)$. As usual the Green's function indicates the influence at time t and position x of a delta source term at position y and time 0. The influence of a source at time s and position y would be given more generally by $G(t,x;s,y)$ for $t > s$. By invariance by translation (in time), we obtain that $G(t,x;s,y) = G(t-s, x-y)$.

Lecture 25. Fourier transform II.

We now look at fundamental properties of the Fourier transform. Let us first consider the shift

$$f(x-\beta) = \int_{\mathbb{R}} e^{-i\omega(x-\beta)} F(\omega) d\omega = \int_{\mathbb{R}} e^{-i\omega x} e^{i\beta\omega} F(\omega) d\omega.$$

As a consequence, we have

$$\mathcal{F}[f(x-\beta)](\omega) = e^{i\beta\omega} \mathcal{F}[f(x)](\omega).$$

This allows us to evaluate the Fourier transform of a derivative. By the above derivation, we have

$$\mathcal{F}\Big[\frac{f(x) - f(x-\beta)}{\beta}\Big](\omega) = \frac{1 - e^{i\beta\omega}}{\beta} \mathcal{F}[f(x)](\omega).$$

In the limit of $\beta \to 0$, we get

$$\mathcal{F}[f'(x)](\omega) = -i\omega \mathcal{F}[f(x)](\omega). \tag{221}$$

The above relation is probably the most important feature of the Fourier transform: it transforms differentiation into a multiplication by $-i\omega$. Obviously, iterating one more time yields

$$\mathcal{F}[f''(x)](\omega) = -\omega^2 \mathcal{F}[f(x)](\omega). \tag{222}$$

So the Fourier transform "diagonalizes" any differential operator with constant coefficients. For non constant coefficients, as in the Sturm Liouville theory, Fourier transforms no longer diagonalize the operator.

Consider an application to the usual PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = f(x), \quad t > 0, \; x \in \mathbb{R}.$$

We verify that

$$\mathcal{F}[\frac{\partial u}{\partial t}] = \frac{\partial}{\partial t}\mathcal{F}[u],$$

since the Fourier transform involves an integration in space. However,

$$\mathcal{F}[k\frac{\partial^2 u}{\partial x^2}] = -k\omega^2\mathcal{F}[u].$$

As a consequence,

$$\frac{\partial}{\partial t}\mathcal{F}[u] = -k\omega^2\mathcal{F}[u], \qquad \text{so that} \qquad \mathcal{F}[u](t,\omega) = \mathcal{F}[u](0,\omega)e^{-kt\omega^2} = \mathcal{F}[f](\omega)e^{-kt\omega^2} \equiv F(\omega)e^{-kt\omega^2}.$$

We have thus shown that

$$u(t,x) = \int_{\mathbb{R}} e^{-i\omega x}F(\omega)e^{-kt\omega^2}d\omega = \mathcal{F}^{-1}[F(\omega)e^{-kt\omega^2}](x). \tag{223}$$

In order to obtain a more explicit expression, we need to understand the inverse Fourier transform of a product. The result is a **convolution** of two functions. More precisely, we have

$$\int_{\mathbb{R}} e^{-ix\omega}F(\omega)G(\omega)d\omega = \int_{\mathbb{R}} e^{-ix\omega}F(\omega)\frac{1}{2\pi}\int_{\mathbb{R}} e^{iy\omega}g(y)dy = \frac{1}{2\pi}\int_{\mathbb{R}} f(x-y)g(y)dy,$$

thanks to the formula for the inverse Fourier transform. Let us define the convolution of f and g as

$$(f*g)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x-y)g(y)dy = \frac{1}{2\pi}\int_{\mathbb{R}} g(x-y)f(y)dy = (g*f)(x). \tag{224}$$

The middle equality is obtained by using the change of variables $y \to (x-y)$. We have thus shown that

$$\mathcal{F}^{-1}[F(\omega)G(\omega)](x) = (f*g)(x), \qquad \text{where} \qquad F = \mathcal{F}[f], \quad G = \mathcal{F}[g]. \tag{225}$$

As a consequence, the solution $u(t,x)$ to the PDE is given by

$$u(t,x) = \Big(\mathcal{F}^{-1}[e^{-kt\omega^2}](x)\Big) * \Big(\mathcal{F}^{-1}[\mathcal{F}[f]](x)\Big)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}^{-1}[e^{-kt\omega^2}](x-y)f(y)dy.$$

We know the explicit inverse Fourier transform of a Gaussian function, so that:

$$\mathcal{F}^{-1}[e^{-kt\omega^2}](x) = \sqrt{\frac{\pi}{kt}}e^{-\frac{x^2}{4kt}}.$$

We thus recover that

$$u(t,x) = \frac{1}{\sqrt{4\pi kt}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}}f(y)dy. \tag{226}$$

Notice that for $f(x) = \delta(x)$ and $F(\omega)$ its Fourier transform, (223) and (226) provide that

$$\mathcal{F}^{-1}[F(\omega)e^{-kt\omega^2}](x) = \frac{1}{2\pi}\mathcal{F}^{-1}[e^{-kt\omega^2}](x),$$

from which we deduce that $F(\omega)$ better be $(2\pi)^{-1}$. This can be easily deduced from the definition of the Fourier transform:

$$\mathcal{F}[\delta(x)](\omega) = \frac{1}{2\pi}, \qquad \mathcal{F}^{-1}[1](x) = 2\pi\delta(x). \tag{227}$$

Note however that we have

$$\mathcal{F}^{-1}[\delta(\omega)](x) = 1, \qquad \mathcal{F}[1](\omega) = \delta(\omega). \tag{228}$$

Parseval relation. If $g^*(x)$ denotes the complex conjugate of $g(x)$ and $G = \mathcal{F}[g]$, then we find that

$$g^*(x) = \int_{\mathbb{R}} e^{ix\omega} G^*(\omega) d\omega \qquad \text{so that} \quad G^*(\omega) = \mathcal{F}[g^*(-x)](\omega).$$

Thus (225) shows that

$$\frac{1}{2\pi} \int_{\mathbb{R}} f(y) g^*(y-x) dy = (f(x) * g^*(-x))(x) = \mathcal{F}^{-1}[F(\omega)G^*(\omega)](x) = \int_{\mathbb{R}} e^{-i\omega x} F(\omega) G^*(\omega) d\omega.$$

If we evaluate the above relation at $x = 0$, we find the Parseval relation

$$\frac{1}{2\pi} \int_{\mathbb{R}} f(x) g^*(x) dx = \int_{\mathbb{R}} F(\omega) G^*(\omega) d\omega, \qquad F = \mathcal{F}[f], \qquad G = \mathcal{F}[g]. \tag{229}$$

With $f = g$, we find

$$\frac{1}{2\pi} \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |F(\omega)|^2 d\omega. \tag{230}$$

This means that the energy of a function $f(x)$ can be expressed in terms of the Fourier transform $F(\omega)$. Up to the factor 2π, the energies in the Fourier domain and in the physical domain agree. This is a remarkable property.

Lecture 26. Fourier transform III.

We consider further applications of the theory of Fourier transforms to partial differential equations. Let us first consider the **non-homogeneous heat equation**

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = q(t,x), \qquad t > 0, \quad x \in \mathbb{R}, \tag{231}$$

with initial conditions $u(0,x) = 0$. Let us define $U(t,\omega)$ and $Q(t,\omega)$ the Fourier transforms of u and q, respectively. Then we obtain, as we have before, the equation in the Fourier domain:

$$\frac{\partial U}{\partial t} + \omega^2 U = Q(t,\omega), \qquad t > 0, \quad \omega \in \mathbb{R}, \tag{232}$$

with the initial condition $U(0,\omega) = 0$. We can solve this non-homogeneous ordinary differential equation:

$$U(t,\omega) = \int_0^t Q(s,\omega) e^{-\omega^2(t-s)} ds.$$

The solution in the physical domain is thus given by

$$u(t,x) = \int_{\mathbb{R}} e^{-i\omega x} \int_0^t Q(s,\omega) e^{-\omega^2(t-s)} ds d\omega = \int_0^t \Big(\int_{\mathbb{R}} e^{-i\omega x} e^{-\omega^2(t-s)} Q(s,\omega) d\omega \Big) ds.$$

Using the inverse Fourier transform of a product, we get

$$u(t,x) = \int_0^t \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}^{-1}[e^{-\omega^2(t-s)}](x-y) q(s,y) dy ds.$$

Using the same Green's function as in the preceding lecture, we obtain

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{|x-y|^2}{4(t-s)}} q(s,y) dy ds = \int_0^t \int_{\mathbb{R}} G(t-s, x-y) q(s,y) dy ds. \tag{233}$$

Laplace equation in a half space. Let us now consider the following half space problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad && x \in \mathbb{R}, \quad y > 0 \\ &u(x,0) = f(x) && x \in \mathbb{R}. \end{aligned} \tag{234}$$

We assume that $u(x,y)$ converges to 0 as $y \to +\infty$. Since the domain is invariant in the x variable, we can define $U(\omega, y)$ as the Fourier transform of $u(x,y)$ in the x variable and obtain the equation

$$\begin{aligned} &-\omega^2 U(\omega,y) + \frac{\partial^2 U}{\partial y^2} = 0, \qquad && \omega \in \mathbb{R}, \quad y > 0 \\ &U(\omega,0) = F(\omega) && \omega \in \mathbb{R}, \end{aligned} \tag{235}$$

where $F = \mathcal{F}_{x\to\omega}[f]$. The only solution bounded at infinity (as $y \to \infty$) of the above ODE is

$$U(\omega,y) = F(\omega)e^{-|\omega|y}.$$

So we have

$$u(x,y) = \int_{\mathbb{R}} e^{-i\omega x} U(\omega,y) d\omega = \int_{\mathbb{R}} e^{-i\omega x} F(\omega) e^{-|\omega|y} d\omega.$$

Using the inverse Fourier transform of a product we find the following convolution

$$u(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}^{-1}[e^{-|\omega|y}](x-z) f(z) dz.$$

Now we have that

$$\mathcal{F}^{-1}[e^{-|\omega|y}](x) = \frac{2y}{x^2+y^2}.$$

Indeed, since $y > 0$,

$$\int_{\mathbb{R}} e^{-i\omega x} e^{-|\omega|y} d\omega = \int_0^{\infty} e^{-i\omega x - \omega y} d\omega + \int_{-\infty}^{0} e^{-i\omega x + \omega y} d\omega = \frac{1}{y+ix} + \frac{1}{y-ix} = \frac{2y}{x^2+y^2}.$$

Consequently, the solution to the Laplace equation (234) in a half space is given by

$$u(x,y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y f(z)}{(x-z)^2 + y^2} dz. \tag{236}$$

Note that for $f(x) = \delta(x)$, we find the Green's function

$$G(x,y) = \frac{1}{\pi} \frac{y}{x^2+y^2}.$$

We verify that $G(x,y)$ converges to $\delta(x)$ as $y \to 0$ so that, as usual, the above Green's function provides the solution at (x,y) for $y > 0$ corresponding to a source located on the boundary at $(0,0)$.

Two dimensional Fourier transform. Let us consider the following heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big), \qquad && (x,y) \in \mathbb{R}^2, \quad t > 0 \\ &u(0,x,y) = f(x,y), && (x,y) \in \mathbb{R}^2. \end{aligned} \tag{237}$$

Since the domain $\mathbb{R}^2$ is invariant by translation both in x and in y, it seems reasonable to assume that the Fourier transform in both of these variables will be useful. Let us first define

$$V(t, \omega_x, y) = \mathcal{F}_{x\to\omega_x}[u(t,x,y)](\omega_x).$$

We find that the latter quantity satisfies the following PDE:

$$\frac{\partial V}{\partial t} = -k\omega_x^2 V + k\frac{\partial^2 V}{\partial y^2}, \qquad (\omega_x, y) \in \mathbb{R}^2, \quad t > 0$$
$$V(0, \omega_x, y) = \mathcal{F}_{x\to\omega_x}[f(x,y)](\omega_x).$$

The derivatives in x have been replaced by multiplications but we still have to solve a PDE (in the variables (t, y)). Let us now define

$$U(t, \omega_x, \omega_y) = \mathcal{F}_{y\to\omega_y}[V(t, \omega_x, y)](\omega_y).$$

Now we find that

$$\begin{aligned} &\frac{\partial U}{\partial t} + k(\omega_x^2 + \omega_y^2)U = 0, && (\omega_x, \omega_y) \in \mathbb{R}^2, \quad t > 0 \\ &U(0, \omega_x, \omega_y) = \mathcal{F}_{y\to\omega_y}[\mathcal{F}_{x\to\omega_x}[f(x,y)]](\omega_x, \omega_y) \equiv F(\omega_x, \omega_y), && (\omega_x, \omega_y) \in \mathbb{R}^2. \end{aligned} \tag{238}$$

Define $\mathbf{x} = (x, y)$ and $\boldsymbol{\omega} = (\omega_x, \omega_y)$. We can define the **two dimensional Fourier transform**

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\omega}}[f(\mathbf{x})](\boldsymbol{\omega}) = \mathcal{F}_{y\to\omega_y}[\mathcal{F}_{x\to\omega_x}[f(x,y)]](\omega_x, \omega_y). \tag{239}$$

We verify that this is also equal to

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\omega}}[f(\mathbf{x})](\boldsymbol{\omega}) = \mathcal{F}_{x\to\omega_x}[\mathcal{F}_{y\to\omega_y}[f(x,y)]](\omega_x, \omega_y).$$

More explicitly, the Fourier transform is thus given by

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\omega}}[f(\mathbf{x})](\boldsymbol{\omega}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{x}\cdot\boldsymbol{\omega}} f(\mathbf{x})d\mathbf{x} \equiv F(\boldsymbol{\omega}) \tag{240}$$

We recall that $d\mathbf{x} = dxdy$. We verify, using the 1D formulas, that the **two dimensional inverse Fourier transform** is given by

$$\mathcal{F}^{-1}_{\boldsymbol{\omega}\to\mathbf{x}}[F(\boldsymbol{\omega})](\mathbf{x}) = \int_{\mathbb{R}^2} e^{-i\boldsymbol{\omega}\cdot\mathbf{x}} F(\boldsymbol{\omega})d\boldsymbol{\omega} \equiv f(\mathbf{x}). \tag{241}$$

Thus, we have defined $U(t, \boldsymbol{\omega}) = \mathcal{F}[u(t, \mathbf{x})](\boldsymbol{\omega})$, where we drop the subscript $\mathbf{x} \to \boldsymbol{\omega}$. We have seen that it verify the equation

$$\frac{\partial U}{\partial t} - k|\boldsymbol{\omega}|^2 U = 0, \qquad U(0, \boldsymbol{\omega}) = F(\boldsymbol{\omega}),$$

whose solution is given by

$$U(t, \boldsymbol{\omega}) = F(\boldsymbol{\omega})e^{-kt|\boldsymbol{\omega}|^2}. \tag{242}$$

Using the inverse Fourier transform of a product:

$$\mathcal{F}^{-1}[F(\omega)G(\omega)] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{y})g(\mathbf{y})d\mathbf{y}, \tag{243}$$

we obtain (as usual) that

$$u(t, \mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}^{-1}[e^{-kt|\boldsymbol{\omega}|^2}](\mathbf{x} - \mathbf{y})f(\mathbf{y})d\mathbf{y}. \tag{244}$$

Now using the 1D result, we find that

$$\int_{\mathbb{R}^2} e^{-i\boldsymbol{\omega}\cdot\mathbf{x}} e^{-k|\boldsymbol{\omega}|^2} d\boldsymbol{\omega} = \sqrt{\frac{\pi}{kt}}e^{-\frac{x^2}{4kt}}\sqrt{\frac{\pi}{kt}}e^{-\frac{y^2}{4kt}} = \frac{\pi}{kt}e^{-\frac{|\mathbf{x}|^2}{4kt}}.$$

Finally, we have the (very important) result

$$u(t, \mathbf{x}) = \int_{\mathbb{R}^2} \frac{1}{4\pi kt}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4kt}} f(y)d\mathbf{y}. \tag{245}$$

Obviously, we can define the two dimensional Green's function

$$G(t, \mathbf{x}) = \frac{1}{4\pi kt}e^{-\frac{|\mathbf{x}|^2}{4kt}}, \tag{246}$$

solution of the two dimensional heat equation with a $\delta(\mathbf{x})$ source term at $\mathbf{x} = (0, 0)$.

Lecture 27. Method of characteristics.

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = f(x), \quad \frac{\partial}{\partial t} u(0,x) = g(x), \tag{247}$$

for $t > 0$ and $x \in \mathbb{R}$.

We verify that

$$\Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big)u = 0, \qquad \Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\Big)u = 0,$$

so that

$$\begin{aligned} \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} &= 0, \qquad & w &= \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}, \\ \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} &= 0, \qquad & v &= \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}. \end{aligned} \tag{248}$$

The first-order partial differential equation

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0, \qquad w(0,x) = h(x), \tag{249}$$

is therefore of interest. Let us now concentrate on this equation; we will come back to the above wave equation later.

We want to solve (249) by the **method of characteristics**. The main idea is to look for solutions of the form

$$w(t, x(t)), \tag{250}$$

where $x(t)$ is a characteristic, solving an ordinary differential equation, and where the rate of change of $\omega(t) = w(t, x(t))$ is prescribed along the characteristic $x(t)$.

Using the chain rule for the expression (250), we obtain:

$$\frac{d}{dt}\Big(w(t,x(t))\Big) = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}.$$

We see that (249) is verified if

$$\frac{dx}{dt} = c, \qquad \frac{d\omega}{dt} \equiv \frac{dw}{dt} = 0. \tag{251}$$

These equations are solved as

$$x(t) = x_0 + ct, \quad w(t, x(t)) = w(0, x_0). \tag{252}$$

This means that the characteristic $x(t)$ is a straight line with speed c and that w remains constant along the characteristic. If we now have an initial condition $w(0,x) = h(x)$, then we deduce that

$$w(t, x_0 + ct) = h(x_0).$$

Since this holds for any x_0, we check that for any $x \in \mathbb{R}$ and $t > 0$, there exists a unique $x_0(t,x) = x - ct$. As a consequence,

$$\boxed{w(t,x) = h(x - ct).} \tag{253}$$

Note that the above equation is well defined, and that it clearly satisfies (249) (check it).

Let us generalize the above procedure to the equation

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} + \alpha v = \beta, \qquad v(0,x) = P(x), \tag{254}$$

still for $t > 0$ and $x \in \mathbb{R}$. We observe that

$$\frac{d}{dt}\Big(v(t,x(t))\Big) = \frac{\partial v}{\partial t} + \frac{dx}{dt}\frac{\partial v}{\partial x},$$

so that the characteristic equations for (255) are

$$\frac{dx}{dt} = -c, \qquad \frac{dv}{dt} + \alpha v = \beta. \tag{255}$$

The solutions are found to be

$$x(t) = x_0 - ct, \qquad v(t, x(t)) = e^{-\alpha t} v(0, x_0) + \frac{\beta}{\alpha}(1 - e^{-\alpha t}). \tag{256}$$

As above, for each (t, x) we can find a unique $x_0(t, x) = x + ct$ so that

$$v(t, x) = e^{-\alpha t} P(x + ct) + \frac{\beta}{\alpha}(1 - e^{-\alpha t}). \tag{257}$$

The above method generalizes to more complicated equations, including for instance α and β functions of t and x. However, it is important to verify that $x_0(t, x)$ is defined uniquely; which is the case when the PDE is linear, but not necessarily the case when the PDE is nonlinear. The reason is the following. We replace the solution of the PDE by the evolution of a characteristic plus the evolution of the field of interest (w or v above) along the characteristic. For each point (t, x), we need to ensure that only one characteristic maps the initial conditions to that point. If two different characteristics join the same point (t, x), then the method provides possibly multiple solutions to the PDE. The method of characteristics then ceases to be valid and a more refined theory is necessary.

Let us consider the quasilinear equation

$$\frac{\partial \rho}{\partial t} + c(t, x; \rho)\frac{\partial \rho}{\partial x} = Q(t, x; \rho), \qquad \rho(0, x) = \rho_0(x). \tag{258}$$

Here the speed $c(t, x; \rho)$ and the source term $Q(t, x, \rho)$ are allowed to depend on ρ so that the above equation is *nonlinear*. Nonetheless the method of characteristics works and provides

$$\frac{dx}{dt} = c(t, x(t); \rho(t, x(t))), \qquad \frac{d\rho}{dt} = Q(t, x(t); \rho(t, x(t))). \tag{259}$$

When the above equations can be solved, they provide solutions to (258). However the solution may no longer be unique. Consider the example with $Q = 0$ and $c = c(\rho)$ and initial condition $\rho(0, x) = \rho_0(x)$. Then,

$$\frac{d\rho}{dt} = 0, \quad \frac{dx}{dt} = c(\rho),$$

so that $x = x_0 + c(\rho)t$ and as a consequence

$$\rho(t, x(t)) = \rho(0, x_0) = \rho(t, x_0 + c(\rho)t) = \rho(t, x_0 + c(\rho_0(x_0))t).$$

If there is a unique $x_0(t, x)$ such that

$$x = x_0 + c(\rho_0(x_0))t, \tag{260}$$

then the solution to the nonlinear PDE is given by

$$\rho(t, x) = \rho_0(x_0(t, x)). \tag{261}$$

We can verify that (260) admits a unique solution when $c(\rho_0(x))$ is a smooth, positive, strictly increasing function of x. Indeed $x_0 + c(\rho_0(x_0))t$ is then a smooth strictly increasing function of x_0 and there cannot be $x_0 \neq x_1$ such that $x_0 + c(\rho_0(x_0))t = x_1 + c(\rho_0(x_1))t$.

However, when $c(\rho(x_0)) > c(\rho(x_1))$ for some $x_0 < x_1$, then there is a time

$$t = \frac{x_1 - x_0}{c(\rho(x_0)) - c(\rho(x_1))} > 0,$$

such that $x_0 + c(\rho_0(x_0))t = x_1 + c(\rho_0(x_1))t$. This means that the *characteristics* emanating from x_0 and x_1 *cross* (and meet at time t). Then the method of characteristics states that $\rho(t, x) = \rho_0(x_0) = \rho_0(x_1)$.

However, the latter two terms may well be different and the method of characteristics then provides a *multivalued* solution, i.e., not a bona fine solution to the PDE (258). The method of characteristics fails to provide a unique solution to the PDE. This is typically the situation when shock waves develop, and additional mathematics needs to be developed. Note that the above time of crossing is positive. We can show that for sufficiently small times, the method of characteristics provides the correct solution to (258) provided that the speed c, the source Q, and the initial condition $\rho_0(x)$ are sufficiently smooth.

D'Alembert principle. Let us come back to the wave equation (247). We have seen that

$$w(t,x) = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = h(x-ct),$$

and that

$$v(t,x) = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = P(x+ct).$$

So we obtain e.g. that

$$2\frac{\partial u}{\partial t} = h(x-ct) + P(x+ct).$$

Define by $-F$ an antiderivative of $h/(2c)$ and by G an antiderivative of $P/(2c)$ and we get that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\Big(F(x-ct) + G(x+ct)\Big),$$

so that

$$u(t,x) = F(x-ct) + G(x+ct), \tag{262}$$

up to a constant that we absorb into one of the antiderivatives. This is the d'Alembert formula. Now, initial conditions show that

$$F(x) + G(x) = f(x), \qquad -c(F'(x) - G'(x)) = g(x),$$

that is,

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(y)dy, \qquad G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(y)dy.$$

The above formula (262) shows that information propagates with finite speed (speed c) for the wave equation. One part propagates with speed c to the right ($F(x-ct)$), the other part to the left ($G(x+ct)$) with the same speed.

The principle of propagation of information with speed c generalizes to higher dimensions. However the method of characteristics does not work for the wave equation in dimensions higher than one. In higher dimensional cases, wave propagation cannot be replaced by information propagating along characteristics.

Conservation of particles and method of characteristics. Recall that the conservation of particles with density $\rho(t,\mathbf{x})$ and velocity $\mathbf{v}(\mathbf{x})$ reads

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\mathbf{v}\rho) = 0, \tag{263}$$

which can be augmented with initial conditions of the form $\rho(0,\mathbf{x}) = \rho_0(\mathbf{x})$. Here $\mathbf{x}\in\mathbb{R}^d$, where d is space dimension. Assuming that the velocity $\mathbf{v}(\mathbf{x})$ is given and sufficiently smooth, we may recast the above equation as

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + (\nabla\cdot\mathbf{v})\rho = 0, \qquad \rho(0,\mathbf{x}) = \rho_0(\mathbf{x}). \tag{264}$$

This (possibly) multi-dimensional equation can now be solved by the method of characteristics. We find that

$$\frac{d}{dt}\rho(t,\mathbf{x}(t)) + (\nabla\cdot\mathbf{v})\rho(t,\mathbf{x}(t)) = 0, \qquad \frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t)). \tag{265}$$

The second equation is a well posed ordinary differential equation, assuming that $\mathbf{v}(\mathbf{x})$ is smooth and well-behaved. For any point $(T, \mathbf{x})$ we can run the characteristic backwards for time T to find $\mathbf{x}_0(T, \mathbf{x})$. More precisely, define

$$\frac{d\mathbf{y}(t)}{dt} = -\mathbf{v}(\mathbf{y}(t)), \qquad \mathbf{y}(0) = \mathbf{x}. \tag{266}$$

When we define $\mathbf{x}_0 = \mathbf{y}(T)$, and verify that the solution to

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

gives $\mathbf{x} = \mathbf{x}(T)$. It remains to solve the other ODE along the characteristic and find

$$\rho(T, \mathbf{x}) = e^{-\int_0^T (\nabla\cdot\mathbf{v})(s,\mathbf{x}(s))ds} \rho(\mathbf{x}_0), \tag{267}$$

where $\mathbf{x}(s)$ is the characteristic such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}$.

The physical interpretation of the above result is the following. When characteristics move closer to each other, then $\nabla \cdot \mathbf{v}$ is negative. Because the volume occupied by the particles shrinks, the density of particles has to increase so that the total number of particles is conserved. When characteristics move away from each other (they diverge!), then the divergence $\nabla \cdot \mathbf{v} > 0$. Because the volume occupied by the particles increases, the density has to decrease so that the total number of particles is conserved.

Recommended exercises for this section: 12.2.2;3;5;10, 12.6.1;2;3;8.